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Almost contact metric 3-structures with torsion

Some preliminaries on almost contact manifolds.

An **almost contact manifold** is a $(2n+1)$-dimensional manifold M endowed with

- a field φ of endomorphisms of the tangent spaces
- a global 1-form η
- a global vector field ξ, called *Reeb vector field*

such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1.$$

Given an almost contact manifold $(M^{2n+1},\varphi,\xi,\eta)$, one can define on $M^{2n+1}\times\mathbb{R}$ an almost complex structure J by setting

$$J(X, f\,d/dt) = (\varphi X - f\xi, \eta(X)d/dt)$$

for all $X\in\Gamma(TM^{2n+1})$ and $f\in C^\infty(M^{2n+1}\times\mathbb{R})$.

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Then (φ,ξ,η) is said to be **normal** if the almost complex structure J is integrable, that is $[J,J] \equiv 0$. This happens if and only if

$$N := [\varphi,\varphi] + 2\eta\otimes\xi \equiv 0.$$

Given an almost contact structure (φ,ξ,η) on M, there exists a Riemannian metric g such that

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all $X,Y \in \Gamma(TM)$.

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If we fix such a metric, (M,φ,ξ,η,g) is called an **almost contact metric manifold** and we can define the *fundamental 2-form* Φ by

$$\Phi(X,Y) = g(X,\varphi Y).$$

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- An almost contact metric manifold such that $N \equiv 0$ and $d\Phi = 0$, $d\eta = 0$ is said to be a **cosymplectic manifold**.

Definition (Blair, *J. Differential Geom. 1967*).

If $\mathrm{d}\Phi = 0$ and $N \equiv 0$ then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a **quasi-Sasakian manifold**.

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An almost contact manifold $(M^{2n+1},\varphi,\xi,\eta)$ is said to be of

- rank $2p$ if $(d\eta)^p \neq 0$ and $\eta\wedge(d\eta)^p=0$ on M^{2n+1}, for some $p \leq n$
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Theorem (Blair, Tanno)
No quasi-Sasakian manifold has even rank.

Remarkable subclasses of quasi-Sasakian manifolds are given by

- **Sasakian manifolds** ($d\eta = \Phi$, maximal rank $2n+1$)
- **cosymplectic manifolds** ($d\eta = 0$, $d\Phi = 0$, minimal rank 1).

3-structures

An **almost contact 3-structure** on a manifold M is given by three distinct almost contact structures (φ_1,ξ_1,η_1), (φ_2,ξ_2,η_2), (φ_3,ξ_3,η_3) on M satisfying the following relations, for an even permutation (i,j,k) of $\{1,2,3\}$,

$$\varphi_k = \varphi_i\varphi_j - \eta_j\otimes\xi_i = -\varphi_j\varphi_i + \eta_i\otimes\xi_j,$$

$$\xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \quad \eta_k = \eta_i\circ\varphi_j = -\eta_j\circ\varphi_i.$$

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One can prove that (Kuo, Udriste)

- $\dim(M) = 4n+3$ for some $n\geq 1$,
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One can prove that (Kuo, Udriste)

- $\dim(M) = 4n+3$ for some $n\geq 1$,
- the structural group of TM is reducible to $Sp(n)\times I_3$.

If each almost contact structure is *normal*, then the 3-structure is said to be **hyper-normal**.

Moreover, there exists a Riemannian metric g compatible with each almost contact structure (φ_i,ξ_i,η_i), i.e. satisfying

$$g(\varphi_i X,\varphi_i Y) = g(X,Y) - \eta_i(X)\eta_i(Y)$$

for each $i \in \{1,2,3\}$.

Then we say that $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ is an **almost 3-contact metric manifold**.

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Remarkable examples of (hyper-normal) almost 3-contact metric manifolds are given by

- **3-Sasakian manifolds** (each structure (φ_i,ξ_i,η_i) is Sasakian)
- **3-cosymplectic manifolds** (each structure (φ_i,ξ_i,η_i) is cosymplectic)
- **3-quasi-Sasakian manifolds** (each structure (φ_i,ξ_i,η_i) is quasi-Sasakian).

“Foliated” 3-structures

Let $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ be an almost 3-contact (metric) manifold. Putting

$$\mathcal{V} := \mathrm{span}\{\xi_1,\xi_2,\xi_3\} \quad \text{and} \quad \mathcal{H} := \ker(\eta_1)\cap\ker(\eta_2)\cap\ker(\eta_3),$$

we have the (orthogonal) decomposition

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p.$$

$\mathcal{V}$ is called *Reeb distribution* (or *vertical distribution*), whereas $\mathcal{H}$ *horizontal distribution*.

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Question (Kuo-Tachibana, 1970)
Is the distribution $\mathcal{V}$ integrable?

The answer is negative, in general.

Example (C. M. - De Nicola - Dileo, *Ann. Glob. Anal. Geom. 2008*)

Let $\mathfrak{g}$ be the 7-dimensional Lie algebra with basis $\{X_1, X_2, X_3, X_4, \xi_1, \xi_2, \xi_3\}$ and Lie brackets given by

$$[X_h, X_k] = [X_h, \xi_i] = 0, \quad [\xi_1, \xi_2] = [\xi_2, \xi_3] = [\xi_3, \xi_1] = X_1.$$

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$ and let us define three tensor fields φ_1, φ_2, φ_3 on G, and three 1-forms η_1, η_2, η_3, by putting, for all $i,j,k \in \{1,2,3\}$, $\varphi_i \xi_j = \varepsilon_{ijk} \xi_k$ and

$$\begin{array}{llll} \varphi_1 X_1 = X_2, & \varphi_1 X_2 = -X_1, & \varphi_1 X_3 = X_4, & \varphi_1 X_4 = -X_3, \\ \varphi_2 X_1 = X_3, & \varphi_2 X_2 = -X_4, & \varphi_2 X_3 = -X_1, & \varphi_2 X_4 = X_2, \\ \varphi_3 X_1 = X_4, & \varphi_3 X_2 = X_3, & \varphi_3 X_3 = -X_2, & \varphi_3 X_4 = -X_1, \end{array}$$

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- (φ_i,ξ_i,η_i) is an almost contact 3-structure on G

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- $(\varphi_i, \xi_i, \eta_i)$ is an almost contact 3-structure on G
- by construction the Reeb distribution is not integrable.

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- by construction the Reeb distribution is not integrable.

Remark

$(G,\varphi_i,\xi_i,\eta_i)$ <u>is not hyper-normal</u> since $N_1(\xi_1,\xi_2) = -X_1+X_2 \neq 0$.

It is known that the Reeb distribution $\mathcal{V} := \mathrm{span}\{\xi_1,\xi_2,\xi_3\}$ is integrable in 3-Sasakian manifolds and in 3-cosymplectic manifolds.

manifold	*space of leaves*	
3-Sasakian	Quaternionic-Kähler	Ishihara (*Kodai Math. Sem. Rep. 1973*) Boyer-Galicki-Mann (*J. Reine Angew. Math. 1994*)
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Question

Does the hyper-normality of the almost contact 3-structure imply the integrability of $\mathcal{V}$?

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Question

Does the hyper-normality of the almost contact 3-structure imply the integrability of $\mathcal{V}$?

Rather surprisingly, the answer is NO.

Example (C. M., *Differential Geom. Appl. 2009*)

Let $\mathfrak{g}$ be the $(4n+3)$-dimensional Lie algebra with basis $\{E_1,\dots,E_{4n}, \xi_1,\xi_2,\xi_3\}$ and Lie brackets defined by

$$[\xi_1,\xi_2] = E_1, \; [\xi_2,\xi_3] = E_{n+1}, \; [\xi_2,\xi_3] = E_{2n+1}, \; [E_h,E_k] = [\xi_i,X_k] = 0.$$

Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. We define on G a left-invariant almost contact 3-structure (φ_i,ξ_i,η_i) by putting $\varphi_i\xi_j = \varepsilon_{ijk}\xi_k$ and

$$\varphi_1 E_h = E_{n+h}, \; \varphi_1 E_{n+h} = -E_h, \; \varphi_1 E_{2n+h} = E_{3n+h}, \; \varphi_1 E_{3n+h} = -E_{2n+h},$$
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and setting $\eta_i(E_k) = 0$ and $\eta_i(\xi_j) = \delta_{ij}$. Then (φ_i,ξ_i,η_i) is a hyper-normal almost contact 3-structure on G though the Reeb distribution is not integrable.

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and setting $\eta_i(E_k) = 0$ and $\eta_i(\xi_j) = \delta_{ij}$. Then (φ_i,ξ_i,η_i) is a hyper-normal almost contact 3-structure on G though the Reeb distribution is not integrable.

Therefore

$$\text{hyper-normality of the 3-structure} \ \not\Rightarrow \ \text{integrability of } \mathcal{V}.$$

Conversely,

$$\text{hyper-normality of the 3-structure} \;\not\Leftarrow\; \text{integrability of } \mathcal{V}.$$

Example

Let $\mathfrak{g}$ be the 7-dimensional Lie algebra with basis $\{X_1,X_2,X_3,X_4,\xi_1,\xi_2,\xi_3\}$ and Lie brackets defined by

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and setting $\eta_i(X_h)=0$ and $\eta_i(\xi_j)=\delta_{ij}$. Then $(G,\varphi_i,\xi_i,\eta_i)$ is an almost 3-contact manifold which is not hyper-normal. Nevertheless, $\mathcal{V}$ is integrable.

Definition

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Theorem (C. M., *Different. Geom. Appl. 2009*)

Let $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ be an almost 3-contact metric manifold. Then any two of the following conditions imply the other one:

- (i) $\mathcal{V}:=\mathrm{span}\{\xi_1,\xi_2,\xi_3\}$ is integrable;
- (ii) each Reeb vector field is an infinitesimal automorphism with respect to the horizontal distribution $\mathcal{H}$;
- (iii) $(\mathcal{L}_{\xi_i} g)|_{H\times V} = 0$ for all $i\in\{1,2,3\}$.

Moreover, if any two, and hence all, of the above conditions hold, then $\mathcal{V}$ defines a totally geodesic foliation of M^{4n+3}.

- The most famous example of foliated almost 3-contact manifolds is given by *3-Sasakian manifolds*. Indeed, in any 3-Sasakian manifold

$$[\xi_i,\xi_j] = 2\xi_k.$$

- Another important class is given by *3-cosymplectic manifolds*, where

$$[\xi_i,\xi_j] = 0.$$

- A more general class is given by *3-quasi-Sasakian manifolds*.

3-quasi-Sasakian manifolds

A **3-quasi-Sasakian manifold** is an almost 3-contact metric manifold $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ such that each structure is quasi-Sasakian, that is for each $i\in\{1,2,3\}$ $N_i\equiv 0$ and $d\Phi_i = 0$, where

$$N_i := [\varphi_i,\varphi_i] + 2\eta_i\otimes\xi_i$$

and

$$\Phi_i(X,Y) := g(X,\varphi_i Y).$$

Some recent results on 3-quasi-Sasaki manifolds are obtained in

- C. M., De Nicola, Dileo, *3-quasi-Sasakian manifolds*, Ann. Glob. Anal. Geom. (2008)
- C. M., De Nicola, Dileo, *The geometry of 3-quasi-Sasakian manifolds*, Internat. J. Math. (2009)

Theorem 1

Let $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ be a 3-quasi-Sasakian manifold. Then the Reeb distribution $\mathcal{V}:=\mathrm{span}\{\xi_1,\xi_2,\xi_3\}$ defines a Riemannian foliation with totally geodesic leaves, and the Reeb vector fields obey to the rule

$$[\xi_i,\xi_j] = c\xi_k,$$

for some $c\in\mathbb{R}$. Moreover, M^{4n+3} is 3-cosymplectic if and only if $c=0$.

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Sub-classes of the 3-quasi-Sasakian manifolds are given by the 3-Sasakian manifolds ($c=2$) e by the 3-cosymplectic manifolds ($c=0$).

Nevertheless there are also examples of 3-quasi-Sasakian manifolds which are neither 3-Sasakian nor 3-cosymplectic.

The rank of a 3-quasi-Sasakian manifold

In a 3-quasi-Sasakian manifold one has, in principle, the three odd ranks r_1, r_2, r_3 associated to the 1-forms η_1, η_2, η_3, since we have three distinct, although related, quasi-Sasakian structures.

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We have proved that $r_1 = r_2 = r_3$.

Theorem 2
Let $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ be 3-quasi-Sasakian manifold. Then the almost contact structures (φ_1,ξ_1,η_1), (φ_2,ξ_2,η_2), (φ_3,ξ_3,η_3) have the same rank, which we call *the rank* of the 3-quasi-Sasakian manifold M^{4n+3}, and

rank(M) = 1 if M is 3-cosymplectic (c=0)
rank(M) = $4l+3$, $l \leq n$, in the other cases ($c \neq 0$)

Furthermore, M is of maximal rank if and only if it is 3-α-Sasakian (i.e. $d\eta_i = \alpha\Phi_i$ for each $i = 1,2,3$).

Theorem 3

Let $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$ with $[\xi_i,\xi_j]=2\xi_k$. Then M^{4n+3} is locally a Riemannian product of a 3-Sasakian manifold $\mathcal{S}^{4l+3}$ and a hyper-Kähler manifold $\mathcal{K}^{4m}$, where $m=n-l$.

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Theorem 4
Every 3-quasi-Sasakian manifold has non-negative scalar curvature

$$\tfrac{1}{2}c^2(2n+1)(4l+3),$$

where $\dim(M)=4n+3$, $\mathrm{rank}(M)=4l+3$ and $[\xi_i,\xi_j]=c\xi_k$.

Furthermore, any 3-quasi-Sasakian manifold is Einstein if and only if it is 3-α-Sasakian (strictly positive scalar curvature) or 3-cosymplectic (Ricci-flat).

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- Such results are peculiar to the 3-quasi-Sasakian setting, since they do not hold in general for a single quasi-Sasakian structure on a manifold M^{2n+1}.

3-structures with torsion

Another class of foliated almost 3-contact manifolds is given by the "almost 3-contact metric manifolds with torsion".

Definition

A linear connection ∇ on a Riemannian manifold (M,g) is said to be a **metric connection with torsion** if $\nabla g = 0$ and the torsion tensor T, defined as

$$T(X,Y,Z) = g(T^{\nabla}(X,Y),Z),$$

is a 3-form.

Riemannian manifolds admitting a metric connection with totally skew-symmetric torsion recently become of interest in Theoretical and Mathematical Physics, especially in

- supersymmetry theories
- supergravity
- string theory

Of particular interest are **hyper-Kähler manifolds with torsion** (HKT) and **quaternionic-Kähler manifolds with torsion** (QKT)

- A **HKT** manifold is a hyper-Hermitian manifold $(M^{4n}, J_1, J_2, J_3, g)$ which admits a metric connection with torsion ∇ such that $\nabla J_1 = \nabla J_2 = \nabla J_3 = 0$.

- Likewise, a **QKT** manifold is an almost quaternionic-Hermitian manifold (M^{4n}, Q, g) admitting a metric connection with torsion ∇ such that $\nabla Q \subset Q$ and

$$T(X,Y,Z) = T(J_iX,J_iY,Z) + T(J_iX,Y,J_iZ) + T(X,J_iY,J_iZ),$$

for all $X,Y,Z \in \Gamma(TM^{4n})$ and $i \in \{1,2,3\}$, where $\{J_1,J_2,J_3\}$ is an admissible basis which locally spans the almost quaternionic structure Q.

- I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) **42** (2006), 5-84.

Question

What is a possible generalization in odd dimension of the notion of hyper-Kähler structure with torsion?

Theorem (Friedrich - Ivanov, *Asian J. Math. 2002*)

An almost contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ admits a metric connection ∇ with totally skew-symmetric torsion T such that $\nabla\xi = \nabla\eta = \nabla\varphi = 0$ if and only if ξ is a Killing vector field and the tensor N' given by

$$N'(X,Y,Z) := g(N(X,Y),Z) = g([\varphi,\varphi](X,Y) + \mathrm{d}\eta(X,Y)\xi, Z)$$

is skew-symmetric. The connection ∇ is explicitly given by

$$g(\nabla_X Y,Z) = g(\nabla^g{}_X Y,Z) + \tfrac{1}{2}T(X,Y,Z)$$

with

$$T = \eta\wedge\mathrm{d}\eta + \mathrm{d}^\varphi\Phi + N' - \eta\wedge(\mathrm{i}_\xi N'),$$

where $\mathrm{d}^\varphi\Phi$ denotes the "φ-twisted" derivative defined by $\mathrm{d}^\varphi\Phi(X,Y,Z) := -\mathrm{d}\Phi(\varphi X,\varphi Y,\varphi Z)$.

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where $d^{\varphi}\Phi$ denotes the "φ-twisted" derivative defined by $d^{\varphi}\Phi(X,Y,Z) := -d\Phi(\varphi X,\varphi Y,\varphi Z)$.

- In particular, if $(M^{2n+1},\varphi,\xi,\eta,g)$ is Sasakian then $N \equiv 0$ and $d\eta = \Phi$ (hence $d^{\varphi}\Phi = 0$), and so

$$T = \eta\wedge d\eta.$$

Using that result, Agricola pointed out that a 3-Sasakian manifold $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ can not admit any metric connection ∇ with totally skew-symmetric torsion such that $\nabla\xi_i = \nabla\eta_i = \nabla\varphi_i = 0$, for each $i\in\{1,2,3\}$.

Indeed by the previous theorem we have that M^{4n+3} admits three connections ∇^1, ∇^2, ∇^3, one for each Sasakian structure $(\varphi_i,\xi_i,\eta_i,g)$, such that

$$\nabla^i\xi_i = \nabla^i\eta_i = \nabla^i\varphi_i = 0 \quad \text{and} \quad T^i = \eta_i\wedge d\eta_i$$

for each $i\in\{1,2,3\}$.

But the problem is that <u>these three connections do not coincide</u> and so the 3-Sasakian structure in question is not preserved by any metric connection with skew-symmetric torsion.

Definition

An **almost 3-contact metric manifold with torsion** is a hyper-normal almost 3-contact metric manifold $(M,\varphi_i,\xi_i,\eta_i,g)$ admitting a linear connection ∇ such that

$$\nabla g = 0, \quad \nabla\eta_1 = \nabla\eta_2 = \nabla\eta_3 = 0, \quad \nabla\xi_1 = \nabla\xi_2 = \nabla\xi_3 = 0,$$

$$(\nabla_X\varphi_1)Y = -c\eta_2(X)\varphi_3 Y^h + c\eta_3(X)\varphi_2 Y^h,$$
$$(\nabla_X\varphi_2)Y = -c\eta_3(X)\varphi_1 Y^h + c\eta_1(X)\varphi_3 Y^h,$$
$$(\nabla_X\varphi_3)Y = -c\eta_1(X)\varphi_2 Y^h + c\eta_2(X)\varphi_1 Y^h,$$

for some $c\in\mathbb{R}$, and whose torsion tensor T satisfies the following conditions:

(i) T is horizontally skew-symmetric,

(ii) $T(X,Y,\xi_i) = T(X,\xi_i,Y) = T(X,\xi_j,\xi_i) = T(\xi_i,\xi_j,X) = 0$ for all $X,Y\in\Gamma(\mathcal{H})$,

(iii) $T(\xi_i,\xi_j,\xi_k) = -c\varepsilon_{ijk}$ for all $i,j,k\in\{1,2,3\}$.

Remark

The conditions

$$(\nabla_X \varphi_1)Y = -\, c\eta_2(X)\varphi_3 Y^h + c\eta_3(X)\varphi_2 Y^h$$
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$$(\nabla_X \varphi_3)Y = -\, c\eta_1(X)\varphi_2 Y^h + c\eta_2(X)\varphi_1 Y^h$$

are equivalent to

$$\nabla\varphi_1 = -c(\eta_2\otimes\varphi_3 - \eta_3\otimes\varphi_2 + (\eta_2\otimes\eta_2+\eta_3\otimes\eta_3)\otimes\xi_1 - \eta_1\otimes\eta_2\otimes\xi_2 - \eta_1\otimes\eta_3\otimes\xi_3)$$
$$\nabla\varphi_2 = -c(\eta_3\otimes\varphi_1 - \eta_1\otimes\varphi_3 - \eta_1\otimes\eta_2\otimes\xi_1 + (\eta_1\otimes\eta_1+\eta_3\otimes\eta_3)\otimes\xi_2 - \eta_3\otimes\eta_2\otimes\xi_3)$$
$$\nabla\varphi_3 = -c(\eta_2\otimes\varphi_3 - \eta_3\otimes\varphi_2 + (\eta_2\otimes\eta_2+\eta_3\otimes\eta_3)\otimes\xi_1 - \eta_1\otimes\eta_2\otimes\xi_2 - \eta_1\otimes\eta_3\otimes\xi_3)$$

- C. M., *3-structures with torsion*, Different. Geom. Appl. **27** (2009), 496–506

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Theorem 1
Let $(M^{4n+3}, \varphi_i, \xi_i, \eta_i, g)$ be a hyper-normal almost 3-contact metric manifold. Then M^{4n+3} is an "almost 3-contact metric manifold with torsion" if and only if

1. $d^{\varphi_1}\Phi_1 = d^{\varphi_2}\Phi_2 = d^{\varphi_3}\Phi_3$ on $\mathcal{H}$,
2. ξ_1, ξ_2, ξ_3 are Killing,
3. the Reeb distribution $\mathcal{V} = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$ is integrable,
4. the tensor fields φ_1, φ_2, φ_3 satisfy the relations

$$\mathcal{L}_{\xi_i}\varphi_j = c\varphi_k.$$

If an "almost 3-contact metric connection with torsion" exists, then it is unique.

Theorem 2

Any almost 3-contact metric manifold with torsion is a *foliated* almost 3-contact manifold. Moreover, the Reeb vector fields obey to the rule

$$[\xi_i,\xi_j] = c\xi_k.$$

The space of leaves (with respect to $\mathcal{V}$) is HKT or QKT according to $c = 0$ or $c \neq 0$, respectively.

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Thus we may divide almost 3-contact metric manifolds with torsion in two classes according to the behavior of the leaves of $\mathcal{V}$: those for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (which corresponds to the case $c \neq 0$) and those for which each leaf of $\mathcal{V}$ is locally an abelian group ($c = 0$).

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- Almost 3-contact metric manifolds with torsion such that $c=2$ are called 3-Sasakian manifolds with torsion.
- Almost 3-contact metric manifolds with torsion such that $c=0$ are called 3-cosymplectic manifolds with torsion.

Corollary 1

The torsion T is totally skew-symmetric if and only if the horizontal distribution $\mathcal{H}$ is integrable.

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Corollary 2
An almost contact metric 3-structure with torsion $(\varphi_i,\xi_i,\eta_i,g,\nabla)$ on M is 3-quasi-Sasakian if and only if the torsion is given by

$$T(X,Y,Z) = c\sum_i \eta_i(X)\Phi_i(Y,Z).$$

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In this case,

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Corollary 1

The torsion T is <u>totally</u> skew-symmetric if and only if the horizontal distribution $\mathcal{H}$ is integrable.

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In this case,

- if $c=0$ then M^{4n+3} is 3-cosymplectic and ∇ coincides with the Levi Civita connection
- if $c=2$ then M^{4n+3} is 3-Sasakian and ∇ coincides with the Biquard connection.

Some open problems

- Classification of foliated almost contact 3-structures
- The class of (foliated) almost 3-contact metric manifolds which are Einstein.
  - Conjecture: the only foliated almost 3-contact metric manifolds which are Einstein are the 3-Sasakian and the 3-cosymplectic manifolds.
  - Example with negative curvature?
- Curvature properties of 3-Sasakian and 3-cosymplectic manifolds with torsion (ongoing paper)

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4. B. Cappelletti Montano, A. De Nicola, *3-Sasakian manifolds, 3-cosymplectic manifolds and Darboux theorem*, J. Geom. Phys. **57** (2007), 2509-20
5. B. Cappelletti Montano, *Curvature of 3-Sasakian manifolds with torsion*, in preparation.

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THANK YOU!

Example

Consider $\mathbb{R}^{4n+3}$ with its global coordinates $x_1,\dots,x_n,y_1,\dots,y_n,u_1,\dots,u_n$, $v_1,\dots,v_n,z_1,z_2,z_3$. Let M be the open submanifold of $\mathbb{R}^{4n+3}$ obtained by removing the points where $\sin(z_2)=0$ and define three vector fields on M

$$\xi_1 := c\partial_1$$
$$\xi_2 := c(\cos(z_1)\cot(z_2)\partial_1 + \sin(z_1)\partial_2 - \cos(z_1)/\sin(z_2)\partial_3)$$
$$\xi_3 := c(-\sin(z_1)\cot(z_2)\partial_1 + \cos(z_1)\partial_2 + \sin(z_1)/\sin(z_2)\partial_3)$$

(where $\partial_i = \partial/\partial z_i$) for some $c \neq 0$, and three 1-forms

$$\eta_1 := c^{-1}(dz_1 + \cos(z_2)dz_3)$$
$$\eta_2 := c^{-1}(\sin(z_1)dz_2 - \cos(z_1)\sin(z_2)dz_3)$$
$$\eta_3 := c^{-1}(\cos(z_1)dz_2 + \sin(z_1)\sin(z_2)dz_3).$$

One has $[\xi_i,\xi_j]=c\xi_k$ and $\eta_i(\xi_j)=\delta_{ij}$.

Define a Riemannian metric g by declaring that the set $\{X_i=\partial/\partial x_i$, $Y_i = \partial/\partial y_i$, $U_i = \partial/\partial u_i$, $V_i = \partial/\partial v_i$, $\xi_1,\xi_2,\xi_3\}$ $(i = 1,\dots,n)$ is a global orthonormal frame.

Moreover, define three tensor fields φ_1, φ_2, φ_3 on M by setting

$$\varphi_i\xi_j = \varepsilon_{ijk}\xi_k$$

$$\varphi_1 X_i = Y_i,\ \ \varphi_1 Y_i = -X_i,\ \ \varphi_1 U_i = V_i,\ \ \varphi_1 V_i = -U_i,$$
$$\varphi_2 X_i = U_i,\ \ \varphi_2 Y_i = -V_i,\ \ \varphi_2 X_3 = -X_1,\ \ \varphi_2 V_i = Y_i,$$
$$\varphi_3 X_i = V_i,\ \ \varphi_3 Y_i = U_i,\ \ \varphi_3 U_i = -Y_i,\ \ \varphi_3 V_i = -X_i.$$

One can prove that $(M,\varphi_i,\xi_i,\eta_i,g)$ is a 3-quasi-Sasakian manifold, which is

- neither 3-cosymplectic, since the Reeb vector fields do not commute,
- nor 3-Sasakian, since it admits Darboux-like coordinates

Theorem (C. M. - De Nicola, *J. Geom. Phys*. *2007*)
A 3-Sasakian manifold can not admit a Darboux-like coordinate system.

For "Darboux-like coordinate system" we mean local coordinates $x_1,\ldots,x_{4n}$, z_1,z_2,z_3 with respect to which, for each $i \in \{1,2,3\}$, $\Phi_i = d\eta_i$ has constant components and $\xi_i = a^1{}_i\,\partial/\partial z_1 + a^2{}_i\,\partial/\partial z_2 + a^3{}_i\,\partial/\partial z_3$, where $a^j{}_i$ are functions depending only on the coordinates z_1,z_2,z_3.

Furthermore, $(M,\varphi_i,\xi_i,\eta_i,g)$ is η-Einstein, i.e. the Ricci tensor is of the form

$$\mathrm{Ric} = ag + b_1\eta_1\otimes\eta_1 + b_2\eta_2\otimes\eta_2 + b_3\eta_3\otimes\eta_3.$$

Indeed one has

$$\mathrm{Ric} = c^2/4\ (\eta_1\otimes\eta_1 + \eta_2\otimes\eta_2 + \eta_3\otimes\eta_3).$$

Thus, differently from 3-Sasakian and 3-cosymplectic geometry, there are 3-quasi-Sasakian manifolds which are not Einstein.

Let us write the explicit expression of the "connection with torsion" stated in the previous theorem.

Since $\mathcal{V}$ is involutive, we can consider the corresponding Bott connection ∇^B. Then we put

$$\nabla_X Y := \begin{cases} (\nabla^1{}_X Y)^h = (\nabla^2{}_X Y)^h = (\nabla^3{}_X Y)^h & \text{if } X, Y \in \Gamma(\mathcal{H}) \\ \nabla^B{}_V Y & \text{if } V \in \Gamma(\mathcal{V}),\ Y \in \Gamma(\mathcal{H}) \\ X(\eta_1(Y))\xi_1 + X(\eta_2(Y))\xi_2 + X(\eta_3(Y))\xi_3 & \text{if } Y \in \Gamma(\mathcal{V}). \end{cases}$$

The complete expression of the torsion is the following:

$$T(X,Y,Z) = d\Phi_i(\varphi_i X, \varphi_i Y, \varphi_i Z), \quad T(\xi_i, X, Y) = d\eta_i(X,Y),$$

$$T(\xi_i, \xi_j, \xi_k) = -c\varepsilon_{ijk}$$

for all $X,Y,Z \in \Gamma(\mathcal{H})$, the remaining terms being zero.

Example of 3-structure with torsion

Let $\mathfrak{g}$ be the 11-dimensional Lie algebra with basis $\{E_1,\dots,E_8,\xi_1,\xi_2,\xi_3\}$ and Lie brackets defined by

$$[E_1,E_2]=-[E_3,E_4]=E_5,\quad [E_1,E_3]=[E_2,E_4]=E_6,\quad [E_1,E_4]=-[E_2,E_3]=E_7,$$

with the remaining brackets zero. Let G be a Lie group whose Lie algebra is $\mathfrak{g}$. Define on G an almost contact metric 3-structure $(\varphi_i,\xi_i,\eta_i,g)$ by putting $\eta_i(E_h)=0$, $\eta_i(\xi_j)=\delta_{ij}$ for all $i,j\in\{1,2,3\}$, $h\in\{1,\dots,8\}$, and

$\varphi_1E_1=E_2$	$\varphi_1E_2=-E_1$	$\varphi_1E_3=E_4$	$\varphi_1E_4=-E_3$	$\varphi_1E_5=E_6$	$\varphi_1E_6=-E_5$
$\varphi_1E_7=E_8$	$\varphi_1E_8=-E_7$	$\varphi_1\xi_1=0$	$\varphi_1\xi_2=\xi_3$	$\varphi_1\xi_3=-\xi_2$	
$\varphi_2E_1=E_3$	$\varphi_2E_2=-E_4$	$\varphi_2E_3=-E_1$	$\varphi_2E_4=E_2$	$\varphi_2E_5=E_7$	$\varphi_2E_6=-E_8$
$\varphi_2E_7=-E_5$	$\varphi_2E_8=E_6$	$\varphi_2\xi_1=-\xi_3$	$\varphi_2\xi_2=0$	$\varphi_2\xi_3=\xi_1$	
$\varphi_3E_1=E_4$	$\varphi_3E_2=E_3$	$\varphi_3E_3=-E_2$	$\varphi_3E_4=-E_1$	$\varphi_3E_5=E_8$	$\varphi_3E_6=E_7$
$\varphi_3E_7=-E_6$	$\varphi_3E_8=-E_5$	$\varphi_3\xi_1=\xi_2$	$\varphi_3\xi_2=-\xi_1$	$\varphi_3\xi_3=0.$	

The Riemannian metric g is defined by requiring that $\{E_1,\dots,E_8,\xi_1,\xi_2,\xi_3\}$ is g-orthonormal.

Definition

Let M^{4n+3} be a smooth manifold of dimension $4n+3$. A **quaternionic-contact structure** (QC-structure) is given by:

- a distribution H of codimension 3 on M^{4n+3}, locally defined by the kernel of a $\mathbb{R}^3$-valued 1-form $\eta=(\eta_1,\eta_2,\eta_3)$, $H=\ker(\eta)$,
- a metric tensor g on H and a local hyper-complex structure $Q=(I_1,I_2,I_3)$ on H (I_s: $H \to H$, $s=1,2,3$), compatible with g, i.e. such that $g(X,I_sY)=d\eta_s(X,Y)$, $s = 1,2,3$, $X,Y \in \Gamma(H)$.

O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000).

Theorem (Biquard)
Let H be a quaternionic-contact structure on M^{4n+3} and let us assume $n>1$. Then there exists a unique distribution V supplementary to H and a unique linear connection ∇ on M^{4n+3} such that

1. V and H are ∇-parallel,
2. $\nabla g = 0$,
3. $\nabla Q \subset Q$,
4. the torsion tensor field T of ∇ satisfies the conditions
   a. for any $X,Y \in \Gamma(H)$, $T(X,Y) = -[X,Y]|_V$
   b. for any $\xi \in \Gamma(V)$, the endomorphism $T_\xi := (X \mapsto (T(X,\xi))|_H) \in (sp(n) \oplus sp(1))^\perp \subset so(4n)$.

The unique connection stated in the theorem is called **Biquard connection**. In dimension 7 its existence was proved, under a further assumption, by Duchemin.

Corollary

Let $(\varphi_i,\xi_i,\eta_i,g)$, $i\in\{1,2,3\}$, be an almost contact metric 3-structure of M^{4n+3} such that each Reeb vector field ξ_i is Killing and is an infinitesimal automorphism with respect to $\mathcal{H}$. Then $(M^{4n+3},\varphi_i,\xi_i,\eta_i,g)$ is a foliated almost 3-contact manifold. More precisely, $\mathcal{V}$ defines a Riemannian foliation of M^{4n+3} with totally geodesic leaves and the Reeb vector fields satisfy

$$[\xi_i,\xi_j] = c\xi_k$$

for any even permutation (i,j,k) of $\{1,2,3\}$ and for some $c\in\mathbb{R}$.

The peculiarity of 3-quasi-Sasakian manifolds is that they are foliated by four canonical Riemannian foliations, namely

	dimension
▶ $\mathcal{V} := \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$	3
▶ $\mathcal{H}_1 := \{X \in \mathcal{H} \mid i_X(d\eta_j) = 0$ for each $j = 1,2,3\}$	$4m$
▶ $\mathcal{H}_1 \oplus \mathcal{V}$	$4m+3$
▶ $\mathcal{H}_2 \oplus \mathcal{V}$, with $\mathcal{H}_2 := \mathcal{H}_1^{\perp} \cap \mathcal{H}$	$4l+3$

where: $4n+3 = \dim(M)$, $4l+3 = \mathrm{rank}(M)$, $m = n - l$.

- The distributions $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{V}$ are mutually orthogonal and one has the following orthogonal decomposition

$$T_pM = \mathcal{H}_{1p} \oplus \mathcal{H}_{2p} \oplus \mathcal{V}_p = \mathcal{H}_p \oplus \mathcal{V}_p.$$

- $\varphi_i(\mathcal{H}_1) \subset \mathcal{H}_1$, $\varphi_i(\mathcal{H}_2) \subset \mathcal{H}_2$ and $\varphi_i(\mathcal{V}) \subset \mathcal{V}$, for each $i \in \{1,2,3\}$.
- $[\xi_i, \mathcal{H}_1] \subset \mathcal{H}_1$, $[\xi_i, \mathcal{H}_2] \subset \mathcal{H}_2$, for each $i \in \{1,2,3\}$.

The results of our study on the “transverse geometry” with respect to those foliations is summarized in the following table:

foliation	*leaves*	*space of leaves*
$\mathcal{V}$	3-dimensional Lie groups $\mathbb{R}^3$ or $SO(3)$	Almost quaternionic-Hermitian
$\mathcal{H}_1$	Hyper-Kähler	3-α-Sasakian
$\mathcal{H}_1 \oplus \mathcal{V}$	3-cosymplectic	Quaternionic-Kähler
$\mathcal{H}_2 \oplus \mathcal{V}$	3-α-Sasakian	Hyper-Kähler