A problem of Roger Liouville

Maciej Dunajski

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

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MD, Paul Tod (2009) arXiv:0901.2261.

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- Path geometry: $y'' = F(x, y, y')$. Douglas (1936).
- When are the paths unparametrised geodesics of some connection Γ on $U \subset \mathbb{R}^2$? Elliminate the parameter in $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c \sim \dot{x}^a$.

$$y'' = A_0(x, y) + A_1(x, y)y' + A_2(x, y)(y')^2 + A_3(x, y)(y')^3, \quad x^a = (x, y).$$

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- When are the paths geodesics of $g = Edx^2 + 2Fdxdy + Gdy^2$?

Projective Structures

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$$\hat{\Gamma}_{ab}^{c} = \Gamma_{ab}^{c} + \delta_a{}^c \omega_b + \delta_b{}^c \omega_a, \qquad a, b, c = 1, 2, \ldots, n$$

for some one–form $\omega = \omega_a dx^a$.

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- A 'forgotten' subject. Goes back to Tracy Thomas (1925), Elie Cartan (1922).
- In two dimensions there is a link with second order ODEs. Projective invariants of $[\Gamma] =$ point invariants of the ODE. Liouville (1889), Tresse (1896), Cartan, ..., Hitchin, Bryant, Tod, Nurowski, Godliński.

Metrisability Problem

A basic unsolved problem in projective differential geometry is to determine the explicit criterion for the metrisability of projective structure

- What are the necessary and sufficient local conditions on a connection Γ^c_{ab} for the existence of a one form ω_a and a symmetric non–degenerate tensor g_{ab} such that the projectively equivalent connection

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- We mainly focus on local metricity: The pair (g, ω) with $\det(g) \neq 0$ is required to exist in a neighbourhood of a point $p \in U$.
- Vastly overdetermined system of PDEs for g and ω: There are $n^2(n+1)/2$ components in a connection, and $(n + n(n+1)/2)$ components in (ω, g). Naively expect $n(n^2 - 3)/2$ conditions on Γ.

Summary of the Results in 2D

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- Counter intuitive - naively expect only one condition (metric $=$ 3 functions of 2 variables, projective structure $=$ 4 functions of 2 variables).

Second order ODEs

- Geodesic equations for $x^a(t) = (x(t), y(t))$

$$\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a\dot{x}^b = v\dot{x}^c.$$

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- Eliminate the parameter t: second order ODE

$$\frac{d^2y}{dx^2} = A_3(x,y)\Big(\frac{dy}{dx}\Big)^3 + A_2(x,y)\Big(\frac{dy}{dx}\Big)^2 + A_1(x,y)\Big(\frac{dy}{dx}\Big) + A_0(x,y)$$

where

$$A_0 = -\Gamma^2_{11}, \quad A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, \quad A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, \quad A_3 = \Gamma^1_{22}.$$

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- This formulation removes the projective ambiguity.

PROLONGATION

- Metric $g = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2$ gives

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\begin{aligned}
A_0 &= (E\partial_y E - 2E\partial_x F + F\partial_x E)\,(EG - F^2)^{-1}/2,\\
A_1 &= (3F\partial_y E + G\partial_x E - 2F\partial_x F - 2E\partial_x G)\,(EG - F^2)^{-1}/2,\\
A_2 &= (2F\partial_y F + 2G\partial_y E - 3F\partial_x G - E\partial_y G)\,(EG - F^2)^{-1}/2,\\
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- Liouville (1889). Relations $(*)$ linearise:

$$E = \psi_1/\Delta, \quad F = \psi_2/\Delta, \quad G = \psi_3/\Delta, \quad \Delta = (\psi_1\psi_3 - {\psi_2}^2)^2.$$

A projective structure $[\Gamma]$ is metrisable on a neighbourhood of a point $p \in U$ iff there exists functions $\psi_i(x, y), i = 1, 2, 3$ defined on a neighbourhood of p such that $\psi_1\psi_3 - {\psi_2}^2$ does not vanish at p and such that the equations

$$\begin{aligned}
\frac{\partial \psi_1}{\partial x} &= \frac{2}{3}A_1\psi_1 - 2A_0\psi_2, \\
\frac{\partial \psi_3}{\partial y} &= 2A_3\psi_2 - \frac{2}{3}A_2\psi_3, \\
\frac{\partial \psi_1}{\partial y} + 2\frac{\partial \psi_2}{\partial x} &= \frac{4}{3}A_2\psi_1 - \frac{2}{3}A_1\psi_2 - 2A_0\psi_3, \\
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hold on the domain of definition.

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1	18	12	6	0
2	30	24	6	0
3	45	40	5	0
4	63	60	3	0
5	84	84	1	$1 = \mathbf{1}$
6	108	112	1	$5 = 3 + \mathbf{2}$
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- 7-jets. The image has codimension 10. 2 relations between the first derivatives of $E_1 = E_2 = 0$ and the second derivatives of the 5th order equation $M = 0$. The system is involutive.

Invariant approach

- Let $\Gamma \in [\Gamma]$. The curvature decomposition

$$[\nabla_a, \nabla_b]X^c = R_{ab}{}^c{}_d X^d, \quad R_{ab}{}^c{}_d = \delta_a^c \mathrm{P}_{bd} X^d - \delta_b^c \mathrm{P}_{ad} X^d + \beta_{ab}\delta_d^c$$

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- If we change the connection in the projective class then

$$\hat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a \omega_b + \omega_a \omega_b, \quad \hat{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a}\omega_{b]}.$$

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- Use ϵ^{ab} to rise indices. Residual freedom $\omega_a = \nabla_a f$

$$\epsilon_{ab} \longrightarrow e^{3f} \epsilon_{ab}, \quad h \longrightarrow e^{wf} h, \qquad \text{projective weight } w.$$

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  for some tensors $\Psi^\alpha = (\sigma^{ab}, \mu^a, \rho)$, where $Y_{abc} = \frac{1}{2}(\nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac})$.

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$$\Psi^\alpha \Sigma_\alpha := 5Y_a\mu^a + (\nabla_a Y_b)\sigma^{ab} = 0. \qquad (**)$$

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- The determinat of the 6 by 6 matrix $\mathcal{F}_2$ gives the 5th order obstruction M - a section of $\Lambda^2(T^*U)^{\otimes 14}$

$$\det(\mathcal{F}_2)([\Gamma])\,(dx \wedge dy)^{\otimes 14}$$

is a projective invariant.

Explicit Invariant: 1746 terms!

$$\det(\mathcal{F}_2) = \Big(Q_{gi} S_{mp} T_{njk} U_{ac} V_{deq} X_{bfhl} - \frac{1}{6} P_p R_m S_{nq} X_{acgi} X_{behk} X_{dfjl}$$
$$- \frac{1}{2} P_p S_{mq} T_{njl} U_{ce} X_{adgk} X_{bfhi} - \frac{1}{2} P_p T_{mgi} T_{njk} U_{ac} V_{deq} X_{bfhl}$$
$$+ \frac{1}{2} P_p R_m T_{ngi} V_{acq} X_{dejk} X_{bfhl} - \frac{1}{2} Q_{gi} R_m S_{np} V_{acq} X_{dejk} X_{bfhl}$$
$$- \frac{1}{2} Q_{gi} R_m T_{njk} V_{acp} V_{deq} X_{bfhl} - \frac{1}{4} Q_{gi} S_{mp} S_{nq} U_{ac} X_{dejk} X_{bfhl}$$
$$\frac{1}{4} Q_{gi} T_{mjk} T_{nhl} U_{ac} V_{dep} V_{bfq} \Big) \epsilon^{ab} \epsilon^{cd} \epsilon^{ef} \epsilon^{gh} \epsilon^{ij} \epsilon^{kl} \epsilon^{mn} \epsilon^{pq},$$

where

$$P_a \equiv 5Y_a, \quad Q_{ab} \equiv 12 Z_{ab}, \quad R_c \equiv 5Y_c, \quad S_{ca} \equiv 5\nabla_a Y_c + 2Z_{ac},$$
$$T_{cab} \equiv 5\nabla_{(a}\nabla_{b)} Y_c + 4\nabla_{(a} Z_{b)c} - 5\mathrm{P}_{ab} Y_c - 15\mathrm{P}_{c(a} Y_{b)}, \quad U_{cd} \equiv Z_{cd},$$
$$X_{cdab} \equiv \nabla_{(a}\nabla_{b)} Z_{cd} - 5(\nabla_{(a}\mathrm{P}_{b)(c}) Y_{d)} - 5\mathrm{P}_{c(a}\nabla_{b)} Y_d - 5\mathrm{P}_{d(a}\nabla_{b)} Y_c$$
$$-\mathrm{P}_{c(a} Z_{b)d} - \mathrm{P}_{d(a} Z_{b)c} + 10 Y_{(a} Y_{b)(cd)}, \quad V_{cda} \equiv \nabla_a Z_{cd} - 5\mathrm{P}_{a(c} Y_{d)}.$$

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- Spinoff: Koenigs Theorem: The space of metrics compatible with a given projective structures can have dimensions $0, 1, 2, 3, 4$ or 6.

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- These three polynomials do not have a common root. We can make the 5th order obstruction vanish, but the two 6th order obstructions E_1, E_2 do not vanish.

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Given a Riemannian manifold (M, g) is there a non-zero function Ω such that $\Omega^2 g$ is Kähler with respect to some complex structure?

- Leads to overdetermined PDEs. Proceed as before: prolong, construct a curvature, restrict its holonomy, find conformal invariants.

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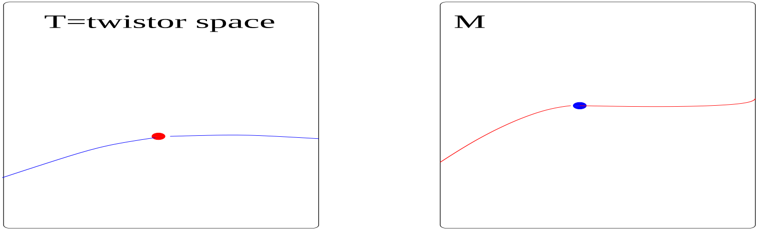
- Theorem (MD, Tod): The metric g is conformal to (para) Kähler iff the projective structure is metrisable.

Twistor Theory

- One-to-one correspondence between holomorphic projective structures $(U, [\Gamma])$ and complex surfaces $\mathbb{T}$ with a family of rational curves.



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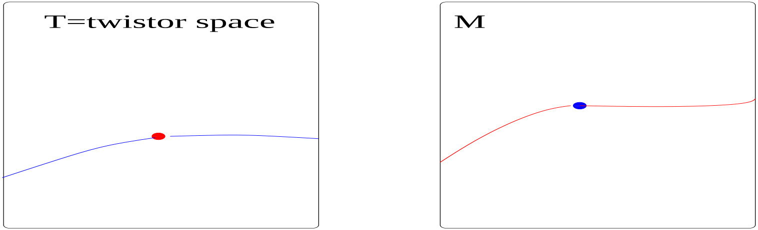

$$\begin{aligned} \text{geodesics} &\longleftrightarrow \text{points} \\ \text{points} &\longleftrightarrow \text{rational curves with normal bundle } \mathcal{O}(1). \end{aligned}$$

- Double fibration $U \longleftarrow \mathbb{P}(TU) \longrightarrow \mathbb{T} = \mathbb{P}(TU)/D_x$, where $D_x = z^a \frac{\partial}{\partial x^a} - \Gamma^c_{ab} z^a z^b \frac{\partial}{\partial z^c}$ is a geodesic spray.

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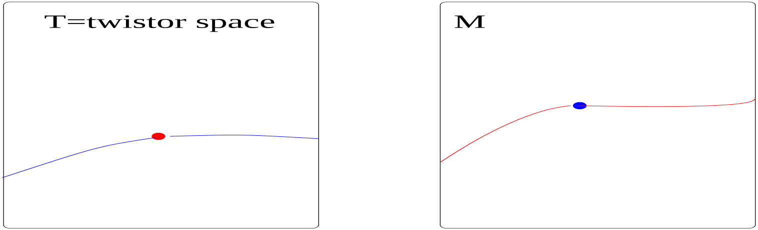

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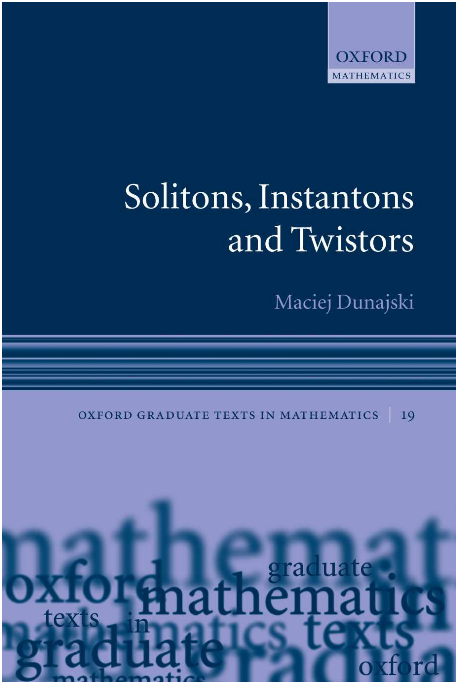
- $(U, [\Gamma])$ is metrisable iff $\mathbb{T}$ is equipped with a preferred section of the line bundle ${\kappa_{\mathbb{T}}}^{-2/3}$, where $\kappa_{\mathbb{T}}$ is the canonical bundle.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