On some cohomological properties of almost complex manifolds

joint with A. Tomassini

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Tamed and calibrated almost complex structures

M: compact oriented $2n$-dimensional manifold.

A symplectic form ω compatible with the orientation is a closed 2-form ω such that ω^n is a volume form compatible with the orientation.

Definition

An almost complex structure J on a symplectic manifold (M, ω) is tamed by ω if $\omega_x(u, Ju) > 0$, $\forall x \in M$ and $\forall u \neq 0 \in T_xM$.

J is calibrated by ω (or ω is compatible with J) if, in addition, $\omega_x(Ju, Jv) = \omega_x(u, v)$, $\forall u, v \in T_xM$.

If J is calibrated by $\omega \Longrightarrow (\omega, J)$ is an almost-Kähler structure $\Rightarrow g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is a J-Hermitian metric.

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ω: a fixed non-degenerate closed 2-form ω on $\mathbb{R}^{2n} = \mathbb{C}^n$.
$\mathcal{J}_c(\omega)$ (resp. $\mathcal{J}_t(\omega)$) = the set of almost-complex structures calibrated (resp. tamed) by ω.

Proposition (Audin)

If on $\mathbb{C}^n$ one considers the standard symplectic structure (J_0, ω), then the map

$$J \mapsto (J + J_0)^{-1} \circ (J - J_0)$$

is a diffeomorphism from $\mathcal{J}_t(\omega)$ (resp. $\mathcal{J}_c(\omega)$) onto the open unit ball in the vector space of (resp. symmetric) matrices L such that $J_0 L = -L J_0$.

Then, if J_0 is calibrated by ω and L is a symmetric matrix such that $||L|| < 1$, $J_0 L = -L J_0$, then

$$(I + L) \circ J_0 \circ (I + L)^{-1}$$

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Symplectic cones

$\mathcal{C}(M)$: symplectic cone of M, i.e. the image of the space of symplectic forms on M compatible with the orientation by the projection $\omega \mapsto [\omega] \in H^2(M, \mathbb{R})$.

T. J. Li e W. Zhang studied the following subcones of $\mathcal{C}(M)$: the J-tamed symplectic cone

$$\mathcal{K}_J^t(M) = \{[\omega] \in H^2(M, \mathbb{R}) \mid \omega \text{ is tamed by } J\}$$

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Theorem (Li, Zhang)

If J is integrable and $\mathcal{K}_J^c(M) \neq \emptyset$, one has

$$\mathcal{K}_J^t(M) = \mathcal{K}_J^c(M) + \left[(H_{\bar\partial}^{2,0}(M) \oplus H_{\bar\partial}^{0,2}(M)) \cap H^2(M,\mathbb{R}) \right],$$
$$\mathcal{K}_J^t(M) \cap \left[H_{\bar\partial}^{1,1}(M) \cap H^2(M,\mathbb{R}) \right] = \mathcal{K}_J^c(M).$$

Problem

Find a relation between $\mathcal{K}_J^t(M)$ and $\mathcal{K}_J^c(M)$ in the case that J is non integrable, related to the question by Donaldson for $n = 2$: if $\mathcal{K}_J^t(M) \neq \emptyset$ for some J, then $\mathcal{K}_J^c(M) \neq \emptyset$ as well?

To solve this problem Li and Zhang introduced the analogous of the previous (real) Dolbeault groups for general almost complex manifolds (M, J).

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$\mathcal{C}^\infty$ pure and full almost complex structures

On (M, J) for the space $\Omega^k(M)_\mathbb{R}$ of real smooth differential k-forms one has:

$$\Omega^k(M)_\mathbb{R} = \bigoplus_{p+q=k} \Omega_J^{p,q}(M)_\mathbb{R},$$

where

$$\Omega_J^{p,q}(M)_\mathbb{R} = \left\{\alpha \in \Omega_J^{p,q}(M) \oplus \Omega_J^{q,p}(M) \,|\, \alpha = \overline{\alpha}\right\}.$$

S: a finite set of pairs of integers. Let

$$\mathcal{Z}_J^S = \bigoplus_{(p,q)\in S} \mathcal{Z}_J^{p,q}, \quad \mathcal{B}_J^S = \bigoplus_{(p,q)\in S} \mathcal{B}_J^{p,q},$$

where $\mathcal{Z}_J^{p,q}$ and $\mathcal{B}_J^{p,q}$ are the spaces of real d-closed (resp. d-exacts) (p,q)-forms.

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There is a natural map

$$\rho_S : \mathcal{Z}_J^S / \mathcal{B}_J^S \to \mathcal{Z}_J^S / \mathcal{B},$$

where $\mathcal{B}$ is the space of d-exact forms.

We will write $\rho_S(\mathcal{Z}_J^S / \mathcal{B}_J^S)$ as $\mathcal{Z}_J^S / \mathcal{B}_J^S$.

Define

$$H_J^S(M)_{\mathbb{R}} = \left\{ [\alpha] \mid \alpha \in \mathcal{Z}_J^S \right\} = \frac{\mathcal{Z}_J^S}{\mathcal{B}}.$$

Then

$$H_J^{1,1}(M)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(M)_{\mathbb{R}} \subseteq H^2(M, \mathbb{R}).$$

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Definition (Li, Zhang)

J is $\mathcal{C}^\infty$ pure and full if and only if

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Calibrated and 4-dimensional case

Proposition (–, Tomassini)

Let ω be a symplectic form on a compact manifold M^{2n}. If J is an almost complex structure on M^{2n} calibrated by ω, then J is $\mathcal{C}^\infty$ pure.

Theorem (Draghici, Li, Zhang)

On a compact manifold M^4 of real dimension 4 any almost complex structure is $\mathcal{C}^\infty$ pure and full.

Problem

Does the previous property hold in higher dimension?

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Example of non $\mathcal{C}^\infty$ pure almost complex structure

A compact manifold of real dimension 6 may admit non $\mathcal{C}^\infty$ pure almost complex structures.

Example

Consider the nilmanifold M^6, compact quotient of the Lie group:

$$\begin{cases} de^j = 0, \quad j = 1, \ldots, 4, \\ de^5 = e^{12}, \\ de^6 = e^{13}. \end{cases}$$

The left-invariant almost complex structure on M^6, defined by

$$\eta^1 = e^1 + ie^2, \quad \eta^2 = e^3 + ie^4, \quad \eta^3 = e^5 + ie^6,$$

is not $\mathcal{C}^\infty$ pure, since one has that

$$[\mathrm{Re}(\eta^1 \wedge \overline{\eta}^2)] = [e^{13} + e^{24}] = [e^{24}] = [\mathrm{Re}(\eta^1 \wedge \eta^2)] = [e^{13} - e^{24}].$$

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$$[\mathrm{Re}(\eta^1 \wedge \overline{\eta}^2)] = [e^{13} + e^{24}] = [e^{24}] = [\mathrm{Re}(\eta^1 \wedge \eta^2)] = [e^{13} - e^{24}].$$

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Example of non $\mathcal{C}^\infty$ pure almost complex structure

A compact manifold of real dimension 6 may admit non $\mathcal{C}^\infty$ pure almost complex structures.

Example

Consider the nilmanifold M^6, compact quotient of the Lie group:

$$\begin{cases} de^j = 0, \quad j = 1, \ldots, 4, \\ de^5 = e^{12}, \\ de^6 = e^{13}. \end{cases}$$

The left-invariant almost complex structure on M^6, defined by

$$\eta^1 = e^1 + ie^2, \quad \eta^2 = e^3 + ie^4, \quad \eta^3 = e^5 + ie^6,$$

is not $\mathcal{C}^\infty$ pure, since one has that

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Pure and full almost complex structures

(M, J) (almost) complex manifold of (real) dimension $2n$.

$\mathcal{E}_k(M)$ the space of k-currents on M, i.e. the topological dual of $\Omega^{2n-k}(M)$.

Since the smooth k-forms can be considered as $(2n-k)$-currents, then

$$H_k(M, \mathbb{R}) \cong H^{2n-k}(M, \mathbb{R}),$$

where $H_k(M, \mathbb{R})$ is the k-th de Rham homology group.

• A k-current is a boundary if and only if it vanishes on the space of closed k-forms.

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On (M, J) for the space of real k-currents $\mathcal{E}_k(M)_{\mathbb{R}}$ one has:

$$\mathcal{E}_k(M)_{\mathbb{R}} = \bigoplus_{p+q=k} \mathcal{E}^J_{p,q}(M)_{\mathbb{R}},$$

where $\mathcal{E}^J_{p,q}(M)_{\mathbb{R}}$ is the space of real k-currents of bidimension (p, q).

S: a finite set of pairs of integers. Let

$$\mathcal{Z}^J_S = \bigoplus_{(p,q)\in S} \mathcal{Z}^J_{p,q}, \quad \mathcal{B}^J_S = \bigoplus_{(p,q)\in S} \mathcal{B}^J_{p,q},$$

where $\mathcal{Z}^J_{p,q}$ and $\mathcal{B}^J_{p,q}$ are the space of real d-closed (resp. boundary) currents of bidimension (p, q).

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Definition (Li, Zhang)

An almost complex structure J is pure if $H^J_{1,1}(M)_{\mathbb{R}} \cap H^J_{(2,0),(0,2)}(M)_{\mathbb{R}} = \{0\}$ or equivalently if $\pi_{1,1}\mathcal{B}_2 \cap \mathcal{Z}^J_{1,1} = \mathcal{B}^J_{1,1}$.
J is full if $H_2(M, \mathbb{R}) = H^J_{1,1}(M)_{\mathbb{R}} + H^J_{(2,0),(0,2)}(M)_{\mathbb{R}}$.

Problem

Relation between $\mathcal{C}^\infty$ pure and full and pure and full?

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Main result

If a 2-form ω on M^{2n} is not necessarily closed but it is only non-degenerate, (M^{2n}, ω) is called *almost symplectic*.

Theorem (–, Tomassini)

Let (M^{2n}, ω) be an almost symplectic compact manifold and J be a $\mathcal{C}^\infty$ pure and full almost complex structure calibrated by ω. Then J is pure.
If, in addition, either $n = 2$ or any class in $H_J^{1,1}(M^{2n})_\mathbb{R}$ ($H_J^{(2,0),(0,2)}(M^{2n})_\mathbb{R}$ resp.) has a harmonic representative in $\mathcal{Z}_J^{1,1}$ ($\mathcal{Z}_J^{(2,0),(0,2)}$ resp.) with respect to the metric induced by ω and J, then J is pure and full.

Remark

• In order to get the pureness of J, it is enough to assume that J is $\mathcal{C}^\infty$ full.
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Sketch of the proof

We start to prove that J is pure, i.e. $\pi_{1,1}\mathcal{B}_2 \cap \mathcal{Z}^J_{1,1} = \mathcal{B}^J_{1,1}$.

Let $T \in \pi_{1,1}\mathcal{B}_2 \cap \mathcal{Z}^J_{1,1} \Rightarrow T = \pi_{1,1}dS$, where S is a real 3-current and $d(\pi_{1,1}dS) = 0$.

We have to show that $T = \pi_{1,1}dS$ is a boundary, i.e. that $T(\alpha) = 0$, for any closed real 2-form α.

If α is exact, then $(\pi_{1,1}dS)(\alpha) = 0$.

If $[\alpha] \neq 0 \in H^2(M^{2n}, \mathbb{R})$, since J is $\mathcal{C}^\infty$ pure and full, we have

$$\alpha = \alpha_1 + \alpha_2 + d\gamma, \text{ with } \alpha_1 \in \mathcal{Z}^{1,1}_J,\ \alpha_2 \in \mathcal{Z}^{(2,0),(0,2)}_J.$$

Then

$$T(\alpha) = (\pi_{1,1}dS)(\alpha) = (\pi_{1,1}dS)(\alpha_1 + \alpha_2) = (dS)(\alpha_1) = 0\,.$$

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• If $n = 2$, let $[T] \in H_2(M^4, \mathbb{R})$; then $\exists$ a smooth closed 2-form α such that $[T] = [\alpha]$.

Since J is $\mathcal{C}^\infty$ full, we have that $[\alpha] = [\alpha_1] + [\alpha_2]$, with $\alpha_1 \in \mathcal{Z}_J^{1,1}$ and $\alpha_2 \in \mathcal{Z}_J^{(2,0),(0,2)}$.

• If $n > 2$, let $[T] \in H_2(M^{2n}, \mathbb{R})$, then $\exists$ a smooth harmonic $(2n-2)$-form β such that $[T] = [\beta]$.

The 2-form $\gamma = *\beta$ defines $[\gamma] \in H^2(M^{2n}, \mathbb{R})$. By the assumption, $\exists$ real harmonic forms $\gamma_1 \in \Omega_J^{1,1}(M^{2n})_{\mathbb{R}}$ and $\gamma_2 \in \Omega_J^{(2,0),(0,2)}(M^{2n})_{\mathbb{R}}$ such that $[\gamma] = [\gamma_1] + [\gamma_2]$.

The $(2n-2)$-forms $\beta_1 = *\gamma_1$ and $\beta_2 = *\gamma_2$ then can be viewed as elements respectively of $\mathcal{Z}^J_{1,1}$ and $\mathcal{Z}^J_{(2,0),(0,2)} \Longrightarrow$ $[T] = [\beta_1] + [\beta_2]$.

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• If $n = 2$, let $[T] \in H_2(M^4, \mathbb{R})$; then $\exists$ a smooth closed 2-form α such that $[T] = [\alpha]$.

Since J is $\mathcal{C}^\infty$ full, we have that $[\alpha] = [\alpha_1] + [\alpha_2]$, with $\alpha_1 \in \mathcal{Z}_J^{1,1}$ and $\alpha_2 \in \mathcal{Z}_J^{(2,0),(0,2)}$.

• If $n > 2$, let $[T] \in H_2(M^{2n}, \mathbb{R})$, then $\exists$ a smooth harmonic $(2n-2)$-form β such that $[T] = [\beta]$.

The 2-form $\gamma = *\beta$ defines $[\gamma] \in H^2(M^{2n}, \mathbb{R})$. By the assumption, $\exists$ real harmonic forms $\gamma_1 \in \Omega_J^{1,1}(M^{2n})_\mathbb{R}$ and $\gamma_2 \in \Omega_J^{(2,0),(0,2)}(M^{2n})_\mathbb{R}$ such that $[\gamma] = [\gamma_1] + [\gamma_2]$.

The $(2n-2)$-forms $\beta_1 = *\gamma_1$ and $\beta_2 = *\gamma_2$ then can be viewed as elements respectively of $\mathcal{Z}^J_{1,1}$ and $\mathcal{Z}^J_{(2,0),(0,2)} \implies [T] = [\beta_1] + [\beta_2]$.

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Link with Hard Lefschetz condition

A symplectic manifold (M^{2n}, ω) satisfies the Hard Lefschetz condition if :

$$\omega^k : \Omega^{n-k}(M^{2n}) \to \Omega^{n+k}(M^{2n}), \alpha \mapsto \omega^k \wedge \alpha$$

induce isomorphisms in cohomology.

Theorem (–, Tomassini)

Let (M^{2n}, ω) be a compact symplectic manifold which satisfies Hard Lefschetz condition and J be a $\mathcal{C}^\infty$ pure and full almost complex structure calibrated by ω. Then J is pure and full.

Problem

Find for $n > 2$ an example of compact symplectic manifold (M^{2n}, ω) which satisfies Hard Lefschetz condition and with an non pure and full almost complex structure calibrated by ω.

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Sketch of the Proof

- If $n = 2$ the result follows by the last Theorem.
- If $n > 2$ J is pure. We have to show that

$$H_2(M^{2n}, \mathbb{R}) = H^J_{1,1}(M^{2n})_{\mathbb{R}} \oplus H^J_{(2,0),(0,2)}(M^{2n})_{\mathbb{R}} .$$

Let $a = [T] \in H_2(M^{2n}, \mathbb{R})$. Then $a = [\alpha]$, where $\alpha \in \Omega^{2n-2}(M^{2n})$ is d-closed.

HL condition $\Rightarrow \exists b \in H^2(M^{2n}, \mathbb{R})$, $b = [\beta]$ such that $a = b \cup [\omega]^{n-2}$, i.e.

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$$[\beta] = [\varphi] + [\psi]\,,$$

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Integrable case

If J is integrable, in general it is not necessarily ($\mathcal{C}^\infty$) pure and full.

If J is an integrable almost complex structure and the Frölicher spectral sequence degenerates at E_1, then J is pure and full [Li, Zhang].

Theorem (–, Tomassini)

If $(M = \Gamma\backslash G, J)$ is a complex parallelizable manifold and $H^2(M, \mathbb{R}) \cong H^2(\mathfrak{g})$, then J is $\mathcal{C}^\infty$ full and it is pure.

$\Rightarrow$ Let (M, J) be a complex parallelizable nilmanifold. Then J is $\mathcal{C}^\infty$ full and it is pure.

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Nakamura manifold

Let G be the solvable Lie group with structure equations

$$(0, e^{12} - e^{45}, -e^{13} + e^{46}, 0, e^{15} - e^{24}, -e^{16} + e^{34}).$$

$G \cong (\mathbb{C}^3, *)$, with $*$ defined in terms of the coordinates $z_j = x_j + ix_{3+j}$ by

$$^t(z_1, z_2, z_3) * {}^t(w_1, w_2, w_3) = {}^t(z_1 + w_1, e^{-w_1} z_2 + w_2, e^{w_1} z_3 + w_3).$$

The Nakamura manifold is the compact quotient $X^6 = \Gamma \backslash G$.

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By de Bartolomeis-Tomassini we have

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\begin{aligned}
H^2(X^6,\mathbb{R}) \quad = \quad & \mathbb{R} < [e^{14}], [e^{26}-e^{35}], [e^{23}-e^{56}], \\
& [\cos(2x_4)(e^{23}+e^{56}) - \sin(2x_4)(e^{26}+e^{35})]\,, \\
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- X^6 has a left-invariant J defined by:

$$\eta^1 = e^1 + ie^4\,,\quad \eta^2 = e^3 + ie^5\,,\quad \eta^3 = e^6 + ie^2$$

calibrated by $\omega = e^{14} + e^{35} + e^{62}$.

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The harmonic forms

$$e^{14},\, e^{26}-e^{35},\, \cos(2x_4)(e^{23}+e^{56})-\sin(2x_4)(e^{26}+e^{35}),$$
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are all of type $(1,1)$ and $e^{23}-e^{56}$ is of type $(2,0)$ $\Rightarrow$ J is pure and full.

- X^6 admits the pure and full bi-invariant complex structure $\tilde{J}$:

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Families in dimension six

Consider the completely solvable Lie algebra $\mathfrak{s} = \mathfrak{sol}(3) \oplus \mathfrak{sol}(3)$ with structure equations

$$(0, -f^{12}, f^{34}, 0, f^{15}, f^{46}).$$

S admits a compact quotient $M^6 = \Gamma \backslash S$ [Fernandez-Gray].

By Hattori's Theorem

$$H^2(M^6, \mathbb{R}) \cong H^*(\mathfrak{s}) = \mathbb{R} < [f^{14}], [f^{25}], [f^{36}] > .$$

J_0 defined by the $(1,0)$-forms

$$\varphi^1 = f^1 + if^4,\ \varphi^2 = f^2 + if^5,\ \varphi^3 = f^3 + if^6.$$

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$$\varphi^1 = f^1 + if^4, \, \varphi^2 = f^2 + if^5, \, \varphi^3 = f^3 + if^6.$$

is almost-Kähler with respect to $\omega = f^{14} + f^{25} + f^{36}$.

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(M^6, J_0, ω) satisfies the Hard Lefschetz condition [Fernandez, Munoz] and $H^2(M^6, \mathbb{R}) = H^{1,1}_{J_0}(M)_{\mathbb{R}}$.

Define the family of almost complex structure

$$J_t = (I + L_t) J_0 (I + L_t)^{-1}$$

with respect to the basis $(f^1, \ldots, f^6)$, where

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad L_t = \begin{pmatrix} 0 & tI \\ tI & 0 \end{pmatrix}, \quad 6t^2 < 1.$$

Then, J_t is a family of ω-calibrated almost complex structures.

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Then, J_t is a family of ω-calibrated almost complex structures.

$\Longrightarrow$ Any J_t is $\mathcal{C}^\infty$ pure.

A basis of $(1,0)$-forms for J_t is

$$\varphi_t^1 = f^1 + i\left(\frac{2t}{(1-t^2)} f^1 + \frac{1+t^2}{1-t^2} f^4\right),$$

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Then J_t is also $\mathcal{C}^\infty$ full.

J_t is actually pure and full, since $\varphi_t^1 \wedge \overline{\varphi}_t^1$, $\varphi_t^2 \wedge \overline{\varphi}_t^2$, $\varphi_t^3 \wedge \overline{\varphi}_t^3$ are harmonic.

The family $\tilde{J}_t$ associated to the basis of $(1,0)$-forms

$$\tilde{\varphi}_t^1 = f^1 + i\left(-2tf^2 + f^4\right), \; \tilde{\varphi}_t^2 = f^2 + if^5, \; \tilde{\varphi}_t^3 = f^3 + if^6$$

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References

T. Draghici, T.J. Li, W. Zhang, Symplectic forms and cohomology of almost complex 4-manifolds, preprint arXiv: 0812.3680, to appear in *Int. Math. Res. Not.*.

A. Fino, A. Tomassini, On some cohomological properties of almost complex manifolds, preprint arXiv: 0807.1800, to appear in *J. of Geom. Anal.*.

T. J. Li, W. Zhang, Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds, preprint arXiv:0708.2520.

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