Split G_2 geometries on solution space of 7th order ODEs

Michał Godliński

Polish Academy of Sciences

Dirac operators and special geometries, 26 September 2009

Based on a work of M.Dunajski, MG and P.Nurowski, in preparation

Idea

$$y_n = F(x, y, y_1, \dots, y_{n-1}), \qquad n \geq 3.$$

We admit contact transformations of variables $(x, y, \dots, y_n)$

$$y = f(x; c_1, \dots, c_n).$$

Space of solutions: M^n parameterized by (c_i).

Idea

$$y_n = F(x, y, y_1, \dots, y_{n-1}), \qquad n \geq 3.$$

We admit contact transformations of variables $(x, y, \dots, y_n)$

$$y = f(x; c_1, \dots, c_n).$$

Space of solutions: M^n parameterized by (c_i).

A point $y_0 \in M^n$: a solution $y = y_0(x)$.

Vectors tangent at y_0: solutions of the linerization around y_0.

Idea

$$y_n = F(x, y, y_1, \ldots, y_{n-1}), \qquad n \geq 3.$$

We admit contact transformations of variables $(x, y, \ldots, y_n)$

$$y = f(x; c_1, \ldots, c_n).$$

Space of solutions: M^n parameterized by (c_i).

A point $y_0 \in M^n$: a solution $y = y_0(x)$.

Vectors tangent at y_0: solutions of the linerization around y_0.

$$y(x) = y_0(x) + \delta(x), \quad \delta_n = \partial_{y_{n-1}} F \cdot \delta_{n-1} + \ldots + \partial_y F \cdot \delta$$

Idea

$$y_n = F(x, y, y_1, \dots, y_{n-1}), \qquad n \geq 3.$$

We admit contact transformations of variables $(x, y, \dots, y_n)$

$$y = f(x; c_1, \dots, c_n).$$

Space of solutions: M^n parameterized by (c_i).

A point $y_0 \in M^n$: a solution $y = y_0(x)$.

Vectors tangent at y_0: solutions of the linerization around y_0.

$$y(x) = y_0(x) + \delta(x), \quad \delta_n = \partial_{y_{n-1}} F \cdot \delta_{n-1} + \dots + \partial_y F \cdot \delta$$

The linearization is trivializable if it may be transformed into $\delta_n(x) = 0$.

Idea

$$y_n = F(x, y, y_1, \ldots, y_{n-1}), \qquad n \geq 3.$$

We admit contact transformations of variables $(x, y, \ldots, y_n)$

$$y = f(x; c_1, \ldots, c_n).$$

Space of solutions: M^n parameterized by (c_i).

A point $y_0 \in M^n$: a solution $y = y_0(x)$.

Vectors tangent at y_0: solutions of the linerization around y_0.

$$y(x) = y_0(x) + \delta(x), \quad \delta_n = \partial_{y_{n-1}} F \cdot \delta_{n-1} + \ldots + \partial_y F \cdot \delta$$

The linearization is trivializable if it may be transformed into $\delta_n(x) = 0$.

Idea

What happens to $T_{y_0}M^7$ if the linearization is trivializable?

Tangent vectors $\leftrightarrow$ $n-1$st degree polynomials in x

$\leftrightarrow$ $n-1$st degree homogeneous polynomials in t and s, $(x = t/s)$.

Idea

What happens to $T_{y_0}M^7$ if the linearization is trivializable?

Tangent vectors $\leftrightarrow$ $n-1$st degree polynomials in x

$\leftrightarrow$ $n-1$st degree homogeneous polynomials in t and s, $(x = t/s)$.

On $T_{y_0}M^n$ acts the group of linear transformations of t and s.

V^i – i-dimensional irreducible representation of $GL(2,\mathbb{R})$.

$$T_{y_0}M^n = S^{n-1}V^2 = V^n.$$

Idea

What happens to $T_{y_0}M^7$ if the linearization is trivializable?

Tangent vectors $\leftrightarrow$ $n-1$st degree polynomials in x

$\leftrightarrow$ $n-1$st degree homogeneous polynomials in t and s, $(x = t/s)$.

On $T_{y_0}M^n$ acts the group of linear transformations of t and s.

V^i – i-dimensional irreducible representation of $GL(2,\mathbb{R})$.

$$T_{y_0}M^n = S^{n-1}V^2 = V^n.$$

Idea

Definition

$GL(2,\mathbb{R})$ geometry on M^n is a reduction of the frame bundle $F\,M^n$ to its $GL(2,\mathbb{R})$-subbundle, where $GL(2,\mathbb{R}) \subset GL(n,\mathbb{R})$ acts irreducibly.

Corollary

An ODE admits a $GL(2,\mathbb{R})$ geometry on its solution space iff the linearizations around its solutions are all trivial.

Idea

Definition

$GL(2,\mathbb{R})$ geometry on M^n is a reduction of the frame bundle $F\,M^n$ to its $GL(2,\mathbb{R})$-subbundle, where $GL(2,\mathbb{R}) \subset GL(n,\mathbb{R})$ acts irreducibly.

Corollary

An ODE admits a $GL(2,\mathbb{R})$ geometry on its solution space iff the linearizations around its solutions are all trivial.

Corollary

A 7-dimensional $GL(2,\mathbb{R})$ geometry uniquely defines a conformal split G_2 geometry, since

$$GL(2,\mathbb{R}) \subset \mathbb{R}_+ \times \tilde{G}_2 \subset CO(3,4).$$

Idea

Definition

$GL(2,\mathbb{R})$ geometry on M^n is a reduction of the frame bundle $F\,M^n$ to its $GL(2,\mathbb{R})$-subbundle, where $GL(2,\mathbb{R}) \subset GL(n,\mathbb{R})$ acts irreducibly.

Corollary

An ODE admits a $GL(2,\mathbb{R})$ geometry on its solution space iff the linearizations around its solutions are all trivial.

Corollary

A 7-dimensional $GL(2,\mathbb{R})$ geometry uniquely defines a conformal split G_2 geometry, since

$$GL(2,\mathbb{R}) \subset \mathbb{R}_+ \times \tilde{G}_2 \subset CO(3,4).$$

Wünschmann, Cartan, Chern, Bryant, Eastwood, Doubrov, Dunajski, Nurowski, Tod, MG,...

Idea

Definition

$GL(2,\mathbb{R})$ geometry on M^n is a reduction of the frame bundle $F\,M^n$ to its $GL(2,\mathbb{R})$-subbundle, where $GL(2,\mathbb{R}) \subset GL(n,\mathbb{R})$ acts irreducibly.

Corollary

An ODE admits a $GL(2,\mathbb{R})$ geometry on its solution space iff the linearizations around its solutions are all trivial.

Corollary

A 7-dimensional $GL(2,\mathbb{R})$ geometry uniquely defines a conformal split G_2 geometry, since

$$GL(2,\mathbb{R}) \subset \mathbb{R}_+ \times \tilde{G}_2 \subset CO(3,4).$$

Wünschmann, Cartan, Chern, Bryant, Eastwood, Doubrov, Dunajski, Nurowski, Tod, MG,...

Jet space J^6

Graph of a function $x \mapsto (x, f(x))$ in the xy-space

lifts to $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$.

J^6 – the space where the lifted curves live.

$(x, y, y_1, y_2, \ldots, y_6)$ – local coordinates in $\mathcal{J}^6$, $\dim J^6 = 8$.

Geometry of J^6 – contact distribution C spanned by all lifted curves. C has rank 2 and it is totally non-integrable

Contact transformations $\equiv$ transformations preserving C.

Jet space J^6

Graph of a function $x \mapsto (x, f(x))$ in the xy-space

lifts to $x \mapsto (x, f(x), f'(x), \dots, f^{(6)}(x))$.

J^6 – the space where the lifted curves live.

$(x, y, y_1, y_2, \dots, y_6)$ – local coordinates in $\mathcal{J}^6$, $\dim J^6 = 8$.

Geometry of J^6 – contact distribution C spanned by all lifted curves. C has rank 2 and it is totally non-integrable

Contact transformations $\equiv$ transformations preserving C.

J^6 and space of solutions

Fix a 7th order ODE $y_7 = F(x, y, y_1, \ldots, y_6)$.

Family of its solutions lift to a congruence in J^6

The lifts: $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$

One solution through any point in J^6.

Contact geometry of 7th order ODEs

J^6 equipped with

i) the contact distribution C,

ii) the congruence of solutions (in C).

J^6 and space of solutions

Fix a 7th order ODE $y_7 = F(x, y, y_1, \ldots, y_6)$.

Family of its solutions lift to a congruence in J^6

The lifts: $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$

One solution through any point in J^6.

Contact geometry of 7th order ODEs

J^6 equipped with

i) the contact distribution C,

ii) the congruence of solutions (in C).

J^6 is a line bundle over the solution space M^7

J^6 and space of solutions

Fix a 7th order ODE $y_7 = F(x, y, y_1, \ldots, y_6)$.

Family of its solutions lift to a congruence in J^6

The lifts: $x \mapsto (x, f(x), f'(x), \ldots, f^{(6)}(x))$

One solution through any point in J^6.

Contact geometry of 7th order ODEs

J^6 equipped with

i) the contact distribution C,

ii) the congruence of solutions (in C).

J^6 is a line bundle over the solution space M^7

Main trick

$$\begin{array}{c} P \\ \downarrow \\ J^6 \end{array}$$

Main trick

$$\begin{array}{c} P,\, \Omega \\ \downarrow \\ J^6 \end{array}$$

Main trick

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R}) & \longrightarrow & P, \Omega \\ & & \downarrow \\ & & J^6 \end{array}$$

Main trick

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R}) & \longrightarrow & P, \Omega \\ & & \downarrow \\ & & J^6 \\ & & \downarrow \\ & & M^7 \end{array}$$

Main trick

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R}) & \longrightarrow & P, \Omega \\ & & \downarrow \\ & & M^7 \end{array}$$

Main trick

$$\begin{array}{ccc} GL(2,\mathbb{R}) & \longrightarrow & P,\,\Omega \\ & & \downarrow \\ & & M^7 \end{array}$$

Main trick

$$\begin{array}{ccc} GL(2,\mathbb{R}) & \longrightarrow & P, \Omega \\ & & \downarrow \\ & & M^7 \end{array}$$

Main trick

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R}) & \longrightarrow & P, \Omega \\ & & \downarrow \\ & & J^6 \end{array}$$

Ω is a $\mathfrak{gl}(2,\mathbb{R}) \oplus_{\cdot} \mathbb{R}^7$-valued Cartan connection. Why? It is a deformation of the trivial case $y_7 = 0$, where $P = GL(2,\mathbb{R}) \ltimes \mathbb{R}^7$, J^6 is a homogeneous space and Ω is the Maurer-Cartan 1-form, $\mathrm{d}\Omega + \Omega \wedge \Omega = 0$

Main trick

$$\begin{array}{ccc} \mathbb{R}\times(\mathbb{R}\ltimes\mathbb{R}) & \longrightarrow & P,\,\Omega \\ & & \downarrow \\ & & J^6 \end{array}$$

Ω is a $\mathfrak{gl}(2,\mathbb{R}) \oplus_{.} \mathbb{R}^7$-valued Cartan connection. Why? It is a deformation of the trivial case $y_7 = 0$, where $P = GL(2,\mathbb{R}) \ltimes \mathbb{R}^7$, J^6 is a homogeneous space and Ω is the Maurer-Cartan 1-form, $\mathrm{d}\Omega + \Omega_\wedge \Omega = 0$

In general $K := \mathrm{d}\Omega + \Omega_\wedge \Omega \neq 0$ contains invariants.

Construction of Ω: Linear conditions on K, Tanaka-Morimoto theory

Main trick

$$\begin{array}{ccc} \mathbb{R}\times(\mathbb{R}\ltimes\mathbb{R}) & \longrightarrow & P,\,\Omega \\ & & \downarrow \\ & & J^6 \end{array}$$

Ω is a $\mathfrak{gl}(2,\mathbb{R})\oplus_{.}\mathbb{R}^7$-valued Cartan connection. Why? It is a deformation of the trivial case $y_7=0$, where $P=GL(2,\mathbb{R})\ltimes\mathbb{R}^7$, J^6 is a homogeneous space and Ω is the Maurer-Cartan 1-form, $\mathrm{d}\Omega+\Omega\wedge\Omega=0$

In general $K:=\mathrm{d}\Omega+\Omega\wedge\Omega\neq 0$ contains invariants.

Construction of Ω: Linear conditions on K, Tanaka-Morimoto theory

Main trick

$$\begin{array}{ccc} GL(2,\mathbb{R}) & \rightarrow & P,\,\Omega \\ & & \downarrow \\ & & M^7 \end{array}$$

How to construct $GL(2,\mathbb{R})$ geometry?

$$R_u^*\Omega = \mathrm{ad}\, u^{-1}\Omega,\ \ u \in GL(2,\mathbb{R}) \iff A^* \lrcorner K = 0,\ \ A \in \mathfrak{gl}(2,\mathbb{R}).$$

$$\Omega = \underbrace{\Gamma}_{\mathfrak{gl}(2,\mathbb{R})} + \underbrace{\theta}_{\mathbb{R}^7}$$

$$\mathrm{d}\theta^i + {\Gamma^i}_j \wedge \theta^j = \tfrac{1}{2} {T^i}_{kl}\theta^k \wedge \theta^l,$$
$$\mathrm{d}{\Gamma^i}_j + {\Gamma^i}_k \wedge {\Gamma^k}_j = \tfrac{1}{2} {R^i}_{jkl}\theta^k \wedge \theta^l.$$

Main trick

$$\begin{array}{ccc} GL(2,\mathbb{R}) & \to & P,\,\Omega \\ & & \downarrow \\ & & M^7 \end{array}$$

How to construct $GL(2,\mathbb{R})$ geometry?

$$R_u^*\Omega = \mathrm{ad}\, u^{-1}\Omega,\ \ u \in GL(2,\mathbb{R}) \iff A^* \lrcorner K = 0,\ \ A \in \mathfrak{gl}(2,\mathbb{R}).$$

$$\Omega = \underbrace{\Gamma}_{\mathfrak{gl}(2,\mathbb{R})} + \underbrace{\theta}_{\mathbb{R}^7}$$

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = \tfrac{1}{2} T^i{}_{kl}\theta^k \wedge \theta^l,$$
$$\mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \tfrac{1}{2} R^i{}_{jkl}\theta^k \wedge \theta^l.$$

Torsion

$$\Lambda^2(V^7) \otimes V^7 = \Lambda^3 V^7 \oplus S^{(2,1)} V^7.$$

$$\Lambda^3 V^7 = V^1 \oplus V^5 \oplus V^7 \oplus V^9 \oplus V^{13}.$$

$$S^{(2,1)} V^7 = V^3 \oplus 2\, V^5 \oplus 2 V^7 \oplus 2 V^9 \oplus 2 V^{11} \oplus V^{13} \oplus V^{15} \oplus V^{17}.$$

Torsion

$$\Lambda^2(V^7) \otimes V^7 = \Lambda^3 V^7 \oplus S^{(2,1)} V^7.$$

$$\Lambda^3 V^7 = V^1 \oplus V^5 \oplus V^7 \oplus V^9 \oplus V^{13}.$$

$$S^{(2,1)} V^7 = V^3 \oplus 2\, V^5 \oplus 2V^7 \oplus 2V^9 \oplus 2V^{11} \oplus V^{13} \oplus V^{15} \oplus V^{17}.$$

$$T = T^{(1)} + T^{(3)} + T^{(5)}.$$

Torsion

$$\Lambda^2(V^7) \otimes V^7 = \Lambda^3 V^7 \oplus S^{(2,1)} V^7.$$

$$\Lambda^3 V^7 = V^1 \oplus V^5 \oplus V^7 \oplus V^9 \oplus V^{13}.$$

$$S^{(2,1)} V^7 = V^3 \oplus 2\, V^5 \oplus 2V^7 \oplus 2V^9 \oplus 2V^{11} \oplus V^{13} \oplus V^{15} \oplus V^{17}.$$

$$T = T^{(1)} + T^{(3)} + T^{(5)}.$$

$T^{(5)}$ lies 'askew'. May we get 'more antisymmetric' torsion?

Torsion

$$\Lambda^2(V^7) \otimes V^7 = \Lambda^3 V^7 \oplus S^{(2,1)} V^7.$$

$$\Lambda^3 V^7 = V^1 \oplus V^5 \oplus V^7 \oplus V^9 \oplus V^{13}.$$

$$S^{(2,1)} V^7 = V^3 \oplus 2\, V^5 \oplus 2V^7 \oplus 2V^9 \oplus 2V^{11} \oplus V^{13} \oplus V^{15} \oplus V^{17}.$$

$$T = T^{(1)} + T^{(3)} + T^{(5)}.$$

$T^{(5)}$ lies 'askew'. May we get 'more antisymmetric' torsion?

This is a unique connection with torsion without blue components.

Torsion

$$\Lambda^2(V^7) \otimes V^7 = \Lambda^3 V^7 \oplus S^{(2,1)} V^7.$$

$$\Lambda^3 V^7 = V^1 \oplus V^5 \oplus V^7 \oplus V^9 \oplus V^{13}.$$

$$S^{(2,1)} V^7 = V^3 \oplus 2\, V^5 \oplus 2V^7 \oplus 2V^9 \oplus 2V^{11} \oplus V^{13} \oplus V^{15} \oplus V^{17}.$$

$$T = T^{(1)} + T^{(3)} + T^{(5)}.$$

$T^{(5)}$ lies 'askew'. May we get 'more antisymmetric' torsion?

This is a unique connection with torsion without blue components.

Towards $\tilde{G}_2$ geometries

$$\mathrm{d}\phi = \lambda * \phi + \frac{3}{4}\tau_4 \wedge \phi + *\tau_3,$$
$$\mathrm{d} * \phi = \tau_4 \wedge * \phi - \tau_2 \wedge \phi.$$

$$\mathcal{X}_1 = V^1, \qquad \lambda \sim T^{(1)}.$$
$$\mathcal{X}_2 = V^3 \oplus V^{11}, \qquad \tau_2 \sim T^{(3)}.$$
$$\mathcal{X}_3 = V^5 \oplus V^9 \oplus V^{13}, \qquad \tau_3 \sim T^{(5)}.$$
$$\mathcal{X}_4 = V^7, \qquad \tau_4 = \frac{4}{7} Tr\Gamma.$$

Fernandez-Gray classes, torsion and contact invariants.

$$T^{(5)} = 0 \quad \Leftrightarrow \quad \text{no } \mathcal{X}_3 \quad \Leftrightarrow \quad F_{66} = 0,$$

$$T^{(3)} = 0 \quad \Leftrightarrow \quad \text{no } \mathcal{X}_2 \quad \Leftrightarrow \quad 21\mathcal{D}F_{66} + 14F_{65} + 15F_6F_{66} = 0,$$

$$T^{(1)} = 0 \quad \Leftrightarrow \quad \text{no } \mathcal{X}_1 \quad \Leftrightarrow \quad \dots$$

Fernandez-Gray classes, torsion and contact invariants.

$$T^{(5)} = 0 \quad \Leftrightarrow \quad \text{no } \mathcal{X}_3 \quad \Leftrightarrow \quad F_{66} = 0,$$
$$T^{(3)} = 0 \quad \Leftrightarrow \quad \text{no } \mathcal{X}_2 \quad \Leftrightarrow \quad 21\mathcal{D}F_{66} + 14F_{65} + 15F_6F_{66} = 0,$$
$$T^{(1)} = 0 \quad \Leftrightarrow \quad \text{no } \mathcal{X}_1 \quad \Leftrightarrow \quad \ldots$$

The same for

$$\mathrm{d}\tau_4 = \mathrm{d}\tau_4^{(3)} + \mathrm{d}\tau_4^{(7)} + \mathrm{d}\tau_4^{(11)}.$$

In particular

$$\mathrm{d}\tau_4^{(11)} = 0 \quad \Leftrightarrow \quad F_{666} = 0.$$

Fernandez-Gray classes, torsion and contact invariants.

$$
\begin{array}{lllll}
T^{(5)} = 0 & \Leftrightarrow & \text{no } \mathcal{X}_3 & \Leftrightarrow & F_{66} = 0, \\
T^{(3)} = 0 & \Leftrightarrow & \text{no } \mathcal{X}_2 & \Leftrightarrow & 21\mathcal{D}F_{66} + 14F_{65} + 15F_6F_{66} = 0, \\
T^{(1)} = 0 & \Leftrightarrow & \text{no } \mathcal{X}_1 & \Leftrightarrow & \ldots
\end{array}
$$

The same for

$$\mathrm{d}\tau_4 = \mathrm{d}\tau_4^{(3)} + \mathrm{d}\tau_4^{(7)} + \mathrm{d}\tau_4^{(11)}.$$

In particular

$$\mathrm{d}\tau_4^{(11)} = 0 \quad \Leftrightarrow \quad F_{666} = 0.$$

Only three conformal geometries of class $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_4$.

1. Holonomy $\tilde{G}_2$ – the flat case of $y_7 = 0$.

Only three conformal geometries of class $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_4$.

1. Holonomy $\tilde{G}_2$ – the flat case of $y_7 = 0$.
2. Class $\mathcal{X}_1 + \mathcal{X}_4$ which contains the nearly-paralel geometry of $SO(3,2)/SO(2,1)$

$$y_7 = 7\frac{y_6 y_4}{y_3} + \frac{49}{10}\frac{y_5^2}{y_3} - 28\frac{y_5 y_4^2}{y_3^2} + \frac{35}{2}\frac{y_4^4}{y_3^3}.$$

Only three conformal geometries of class $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_4$.

1. Holonomy $\tilde{G}_2$ – the flat case of $y_7 = 0$.
2. Class $\mathcal{X}_1 + \mathcal{X}_4$ which contains the nearly-paralel geometry of $SO(3,2)/SO(2,1)$

$$y_7 = 7\frac{y_6 y_4}{y_3} + \frac{49}{10}\frac{y_5^2}{y_3} - 28\frac{y_5 y_4^2}{y_3^2} + \frac{35}{2}\frac{y_4^4}{y_3^3}.$$

3. Class $\mathcal{X}_2 + \mathcal{X}_4$ which contains an almost parallel geometry with at least 8 symmetries.

$$y_7 = \frac{21}{5}\frac{y_6 y_5}{y_4} - \frac{84}{25}\frac{y_5^3}{y_4^2}.$$

Only three conformal geometries of class $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_4$.

1. Holonomy $\tilde{G}_2$ – the flat case of $y_7 = 0$.
2. Class $\mathcal{X}_1 + \mathcal{X}_4$ which contains the nearly-paralel geometry of $SO(3,2)/SO(2,1)$

$$y_7 = 7\frac{y_6 y_4}{y_3} + \frac{49}{10}\frac{y_5^2}{y_3} - 28\frac{y_5 y_4^2}{y_3^2} + \frac{35}{2}\frac{y_4^4}{y_3^3}.$$

3. Class $\mathcal{X}_2 + \mathcal{X}_4$ which contains an almost parallel geometry with at least 8 symmetries.

$$y_7 = \frac{21}{5}\frac{y_6 y_5}{y_4} - \frac{84}{25}\frac{y_5^3}{y_4^2}.$$