Twistor spinors and generic rank 2-distributions on 5-manifolds

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Overview

Outline

Motivation from the point of view of conformal geometry

- Two pseudo-Riemannian metrics g and $\hat{g}$ of signature (p, q) are *conformally related* if there is an $f \in \mathrm{C}^\infty(M)$ with $\hat{g} = e^{2f} g$. The corresponding equivalence class of metrics is a ray subbundle $\mathbf{C} \subset \Gamma(S^2 T^* M)$, which we call a *conformal structure*.
- The study of conformal structures brings new obstacles compared to Riemannian geometry, since there is no unique torsion-free principal connection form on the conformal frame bundle $\mathcal{G}_0 \to M$.
- Operators and objects which are defined in terms of the Riemannian data of a $g \in \mathbf{C}$ but don't depend on the particular choice of representative metric are called *conformally invariant*.

Motivation from the point of view of conformal geometry

- In this talk we dicuss how another geometric structure, a generic distribution $\mathbf{D} \subset TM$, gives rise to a conformal structure $\mathbf{C_D}$ of signature $(2,3)$ plus a conformal object, namely a twistor spinor.
- The discovery that one has a conformal class of metrics $\mathbf{C_D}$ for a generic distribution $\mathbf{D}$ is due to [P. Nurowski, Journ. Geom. Physics (2005)].
- We will describe $\mathbf{D} \rightsquigarrow \mathbf{C_D}$ as a particular case of a *Fefferman-type construction*, which is a powerful tool for *parabolic geometries*.

Motivation from the point of view of conformal geometry

- This description of $\mathbf{D} \rightsquigarrow \mathbf{C_D}$ is used to obtain relations to *conformal holonomy* and existence of a well defined conformal object which encodes the distribution $\mathbf{D}$, namely a twistor spinor.
- Finally, we use this twistor spinor to *decompose symmetries* of the conformal structure $\mathbf{C}$.
- We make extensive use of techniques for *parabolic geometries*, in particular we employ *tractor calculus* and the description of conformal objects as kernels of *BGG-operators*.

Outline

Generic distributions

- For two subbundles $\mathbf{D}_1 \subset TM$ and $\mathbf{D}_2 \subset TM$ we define

$$[\mathbf{D}_1, \mathbf{D}_2]_x := \mathrm{span}(\{[\xi, \eta]_x \ : \ \xi \in \Gamma(\mathbf{D}_1), \eta \in \Gamma(\mathbf{D}_2)\}).$$

- $\mathbf{D}$ is a *generic distribution* if $\mathbf{D}^2 := [\mathbf{D}, \mathbf{D}] \subset TM$ is a subbundle of constant rank 3 and $\mathbf{D}^3 := [\mathbf{D}^2, \mathbf{D}^2] = TM$. These are distributions of maximal growth vector $(2, 3, 5)$ in each point.

D and $\mathbf{C_D}$ on $S^2 \times S^3$

- There is a well known generic rank 2-distribution $\mathbf{D} \subset TM$ on $M = S^2 \times S^3$, which encodes the system of a ball rolling without splipping or twisting on another ball [Montgomery-Bor, Enseign.Mathem. (2009)].
- The automorphism group of this (oriented) distribution is the full Lie group G_2 - which in this talk will always denote the unique connected Lie group with fundamental group $\mathbb{Z}_2$ and Lie algebra the split real form $\mathfrak{g}_2$ of the exceptional complex Lie group $\mathfrak{g}_2^{\mathbb{C}}$.

D and $\mathbf{C_D}$ on $S^2 \times S^3$

- $S^2 \times S^3$ also carries a conformal structure **C** of signature $(2,3)$ with large automorphism group, which has representative $(g_2, -g_3)$; here g_2 and g_3 are the canonical round metrics on the 2- resp. 3-sphere.
- The structure group of $(S^2 \times S^3, \mathbf{C})$ is $\mathrm{CO}(2,3) = \mathbb{R}_+ \times \mathrm{O}(2,3)$. Fixing all orientations, this reduces to the connected group $\mathbb{R}_+ \times \mathrm{SO}(2,3)_o$; fixing also the canonical spin-structure of this space we get the structure group $\mathbb{R}_+ \times \mathrm{Spin}(2,3)$ of *conformal spin* structures of signature $(2,3)$.
- The group of all conformal maps $\{f : f^*\mathbf{C} = \mathbf{C}\}$ preserving this spin structure is then $\mathrm{Spin}(3,4)$, and $S^2 \times S^3$ can be realized as $\mathrm{Spin}(3,4)/\tilde{P}$ with $\tilde{P}$ the stabilizer of an isotropic ray in the standard representation of $\mathrm{Spin}(3,4)$ on $\mathbb{R}^{3,4} = \mathbb{R}^7$.

$\mathbf{D}$ and $\mathbf{C}_{\mathbf{D}}$ on $S^2 \times S^3$

- It has been observed by [I. Kath, Habil (1999)] that $G_2 \hookrightarrow \mathrm{Spin}(3,4)$ as the stabilizer of an arbitrary non-isotropic spinor $\mathbf{X} \in \Delta_{\mathbb{R}}^{3,4} \cong \mathbb{R}^{4,4}$.
- With $P = \mathrm{Spin}(3,4) \cap \tilde{P}$ one then has $G_2/P = \mathrm{Spin}(3,4)/\tilde{P}$.
- We can regard

$$(G_2/P = S^2 \times S^3, \mathbf{D}) \rightsquigarrow (\mathrm{Spin}(3,4)/\tilde{P} = S^2 \times S^3, \mathbf{C})$$

  as going to a 'weaker' geometric structure: The automorphism group increases from G_2 to $\mathrm{Spin}(3,4)$.
- This process $\mathbf{D} \rightsquigarrow \mathbf{C}$ generalizes: Given a 5-dimensional manifold M endowed with an (orientable) generic distribution $\mathbf{D}$ one obtains a conformal spin structure of signature $(2,3)$.
- This is based on the Cartan geometric description of generic distributions and conformal structures:

Cartan's description of geometric structures

Let G be a real Lie group and $P \subset G$ a closed subgroup; The Lie algebras of G, P are denoted $\mathfrak{g}, \mathfrak{p}$.

Definition

Let M be a smooth manifold. A Cartan geometry of type (G, P) on M consists of of a P-principal bundle $\mathcal{G} \to M$ endowed with a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which satisfies the following properties:

1. ω is P-equivariant.
2. ω reproduces fundamental vector fields.
3. ω provides a trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$.

It follows from this definition that $TM = \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$.

Definition

The *curvature* $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of ω is defined by

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$.
It is horizontal and P-equivariant and thus factorizes to an

$$\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$$

valued two form $K \in \Omega^2(M, \mathcal{A}M)$.

Cartan's description of geometric structures

- In the case where P is a parabolic subgroup of a semi-simple Lie group G, which is the case for the groups $P \subset G_2$ and $\tilde{P} \subset \mathrm{Spin}(3,4)$ discussed above, one calls $(\mathcal{G}, \omega)$ a *parabolic geometry*.
- This class of geometries is particularly important because it comes with canonical *regularity* and *normality* conditions on ω resp. its curvature K.
- Parabolic geometries which satisfy these conditions are equivalent (in the categorical sense) with underlying geometric structures. The cases of interest to us are:

Equivalent description of distributions and conformal structures as parabolic geometries

Theorem

Oriented generic rank 2-distributions of 5-manifolds can be equivalently described as regular, normal parabolic geometries of type (G_2, P).

Theorem

Conformal spin structures of signature (p, q) *can be equivalently described as regular, normal parabolic geometries of type* $(\mathrm{Spin}(p+1, q+1), \tilde{P})$, *with* $\tilde{P} \subset \mathrm{Spin}(p+1, q+1)$ *the stabilizer of an isotropic ray in* $\mathbb{R}^{p+1,q+1}$.

Evidently we should tell here how this correspondence comes about. At least, to see how the parabolic geometries define underlying geometric structures can be explained in a reasonable time, but would already demand too many additional definitions at this point. We will just be glad that this identification exists and *use it*.

Extension of structure group for parabolic geometries

- Given a Cartan geometry $(\mathcal{G}, \omega)$, of type (G, P), one has that $\mathcal{G}$ is a P principal bundle over the underlying manifold M.
- In this talk we employ two kinds of extension of structure group - one of these is purely technical and intrinsic to a given parabolic geometry, used to form a real principal bundle connection. The second kind produces a different kind of geometry on the underlying manifold M.
- We begin by the first kind, used to define the *holonomy* of $(\mathcal{G}, \omega)$:

First kind of extension of structure group: forming a principal bundle connection form from the Cartan connection form

- Given a parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P), we remark that $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is not a principal connection form on $\mathcal{G}$ since it is $\mathfrak{g}$ valued on a P-principal bundle; this however, can be mended easily:
- We define $\hat{\mathcal{G}} := \mathcal{G} \times_P G$, which is the P associated bundle to $\mathcal{G}$ defined via restriction of the natural left action of G on itself to the action of P on G.
- Then there is canonical embedding $\mathcal{G} \hookrightarrow \hat{\mathcal{G}}$, and one has $\omega \in \Gamma(T^*\mathcal{G} \otimes \mathfrak{g}) \subset \Gamma(T^*\hat{\mathcal{G}} \otimes \mathfrak{g})$.
- One can extend ω to an element in $\Gamma\big((T^*\hat{\mathcal{G}})_{|\mathcal{G}} \otimes \mathfrak{g}\big)$ by demanding that fundamental vector fields $\zeta_X(u) := \frac{d}{dt}_{|t=0} u \cdot \exp(tX)$ are reproduced.
- By equivariant extension of the resulting form, one obtains a *principal connection form* $\hat{\omega} \in \Omega^1(\hat{\mathcal{G}}, \mathfrak{g})$; This form will soon play an important role.

Second kind of extension of structure group: The Fefferman-type construction $\mathbf{D} \rightsquigarrow \mathbf{C_D}$

- Using the embedding $G_2 \subset \mathrm{Spin}(3,4)$ and the fact that the parabolic subgroup $P \subset G_2$ is just the intersection $G_2 \cap \tilde{P}$ one can define an *extension functor* from Cartan geometries of type (G_2, P) to geometries of type $(\mathrm{Spin}(3,4), \tilde{P})$:
- Let $(\mathcal{G}_\mathbf{D}, \omega_\mathbf{D})$, $\mathcal{G}_\mathbf{D} \to M$, $\omega_\mathbf{D} \in \Omega^1(\mathcal{G}_\mathbf{D}, \mathfrak{g}_2)$ be a parabolic geometry of type (G_2, P), which shall be regular and normal, and therefore equivalent to an underlying generic rank 2-distribution $\mathbf{D}$ on M.
- Define $\mathcal{G}_\mathbf{C} := \mathcal{G}_\mathbf{D} \times_P \tilde{P}$, i.e., we extend the structure group from P to $\tilde{P}$. Then, similarly to above, $\omega_\mathbf{D} \in \Omega^1(\mathcal{G}_\mathbf{D}, \mathfrak{g}_2)$ uniquely extends to a $\mathfrak{so}(3,4)$-valued Cartan connection form $\omega_\mathbf{C} \in \Omega^1(\mathcal{G}_\mathbf{C}, \mathfrak{so}(3,4))$.

The Fefferman-type construction $\mathbf{D} \rightsquigarrow \mathbf{C_D}$ and holonomy reduction

Proposition

$(\mathcal{G}_{\mathbf{C}}, \omega_{\mathbf{C}})$ *is a regular, normal parabolic geometry of type* $(\mathrm{Spin}(3,4), \tilde{P})$*, and thus induces a conformal spin structure* $\mathbf{C_D}$ *of signature* $(2,3)$ *on* M*.*

- In particular, this implies that Nurowski's conformal structure associated to an *orientable* generic distribution $\mathbf{D}$ carries a canonical spin structure.
- The important point in having *normality* of $\omega_{\mathbf{C}}$ is that this implies strong relations between $\mathbf{D}$ and $\mathbf{C_D}$:
- Given the parabolic structure bundles $\mathcal{G}_{\mathbf{D}}$ and $\mathcal{G}_{\mathbf{C}}$ of the generic distribution and the conformal structure, we can form the the extended bundles $\hat{\mathcal{G}}_{\mathbf{D}} := \mathcal{G}_{\mathbf{D}} \times_P \mathcal{G}$ and $\hat{\mathcal{G}}_{\mathbf{C}} := \mathcal{G}_{\mathbf{C}} \times_{\tilde{P}} \mathrm{Spin}(3,4)$, which carry the principal connection forms $\hat{\omega}_{\mathbf{D}}$ and $\hat{\omega}_{\mathbf{C}}$.

The Fefferman-type construction $\mathbf{D} \rightsquigarrow \mathbf{C_D}$ and holonomy reduction

- Now $\hat{\omega}_{\mathbf{C}}$ depends only on the conformal structure $(M, \mathbf{C})$, and thus gives rise to a well defined *conformal holonomy*

$$\mathsf{Hol}(\mathbf{C}) := \mathsf{Hol}(\hat{\omega}_{\mathbf{C}}) \subset \mathrm{Spin}(3,4).$$

- The construction shows that one obtains a holonomy reduction of principal bundles $(\hat{\omega}_{\mathbf{D}}, \hat{\omega}_{\mathbf{D}}) \hookrightarrow (\hat{\omega}_{\mathbf{C}}, \hat{\omega}_{\mathbf{C}})$ from $\mathsf{Hol}(\hat{\omega}_{\mathbf{C}}) \subset \mathrm{Spin}(3,4)$ to $\mathsf{Hol}(\hat{\omega}'_{\mathbf{D}}) \subset G_2$.
- Thus, for every (orientable) generic distribution $\mathbf{D}$ one has for the holonomy of the induced conformal spin structure $\mathsf{Hol}(\mathbf{C_D}) \subset G_2 \subset \mathrm{Spin}(3,4)$.

Holonomy reduction for parabolic geomeries

- Naturally, one now asks wheter any given conformal spin structure $\mathbf{C}$ of signature $(2,3)$ is already induced by a generic distribution if the necessary condition $\mathrm{Hol}(\mathbf{C}) \subset G_2$ is satisfied.
- This works: one employs a reduction procedure for parabolic geometries:
- Starting from $(M, \mathbf{C})$, one has the equivalent description as $(\mathcal{G}_{\mathbf{C}}, \omega_{\mathbf{C}})$ and knows by assumption that $(\hat{\mathcal{G}}_{\mathbf{C}}, \hat{\omega}_{\mathbf{C}})$ reduces to a G_2-principal bundle $\bar{G} \hookrightarrow \hat{G}_{\mathbf{C}}$, $\bar{\omega} \in \Omega^1(\bar{G}, \mathfrak{g}_2)$.
- Then $\bar{G}$ is shown to intersect transversally with $\mathcal{G}_{\mathbf{C}}$ in a $P \subset G_2$-principal bundle $\mathcal{G} \subset \mathcal{G}_{\mathbf{C}}$, and $\bar{\omega}$ can by seen to restrict to a (G_2, P)-Cartan connection form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g}_2)$.

Holonomy reduction for parabolic geomeries

The surprising fact now is that the normality of the $(\mathrm{Spin}(3,4), \tilde{P})$-geometry already implies the normality of the constructed (G_2, P)-structure $(\mathcal{G}, \omega)$, on M, and one obtains:

Theorem (M.H.-K.Sagerschnig, SIGMA (2009))

Let $(M, \mathbf{C})$ *be a conformal structure of signature* $(2,3)$ *with* $\mathrm{Hol}(\mathbf{C}) \subset G_2 \subset \mathrm{Spin}(3,4)$. *Then* $\mathbf{C}$ *is canonically associated to a generic rank two distribution* $\mathbf{D}$.

Evidently one now wants to describe the reduction $\mathrm{Hol}(\mathbf{C}) \subset G_2$ in terms of reasonable conformal data on $(M, \mathbf{C})$.

Outline

Tractor bundles for parabolic geometries

- We already mentioned that the lack of a unique torsion-free connection for a conformal geometry complicates canonical (differential) constructions.
- The Cartan geometry $(\mathcal{G}_{\mathbf{C}}, \omega_{\mathbf{C}})$ provides a substitute via the $\mathfrak{so}(3,4)$-valued 1-form $\omega_{\mathbf{C}}$, which was extended canonically to a $\tilde{P}$-principal connection form on the extended bundle $\mathcal{G}'_{\mathbf{C}} := \mathcal{G}_{\mathbf{C}} \times_{\tilde{P}} \mathrm{Spin}(3,4)$.
- Then, every finite dimensional, real $\mathrm{Spin}(3,4)$-representation V gives rise to an adjoint *tractor bundle*

$$\mathbf{V} := \mathcal{G}_{\mathbf{C}} \times_{\tilde{P}} V = \hat{\mathcal{G}}_{\mathbf{C}} \times_{\mathrm{Spin}(3,4)} V$$

endowed with its canonical *tractor connection*.

Construction/description of invariant differential operators via tractor bundles

- An important application of $\mathbf{V}$ together with its tractor connection is the construction of (conformally) invariant differential operators.
- There is a natural *tractor homology* produced by the *Kostant co-differential* $\partial^* : \Omega^{k+1}(M, \mathbf{V} \to \Omega^k(M, \mathbf{V})$, $\partial^* \circ \partial^* = 0$. Thus, there exists a algebraic theory in the background of the constructions which will follow, but we don't discuss this here.
- The section space of the first and second homologies of ∂^* are denoted $\mathcal{H}_0$ and $\mathcal{H}_1$.
- $\mathcal{H}_0$ is a quotient of $\Gamma(\mathbf{V})$, and we have the canonical surjection $\Pi_0 : \Gamma(\mathbf{V}) \to \mathcal{H}_0$.
- The goal now is to *factorize* the connection

$$\nabla : \Gamma(\mathbf{V}) \to \Omega^1(M, \mathbf{V})$$

to an operator

$$\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1 :$$

The first BGG-operator Θ_0

- The general *BGG-machinery* as developed by [Čap-Slovăk-Souček, Ann. of Math. (2001)] and simplified by [Calderbank-Diemer, (J. Reine u. Angew. Math.) (2001)] describes the *first BGG-operator* Θ_0 as the composition

$$\Theta_0 = \Pi_1 \circ \nabla \circ L_0.$$

- The middle operator is just the first order operator ∇, and Π_1 is simply a sub-quotient projection map, i.e., it maps a subbundle of $\Omega^1(M, \mathbf{V})$ onto $\mathcal{H}_1$.
- The important term in the above formula is $L_0 : \mathcal{H}_0 \to \Gamma(\mathbf{V})$, which takes a section $\sigma \in \mathcal{H}_0$ and maps it to a tractor section $s = L_0\sigma \in \Gamma(\mathbf{V})$.
- L_0 is a *differential splitting operator* of the canonical surjection $\Pi_0 : \Gamma(\mathbf{V}) \to \mathcal{H}_0$. I.e.: $\Pi_0 \circ L_0 = \mathrm{id}_{\mathcal{H}_0}$.

The kernel of Θ_0 and parallel sections of $\mathbf{V}$

- We will be interested in solutions of equations of

$$\Theta_0(\sigma) = 0, \sigma \in \mathcal{H}_0.$$

- An important fact which immediately follows from its construction and relates solutions of the above natural geometric equations to (conformal) holonomy is:

Lemma

Via $\Pi_0 : \Gamma(\mathbf{V}) \to \mathcal{H}_0$, ∇-parallel sections of the tractor bundle $\mathbf{V}$ project into the kernel of Θ_0.

- If, conversely, also every element of $\ker \Theta_0 \subset \Gamma(\mathbf{H}_0)$ splits into a ∇-parallel section of $\mathbf{V}$ we say that ∇ is the *prolongation connection* of Θ_0.
- This is the case for the case of the conformal *standard*- and *spin*-tractor bundle:

The standard tractor bundle of conformal geometry

- Taking the standard representation on $\mathbb{R}^{3,4} = \mathbb{R}^7$ of $\mathrm{Spin}(3,4)$, the corresponding associated tractor bundle is the *standard tractor bundle* **S** of conformal geometry.
- One calculates that with respect to a choice of metric $g \in \mathbf{C}$, which has Levi-Civita connection D, its first BGG-operator is

$$\Theta^g : \mathrm{C}^\infty(M) \to \Gamma(S^2_0 T^* M),$$
$$\Theta^g(\sigma) = (DD\sigma + \mathrm{P}^g \sigma) + \frac{1}{n}(\triangle\sigma - \mathrm{tr}_{(1,2)} \mathrm{P}^g \sigma) g.$$

  Here

$$\mathrm{P}^g := \frac{1}{n-2}\big(\mathrm{Ric}^g - \frac{\mathrm{Sc}^g}{2(n-1)} g\big)$$

  is the Schouten-tensor; $S^2_0 T^* M$ denotes symmetric, trace-free bilinear forms on TM. The convention for the Laplace operator is $\triangle := -\mathrm{tr}_{(1,2)} \circ D^2$.
- For $\sigma \in \mathrm{C}^\infty(M, \mathbb{R}_+)$ one has $\Theta^g(\sigma) = 0$ iff $\sigma^{-2} g$ is Einstein.

The conformal spin tractor bundle

- Taking the associated bundle to the 8-dimensional, real spin representation $\Delta_{\mathbb{R}}^{3,4} \cong \mathbb{R}^{4,4}$ of $\mathrm{Spin}(3,4)$, one obtains the conformal *spin tractor* bundle $\mathbf{\Sigma}$.
- Let $\mathbf{\Delta}$ be the real, conformal spin bundle of rank 4, with Clifford symbol $\gamma \in \Gamma(T^*M \otimes \mathrm{End}(\mathbf{\Delta}))$ and $\not{D} : \Gamma(\mathbf{\Delta}) \to \Gamma(\mathbf{\Delta})$ its Dirac operator.
- With respect to a metric $g \in \mathbf{C}$ the first BGG-operator is the *twistor operator*

$$\Gamma(\mathbf{\Delta}) \to \Gamma(T^*M \otimes \mathbf{\Delta}),$$
$$\chi \mapsto D\chi + \frac{1}{n}\gamma \otimes \not{D}\chi,$$

which projects the Levi-Civita derivative of a spinor to the kernel of the Clifford multiplication.

Characterization of G_2-holonomy in terms of twistor spinors

- It turns out that characterization of $\mathrm{Hol}(\mathbf{C}) \subset G_2$ via a twistor spinor is very simple: The real 4-dimensional spin representation $\Delta_{\mathbb{R}}^{2,3}$ carries a non-degenerate skew-symmetric bilinear form which can be related to the symmetric $(4,4)$-form on $\Delta_{\mathbb{R}}^{3,4}$.
- Now via the first BGG-splitting operator a twistor spinor $\chi \in \Gamma(\mathbf{\Delta})$ is equivalent to a parallel spin tractor $\mathbf{X} \in \Gamma(\Sigma)$. But $\mathbf{X}$ corresponds to an holonomy-invariant element in $\Delta_{\mathbb{R}}^{3,4}$.

Characterization of G_2-holonomy in terms of twistor spinors

- We already know that the stabilizer of an *non-null* element $X \in \Delta_{\mathbb{R}}^{3,4} \cong \mathbb{R}^{4,4}$ is (conjugate to) G_2.
- The condition of $\mathbf{X}$ being non-null can be related to a condition on χ, and one obtains:
- Let $\not{D} : \Gamma(\mathbf{\Delta}) \to \Gamma(\mathbf{\Delta})$ be the Dirac operator, then

Theorem (M.H., Thesis (2009))

Let $(M, \mathbf{C})$ be a conformal spin manifold of signature $(2,3)$ and β the skew-symmetric form on the 4-dimensional real spin bundle $\mathbf{\Delta}$. Then $\mathbf{C}$ is induced from a generic rank 2-distribution iff there is a twistor spinor $\chi \in \Gamma(\mathbf{\Delta})$ with non-vanishing $\beta(\chi, \not{D}\chi)$.

Outline

Decomposition of infinitesimal automorphisms

- We can relate the symmetries of a generic distribution with those of the induced conformal structure:
- A vector field $\xi \in \mathfrak{X}(M)$ is a symmetry of $\mathbf{D}$ if $\mathcal{L}_\xi(\eta) \in \Gamma(\mathbf{D})$ for all $\eta \in \Gamma(\mathbf{D})$.
- A vector field $\xi \in \mathfrak{X}(M)$ is said to be a *conformal Killing field* if it preserves the conformal structure $\mathbf{C_D}$: for every representative metric g there is an $f \in \mathrm{C}^\infty(M)$ with $\mathcal{L}_\xi g = fg$.
- Since the construction $\mathbf{D} \rightsquigarrow \mathbf{C_D}$ is functorial, one has an inclusion of symmetries of $\mathbf{D}$ into the conformal Killing fields, we write

$$\mathrm{sym}(\mathbf{D}) \hookrightarrow \mathrm{cKf}(\mathbf{C_D}).$$

Decomposition of infinitesimal automorphisms

- It follows from the description of infinitesimal automorphisms of parabolic geometries [Čap, JEMS (2008)] that the first BGG-operators of the adjoint tractor bundles $\mathcal{A}_{\mathbf{D}}M := \mathcal{G}_{\mathbf{D}} \times_P \mathfrak{g}_2$ and $\mathcal{A}_{\mathbf{C}}M := \mathcal{G}_{\mathbf{C}} \times_{\tilde{P}} \mathfrak{so}(3,4)$ describe the symmetries of **D** and the conformal Killing fields of **C**.
- Now as a G_2-module, $\mathfrak{so}(3,4)$ decomposes into $\mathbb{R}^{3,4} \oplus \mathfrak{g}_2$. This implies a decomposition of the conformal adjoint tractor bundle $\mathcal{A}_{\mathbf{C}}M$ into **S** and $\mathcal{A}_{\mathbf{D}}M$.
- This decomposition is compatible with the prolongation connections on the respective bundles. Via explicit formulas for BGG-splitting operators this yields the following decomposition theorem:

Theorem (Decomposition of conf. Killing fields via a twistor spinor)

Let $\mathbf{C_D}$ be the conformal $(2,3)$-structure induced by a generic 2-distribution $\mathbf{D} \subset TM$. Every conformal Killing field decomposes into a symmetry of the distribution $\mathbf{D}$ and another part corresponding to an Einstein scale (which may have a singularity set). Via the canonical twistor spinor $\chi \in \Gamma(\mathbf{\Delta})$ this decomposition can be made explicit:

- *An Einstein scale $\sigma \in \mathrm{C}^\infty(M)$ corresponds to the Killing field $\xi \in \mathfrak{X}(M)$ defined by the relation*

$$g(\xi, \eta) = \beta(\frac{2}{5}\sigma \not{D}\chi + \gamma(D\sigma)\chi, \gamma(\eta)\chi)$$

  for all $\eta \in \mathfrak{X}(M)$.

- *The Einstein scale part $\sigma \in \mathrm{C}^\infty(M)$ of a Killing field $\xi \in \mathfrak{X}(M)$ is given by*

$$\sigma = \beta(\frac{4}{5}\gamma(\xi)\not{D}\chi + \gamma(D\xi)\chi, \chi).$$