The eta invariant and equivariant bordism of flat manifolds

Ricardo Podestá
(Universidad Nacional de Córdoba, Argentina)

Dirac operators and special geometries

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- Joint work with **Peter Gilkey** and **Roberto Miatello**
- *"The eta invariant and equivariant bordism of flat manifolds with cyclic holonomy group of odd prime order"*, preprint

First restriction

the true title of the talk

Eta series and eta invariants of $\mathbb{Z}_p$-manifolds

by R.P., etc

Outline

Settings

General setting

- M = (compact) Riemannian manifold
- $E \to M$ = vector bundle of M
- $D : \Gamma^\infty(E) \to \Gamma^\infty(E)$ = elliptic differential operator

Our interest

- M = compact flat manifold
- D = twisted spin Dirac operator

  [but also Laplacians and Dirac-type operators]

Spectrum

Let M be a compact Riemannian manifold

Definition

The **spectrum** of D on M is the set

$$Spec_D(M) = \{\lambda \in \mathbb{R} \,:\, Df = \lambda f,\ f \in \Gamma^\infty(E)\} = \{(\lambda, d_\lambda)\}$$

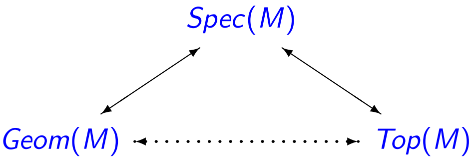
of eigenvalues counted with multiplicities

- $Spec_D(M) \subset \mathbb{R}$ is discrete
- $0 \le |\lambda_1| \le \cdots \le |\lambda_n| \nearrow \infty$
- $d_\lambda = \dim H_\lambda < \infty, \qquad H_\lambda = \lambda$-eigenvalue

Spectral geometry

Goal: to study

- $Spec(M)$
- relations between $Spec(M)$ with $Geom(M)$ and $Top(M)$



Some problems

Some problems of (our) interest

1. Computation of the spectrum
2. Isospectrality
3. **Spectral asymmetry** (this talk)

Definition

$Spec_D(M)$ is **asymmetric** $\Leftrightarrow$ $\exists\lambda \neq 0$ such that $d_\lambda \neq d_{-\lambda}$

Eta series

To study this phenomenon Atiyah-Patodi-Singer '73 introduced

- The **eta series**:

$$\eta_D(s) = \sum_{\lambda \neq 0} \frac{sgn(\lambda)}{|\lambda|^s} = \sum_{\lambda \in \mathcal{A}} \frac{d_\lambda^+ - d_\lambda^-}{|\lambda|^s} \qquad Re(s) > \tfrac{n}{d}$$

  where $n = \dim M$, $d = \text{ord } D$

- has a meromorphic continuation to $\mathbb{C}$ called the **eta function**, also denoted by $\eta_D(s)$, with (possible) simple poles in $\{s = n - k : k \in \mathbb{N}_0\}$

Eta invariants

- the **eta invariant**:

$$\eta_D = \eta_D(0)$$

It is not a trivial fact that $\eta(0) < \infty$
[APS '76, *n* odd], [Gilkey '81, *n* even]

- the **reduced eta invariant**:

$$\bar{\eta}_D = \frac{\eta_D + \dim\ker D}{2} \mod \mathbb{Z}$$

Relation with Index Theorems

- For M closed, the Index Theorem of APS states

$$\operatorname{Ind}(D) = \int_M \alpha_0$$

- For M with boundary $\partial M = N$ (under certain boundary conditions)

$$\underbrace{\operatorname{Ind}(D)}_{top} = \underbrace{\int_M \alpha_0}_{geom} - \underbrace{\bar{\eta}_{D_N}}_{spec}$$

Relation with Index Theorems

M with boundary N

- D = Dirac operator

$$\mathrm{Ind}(D) = \int_M \hat{A}(p) - \frac{\eta_{D_N} + h}{2}$$

where $h = \dim\ker D_N$

- D = signature operator, $\dim M = 4k$

$$\mathrm{Sign}(D) = \int_M L(p) - \eta_{D_N}$$

Our interest

In general

To study questions in Riemannian and spectral geometry using

- M = compact flat manifold
- D = Laplacians or Dirac-type operators

In particular (this talk)

- $M = \mathbb{Z}_p$-manifolds
- D_ℓ = twisted spin Dirac operator

Particular setting and notations

From now on we consider

- p = odd prime in $\mathbb{Z}$
- M = compact flat manifold with holonomy group $F \simeq \mathbb{Z}_p$
- ε = spin structure on M
- ρ_ℓ = character of $\mathbb{Z}_p$, $0 \leq \ell \leq p-1$
- D_ℓ = Dirac operator twisted by ρ_ℓ

Problems considered

Spectral asymmetry (this talk)

for any (M, ε) compute:

1. the eta series $\eta_\ell(s)$ associated to D_ℓ
2. the reduced eta invariants $\bar{\eta}_\ell$
3. the relative eta invariants $\bar{\eta}_\ell - \bar{\eta}_0$

Bordism groups

in addition, can we say something about
the *reduced equivariant spin bordism group* $\widetilde{\mathrm{M}}\,\mathrm{Spin}_n(B\mathbb{Z}_p)$?

Compact flat manifolds

- A **flat manifold** is a Riemannian manifold with $K \equiv 0$
- Any compact flat n-manifold M is isometric to

$$M_\Gamma = \Gamma \backslash \mathbb{R}^n, \qquad \Gamma \simeq \pi_1(M)$$

where Γ is a *Bieberbach group*, i.e.
a **discrete, cocompact, torsion-free** subgroup of

$$\mathrm{I}(\mathbb{R}^n) \simeq \mathrm{O}(n) \rtimes \mathbb{R}^n$$

- $\gamma \in \Gamma \;\Rightarrow\; \gamma = BL_b$, with $B \in \mathrm{O}(n)$, $b \in \mathbb{R}^n$ and

$$BL_b \cdot CL_c = BCL_{C^{-1}b+c}$$

Algebraic properties

- The map

$$r : \mathrm{I}(\mathbb{R}^n) \to \mathrm{O}(n) \qquad BL_b \mapsto B$$

induces the exact sequence

$$0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1$$

- Λ = lattice of $\mathbb{R}^n$ (the lattice of pure translations)
- $F \simeq \Lambda\backslash\Gamma \subset \mathrm{O}(n)$ is finite, called the **holonomy group** of Γ
- One says that M is an F-**manifold**
- fact:

$$n_B := \dim \left(\mathbb{R}^n\right)^B \geq 1 \qquad \forall\, BL_b \in \Gamma$$

The first non-trivial example

A $\mathbb{Z}_2$-manifold in dimension 2

- The Klein bottle:

$$K^2 = \langle \left[\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right] L_{\frac{e_2}{2}}, L_{e_1}, L_{e_2} \rangle \backslash \mathbb{R}^2$$

- where

$$\Lambda = \mathbb{Z}^2, \qquad F \simeq \langle \left[\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right] \rangle \simeq \mathbb{Z}_2$$

Holonomy representation

- The action by conjugation on Λ by $F \simeq \Lambda\backslash\Gamma$

$$B\, L_\lambda\, B^{-1} = L_{B\lambda}$$

defines the **integral holonomy representation**

$$\rho : F \to GL_n(\mathbb{Z})$$

- This ρ is far from determining a flat manifold uniquely
- There are (already in dim 4) non-homeomorphic orientable flat manifolds $M_\Gamma, M_{\Gamma'}$ with the same integral holonomy representation, i.e.

$$\rho_\Gamma = \rho_{\Gamma'} \qquad \text{but} \qquad M_\Gamma \not\simeq M_{\Gamma'}$$

Geometric properties

Bieberbach theorems

- $T_\Lambda \to M_\Gamma, \qquad M_\Gamma = T_\Lambda/F = (\mathbb{R}^n/\Lambda)/(\Gamma/\Lambda)$
- diffeomorphic $\Leftrightarrow$ homeomorphic $\Leftrightarrow$ homotopically equivalent

$$M_\Gamma \simeq M_{\Gamma'} \quad \Leftrightarrow \quad \Gamma \simeq \Gamma' \quad \Leftrightarrow \quad \pi_n(M_\Gamma) = \pi_n(M'_\Gamma)$$

  since $\pi_n(M_\Gamma) = 0$ for $n \geq 2$
- In each dimension, there is a finite number of affine equivalent classes of compact flat manifolds

Geometric properties

- Every finite group can be realized as the holonomy group of a compact flat manifold [Auslander-Kuranishi '57]

- Every compact flat manifold bounds, i.e.,
  if M^n is a compact flat manifold,
  then there is a N^{n+1} such that $\partial N = M$ [Hamrick-Royster '82]

$\mathbb{Z}_p$-manifolds

We will now describe the $\mathbb{Z}_p$-manifolds M_Γ

- M_Γ satisfies

$$0 \to \Lambda \simeq \mathbb{Z}^n \to \Gamma \to \mathbb{Z}_p \to 1$$

- M_Γ can be thought to be constructed by

integral representations of $\mathbb{Z}_p = \mathbb{Z}[\mathbb{Z}_p]$-modules

- $\mathbb{Z}_p$-modules were classified by Reiner [Proc AMS '57]
- $\mathbb{Z}_p$-manifolds were classified by Charlap [Annals Math '65]
- We won't need Charlap's classification, just Reiner's

Reiner $\mathbb{Z}_p$-modules

Any $\mathbb{Z}_p$-module is of the form

$$\Lambda(a, b, c, \mathfrak{a}) := \mathfrak{a} \oplus (a-1)\,\mathcal{O} \oplus b\,\mathbb{Z}[\mathbb{Z}_p] \oplus c\,Id$$

where

- $a, b, c \in \mathbb{N}_0$, $a + b > 0$
- ξ = primitive p^{th}-root of unity
- $\mathcal{O} = \mathbb{Z}[\xi]$ = ring of algebraic integers in $\mathbb{Q}(\xi)$
- $\mathfrak{a}$ = ideal in $\mathcal{O}$
- $\mathbb{Z}[\mathbb{Z}_p]$ = group ring over $\mathbb{Z}$
- Id = trivial $\mathbb{Z}_p$-module

$\mathbb{Z}_p$-actions

- The actions on the modules are given by **multiplication by** ξ
- In matrix form, the action of ξ on $\mathcal{O}$ and $\mathbb{Z}[\mathbb{Z}_p]$ are given by

$$C_p = \begin{pmatrix} 0 & & & & -1 \\ 1 & 0 & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 0 & -1 \\ & & & 1 & -1 \end{pmatrix} \in GL_{p-1}(\mathbb{Z}), \qquad J_p = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \in GL_p(\mathbb{Z})$$

- The action on $\mathfrak{a}$ is given by $C_{p,a} \in GL_{p-1}(\mathbb{Z})$ with $C_{p,a} \sim C_p$
- $n_{J_p} = 1$, $n_{C_p} = n_{C_{p,a}} = 0$

Properties of $\mathbb{Z}_p$-manifolds

Proposition

Let $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ be a $\mathbb{Z}_p$-manifold with $\Gamma = \langle \gamma, \Lambda \rangle$, $\gamma = BL_b$. Then

- $(BL_b)^p = L_{b_p}$ *where* $b_p = \sum_{j=0}^{p-1} B^j b \in L_\Lambda \smallsetminus (\sum_{j=0}^{p-1} B^j)\Lambda$
- *As a* $\mathbb{Z}_p$*-module,* $\Lambda \simeq \Lambda(a, b, c, \mathfrak{a})$, *with* $c \geq 1$ *and*

$$n = a(p-1) + bp + c$$

- a, b, c *are uniquely determined by the* $\simeq$ *class of* Γ
- Γ *is conjugate in* $I(\mathbb{R}^n)$ *to a Bieberbach group* $\tilde{\Gamma} = \langle \tilde{\gamma}, \Lambda \rangle$ *with* $\tilde{\gamma} = BL_{\tilde{b}}$ *where* $B\tilde{b} = \tilde{b}$ *and* $\tilde{b} \in \frac{1}{p}\Lambda \smallsetminus \Lambda$

Properties of $\mathbb{Z}_p$-manifolds

Proposition (continued)

- $n_B = 1 \Leftrightarrow (b, c) = (0, 1)$ *and in this case* $\gamma = BL_b$ *can be chosen so that* $b = \frac{1}{p} e_n$
- *One has*

$$H_1(M_\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^{b+c} \oplus \mathbb{Z}_p^a$$
$$H^1(M_\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^{b+c}$$

  and hence $n_B = b + c = \beta_1$
- M_Γ *is orientable*

The models

For our purposes, **it will suffice to work with the "models"**

$$M_{p,a}^{b,c}(\mathfrak{a}) = \langle BL_{\frac{e_n}{p}}, \Lambda_{p,a}^{b,c}(\mathfrak{a})\rangle \setminus \mathbb{R}^n$$

where

$$\Lambda_{p,a}^{b,c}(\mathfrak{a}) = X_{\mathfrak{a}} L_{\mathbb{Z}^n} X_{\mathfrak{a}}^{-1} = X_{\mathfrak{a}} \mathbb{Z}^{n-c} \overset{\perp}{\oplus} \mathbb{Z}^c$$

for some $X_{\mathfrak{a}} \in GL_n(\mathbb{R})$

The models

and

$$B = \mathrm{diag}(\underbrace{B_p, \ldots, B_p}_{a+b}, \underbrace{1, \ldots, 1}_{b+c})$$

with

$$B_p = \begin{pmatrix} B(\frac{2\pi}{p}) & & & \\ & B(\frac{2\cdot 2\pi}{p}) & & \\ & & \ddots & \\ & & & B(\frac{2q\pi}{p}) \end{pmatrix} \qquad q = [\tfrac{p-1}{2}]$$

$$B(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad t \in \mathbb{R}$$

Exceptional $\mathbb{Z}_p$-manifolds

- In Charlap's classification there is a distinction between *exceptional* and *non-exceptional* $\mathbb{Z}_p$-manifolds
- A $\mathbb{Z}_p$-manifold is called **exceptional** if

$$\Lambda \simeq \Lambda(a, 0, 1, \mathfrak{a})$$

- We will use exceptional $\mathbb{Z}_p$-manifolds $M_{p,a}^{0,1}(\mathfrak{a})$ of dim

$$n = a(p-1) + 1 \quad (\therefore \text{ odd})$$

Example: the "tricosm"

- It is the only 3-dimensional $\mathbb{Z}_3$-manifold
- It is exceptional: $M_{3,1} = M_{3,1}^{0,1}(\mathcal{O})$, with $\mathcal{O} = \mathbb{Z}[\frac{2\pi i}{3}]$
- As a $\mathbb{Z}_3$-module, $\Lambda \simeq \mathbb{Z}[e^{\frac{2\pi i}{3}}] \oplus \mathbb{Z}$
- with $\mathbb{Z}_3$-(integral) action given by $C = \begin{pmatrix} 0 & -1 & \\ 1 & -1 & \\ & & 1 \end{pmatrix}$
- Thus

$$M_{3,1} = \langle BL_{\frac{e_3}{3}}, L_{f_1}, L_{f_2}, L_{e_3} \rangle \backslash \mathbb{R}^3$$

with

$$B = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & \\ \sqrt{3}/2 & -1/2 & \\ & & 1 \end{pmatrix} \in \mathrm{SO}(3)$$

where f_1, f_2, e_3 is a $\mathbb{Z}$-basis of $\Lambda_{3,1} = X\mathbb{Z}^2 \oplus \mathbb{Z}$ and $X \in \mathrm{GL}_3(\mathbb{R})$ is such that $X\, C\, X^{-1} = B$

Spin group and maximal torus

- The **spin group** Spin(n) is the universal covering of SO(n)

$$\pi : \mathrm{Spin}(n) \xrightarrow{2} \mathrm{SO}(n) \qquad n \geq 3$$

- A maximal torus of Spin(n) is given by

$$T = \{x(t_1, \ldots, t_m) : t_1, \ldots, t_m \in \mathbb{R}, m = [\tfrac{n}{2}]\}$$

$$x(t_1, \ldots, t_m) := \prod_{j=1}^{m} (\cos t_j + \sin t_j \, e_{2j-1} e_{2j})$$

where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$

Spin group and maximal torus

- Notation:

$$x_a(t_1, t_2, \ldots, t_q) := x(\underbrace{t_1, t_2, \ldots, t_q}_{1}, \ldots, \underbrace{t_1, t_2, \ldots, t_q}_{a}) \qquad a \in \mathbb{N}$$

- A maximal torus in $\mathrm{SO}(n)$ is given by

$$T_0 = \{x_0(t_1, \ldots, t_m) \ : \ t_1, \ldots, t_m \in \mathbb{R}\}$$

$$x_0(t_1, \ldots, t_m) := diag\big(B(t_1), \ldots, B(t_m), \text{“}1\text{”}\big)$$

- The restriction map $\pi : T \to T_0$ duplicates angles

$$x(t_1, \ldots, t_m) \mapsto x_0(2t_1, \ldots, 2t_m)$$

Spin representations

The **spin representation** of $\mathrm{Spin}(n)$ is the restriction (L_n, S_n) of any irreducible representation of $Cliff(\mathbb{C}^n)$

- $\dim_{\mathbb{C}} \mathsf{S}_n = 2^{[n/2]}$
- (L_n, S_n) is irreducible if n is odd
- (L_n, S_n) is reducible if n is even, $\mathsf{S}_n = \mathsf{S}_n^+ \oplus \mathsf{S}_n^-$
- $L_n^\pm := L_{n|\mathsf{S}_n^\pm}$ are the **half-spin representations**

Characters of spin representations

Characters of L_n, $L_n^{\pm}$ are known on the maximal torus

Lemma (Miatello-P, TAMS '06)

$$\chi_{L_n}(x(t_1,\ldots,t_m)) = 2^m \prod_{j=1}^{m} \cos t_j$$

$$\chi_{L_n^{\pm}}(x(t_1,\ldots,t_m)) = 2^{m-1}\Big(\prod_{j=1}^{m} \cos t_j \pm i^m \prod_{j=1}^{m} \sin t_j\Big)$$

where $m = [n/2]$

Spin structures

Let

- M = orientable Riemannian manifold
- $B(M) = \mathrm{SO}(n)$-principal bundle of oriented frames on M

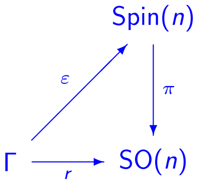
A **spin structure** on M is

- an equivariant double covering $p : \tilde{B}(M) \to B(M)$
- $\tilde{B}(M)$ is a $\mathrm{Spin}(n)$-principal bundle of M, i.e.

$$\begin{array}{ccccc} \tilde{B}(M) & \xrightarrow{\ \cdot\ } & \tilde{B}(M) & & \\ \Big\downarrow p & & \Big\downarrow p & \searrow \tilde{\pi} & \\ B(M) & \xrightarrow[\ \cdot\ ]{} & B(M) & \xrightarrow[\ \pi\ ]{} & M \end{array}$$

Spin structures on compact flat manifolds

- The spin structures on M_Γ are in a 1–1 correspondence with group homomorphisms ε commuting the diagram



Spin structures on compact flat manifolds

- Let M_Γ be a $\mathbb{Z}_p$-manifold, $\Gamma = \langle \gamma, \Lambda = \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n \rangle$. Then ε is determined by

$$\varepsilon(\gamma) \qquad \text{and} \qquad \delta_j := \varepsilon(L_{f_j}) \in \{\pm 1\} \qquad 1 \leq j \leq n$$

- $\exists$ necessary and sufficient conditions on $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ for defining a spin structure on M_Γ when $F \simeq \mathbb{Z}_2^k$ or $F \simeq \mathbb{Z}_n$

  [Miatello-P, MZ '04]

Spin structures on flat manifolds

- Not every flat manifold is spin [Vasquez '70]
- Flat tori are spin [Friedrich '84]
- $\mathbb{Z}_2^k$-manifolds are not spin (in general) but

  $\mathbb{Z}_2$-manifolds are always spin [Miatello-P '04]

Spin structures on $\mathbb{Z}_p$-manifolds

Existence

- every F-manifold with $|F|$ odd is spin (Vasquez, JDG '70)
- thus every $\mathbb{Z}_p$-manifold is spin

Number

- if M is spin, the spin structures are classified by $H^1(M, \mathbb{Z}_2)$
- If M is a $\mathbb{Z}_p$-manifold, since $H^1(M, \mathbb{Z}_2) \simeq \mathbb{Z}_2^{b+c}$,

$$\#\{\text{spin structures of } M\} = 2^{b+c} = 2^{\beta_1}$$

Spin structures on the models $M_{p,a}^{b,c}(\mathfrak{a})$

Proposition

A $\mathbb{Z}_p$-manifold M admits exactly 2^{β_1} spin structures, only one of which is of trivial type.

If $M = M_{p,a}^{b,c}(\mathfrak{a})$, its 2^{b+c} spin structures are explicitly given by

$$\varepsilon_{|\Lambda} = \big(\underbrace{1,\ldots,1}_{a(p-1)},\underbrace{\delta_1,\ldots,\delta_1}_{p},\ldots,\underbrace{\delta_b,\ldots,\delta_b}_{p},\delta_{b+1},\ldots,\delta_{b+c-1},(-1)^{h+1}\big)$$

$$\varepsilon(\gamma) = (-1)^{(a+b)[\frac{q+1}{2}]+h+1}\, x_{a+b}\big(\tfrac{\pi}{p},\tfrac{2\pi}{p},\ldots,\tfrac{q\pi}{p}\big)$$

Note: here $\varepsilon_{|\Lambda} = \big(\varepsilon(L_{f_1}),\ldots,\varepsilon(L_{f_n})\big) \in \{\pm 1\}^n$

Spin structures on exceptional $\mathbb{Z}_p$-manifolds

Remark

If M is an exceptional $\mathbb{Z}_p$-manifold, i.e. $M \simeq M_{p,a}^{0,1}(\mathfrak{a})$, then M has only 2 spin structures $\varepsilon_1, \varepsilon_2$ given by

$$\varepsilon_{h|\Lambda} = \big(1, \ldots, 1, (-1)^{h+1}\big)$$

$$\varepsilon_h(\gamma) = (-1)^{a[\frac{q+1}{2}]+h+1}\, x_a\big(\tfrac{\pi}{p}, \tfrac{2\pi}{p}, \ldots, \tfrac{q\pi}{p}\big)$$

with $h = 1, 2$. In particular, ε_1 is of trivial type

Twisted Dirac operators on flat manifolds

- Let (M_Γ, ε) = compact flat spin n-manifold
  $\rho : \Gamma \to U(V)$ = unitary representation such that $\rho_{|\Lambda} = 1$
- The **spin Dirac operator twisted by** ρ is

$$D_\rho = \sum_{i=1}^{n} L_n(e_i)\, \frac{\partial}{\partial x_i}$$

where $\{e_1, \ldots, e_n\}$ is an o.n.b. of $\mathbb{R}^n$

Twisted Dirac operators on flat manifolds

D_ρ acts on smooth sections of the spinor bundle

$$D_\rho : \Gamma^\infty(\mathcal{S}_\rho(M_\Gamma, \varepsilon)) \to \Gamma^\infty(\mathcal{S}_\rho(M_\Gamma, \varepsilon))$$

where

$$\mathcal{S}_\rho(M_\Gamma, \varepsilon) = \Gamma \backslash (\mathbb{R}^n \times (\mathsf{S}_n \otimes V)) \to \Gamma \backslash \mathbb{R}^n$$

$$\gamma \cdot (x, \omega \otimes v) = \big(\gamma x, L\big(\varepsilon(\gamma)\big)(\omega) \otimes \rho(\gamma) v\big)$$

Spectrum of D_ρ on compact flat manifolds

- The spectrum of D_ρ on (M_Γ, ε) is

$$Spec_{D_\rho}(M_\Gamma, \varepsilon) = \{ (\pm 2\pi\mu, d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon)) : \mu = ||v||, v \in \Lambda^*_\varepsilon \}$$

where

$$\Lambda^*_\varepsilon = \{ u \in \Lambda^* : \varepsilon(L_\lambda) = e^{2\pi i \lambda \cdot u} \quad \forall \lambda \in \Lambda \}$$

Theorem (Miatello-P, TAMS '06)

The multiplicities of $\lambda = \pm 2\pi\mu$ *are given by*

(i) *for* $\mu > 0$*:*

$$d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon) = \frac{1}{|F|} \sum_{\gamma = BL_b \in \Lambda\backslash\Gamma} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\, \chi_{L_{n-1}^{\pm\sigma(u,x_\gamma)}}(x_\gamma)$$

with $(\Lambda^*_{\varepsilon,\mu})^B = \{v \in \Lambda^*_\varepsilon \,:\, Bv = v, ||v|| = \mu\}$

(ii) *for* $\mu = 0$*:*

$$d_{\rho,0}(\Gamma,\varepsilon) = \begin{cases} \frac{1}{|F|} \sum_{\gamma \in \Lambda\backslash\Gamma} \chi_\rho(\gamma)\, \chi_{L_n}(\varepsilon(\gamma)) & \varepsilon_{|\Lambda} = 1 \\ 0 & \varepsilon_{|\Lambda} \neq 1 \end{cases}$$

Eta series of flat manifolds

★ For $\eta_{D_\rho}(s)$ we have

- a general expression for arbitrary compact flat manifolds
- an explicit formula for:
  - $\mathbb{Z}_2^k$-manifolds
  - a family of $\mathbb{Z}_4$-manifolds
  - $\mathbb{Z}_p$-manifolds in the untwisted case

([Miatello-P, TAMS '06, PAMQ '08], [P, Rev UMA '05])

★ **We will compute $\eta_{D_\ell}(s)$ for any $\mathbb{Z}_p$-manifold**

Notations

From now on we consider

- $p = 2q + 1$ an odd prime
- $M = \mathbb{Z}_p$-manifold of dim n
- $\varepsilon_h =$ spin structure on M, $1 \leq h \leq 2^{b+c}$
- For $0 \leq \ell \leq p - 1$, the characters

$$\rho_\ell : \mathbb{Z}_p \to \mathbb{C}^* \qquad k \mapsto e^{\frac{2\pi i k \ell}{p}}$$

- $D_\ell =$ Dirac operator twisted by ρ_ℓ
- $d^{\pm}_{\ell,\mu,h} := d^{\pm}_{\rho_\ell,\mu}(M, \varepsilon_h)$

The eta series for $\mathbb{Z}_p$-manifolds

- Recall that

$$\eta_{\ell,h}(s) = \sum_{\pm 2\pi\mu \in \mathcal{A}} \frac{d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}}{(2\pi\mu)^s}$$

- Although the expressions for $d^{\pm}_{\ell,\mu,h}$ are not explicit, the differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$ can be computed

An important reduction

- For flat manifolds, by a result in [Miatello-P, TAMS '06],

$$n_B > 1 \quad \forall\, BL_b \in \Gamma \quad \Rightarrow \quad Spec_D(M) \text{ is symmetric}$$

thus

$$d^{+}_{\ell,\mu,h} = d^{-}_{\ell,\mu,h} \quad \Rightarrow \quad \eta_D(s) \equiv 0$$

- For $\mathbb{Z}_p$-manifolds, since $n_B = 1 \Leftrightarrow (b, c) = (0, 1)$ then

$\eta(s) \equiv 0$ for **non-exceptional** $\mathbb{Z}_p$-manifolds

An important reduction

We can focus on exceptional $\mathbb{Z}_p$-manifolds

- Thus, it **suffices** to compute

$$d^{+}_{\ell,\mu,h} - d^{-}_{\ell,\mu,h}, \qquad \eta_{\ell,h}(s), \qquad \eta_{\ell,h}$$

for the **exceptional** $\mathbb{Z}_p$-manifolds only

- In particular,

we can assume that $M = M^{0,1}_{p,a}(\mathfrak{a})$

(i.e. $b = \frac{1}{p} e_n$)

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Key lemma

For an exceptional $\mathbb{Z}_p$-manifold (M, ε_h) we have

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = \kappa_{p,a} \sum_{k=1}^{p-1} (-1)^{k(h+1)} \left(\tfrac{k}{p}\right)^a e^{\frac{2\pi i k\ell}{p}} \sin\left(\tfrac{2\pi\mu k}{p}\right)$$

where

$$\kappa_{p,a} = (-1)^{\left(\frac{p^2-1}{8}\right)a+1} \, i^{m+1} \, 2 \, p^{\frac{a}{2}-1}$$

Sketch of proof I

- Apply the general multiplicity formula to this case

$$d^{\pm}_{\ell,\mu,h} = \frac{1}{p} \sum_{k=0}^{p-1} e^{\frac{2\pi i\ell k}{p}} \sum_{u \in (\Lambda^*_{\varepsilon_h,\mu})^{B^k}} e^{-2\pi i u\cdot b_k} \chi_{L_{n-1}}^{\pm\sigma(u,x_{\gamma^k})}(\varepsilon_h(\gamma^k))$$

- note that $(\Lambda^*_{\varepsilon_h})^{B^k} = \mathbb{R}e_n$ and hence

$$(\Lambda^*_{\varepsilon_h,\mu})^{B^k} = \{\pm\mu e_n\}$$

Sketch of proof II

- Thus, we get

$$d^{\pm}_{\ell,\mu,h} = \frac{1}{p}\left(2^{m-1}|\Lambda^*_{\varepsilon_h,\mu}| + \sum_{k=1}^{p-1} e^{\frac{2\pi i k\ell}{p}} S^{\pm}_{\mu,h}(k)\right)$$

where

$$S^{\pm}_{\mu,h}(k) := e^{\frac{-2\pi i \mu k}{p}} \chi_{L^{\pm}_{n-1}}(\varepsilon_h(\gamma^k)) + e^{\frac{2\pi i \mu k}{p}} \chi_{L^{\mp}_{n-1}}(\varepsilon_h(\gamma^k))$$

(only 2-terms sums)

Sketch of proof III

- Note that

$$\varepsilon_h(\gamma^k) = (-1)^{s_{h,k}} \, x_a\Big(\frac{k\pi}{p}, \frac{2k\pi}{p}, \ldots, \frac{qk\pi}{p}\Big)$$

for $1 \le k \le p$, where

$$s_{h,k} := k([\tfrac{q+1}{2}]a + h + 1)$$

- Compute

$$\chi_{L^{\pm}_{n-1}}(\varepsilon_h(\gamma^k)) = (-1)^{s_{h,k}} \, 2^{m-1} \Big\{ \Big(\prod_{j=1}^{q} \cos(\tfrac{jk\pi}{p}) \Big)^a \pm i^m \Big(\prod_{j=1}^{q} \sin(\tfrac{jk\pi}{p}) \Big)^a \Big\}$$

- compute the blue trigonometric products

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Proposition

Let (M, ε_h) be an exceptional $\mathbb{Z}_p$-manifold. Put $r = [\frac{n}{4}]$.

(i) *If* a **is even** *then*

$$d^+_{0,\mu,h} - d^-_{0,\mu,h} = 0$$

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = \begin{cases} \pm(-1)^r p^{\frac{a}{2}} & p \mid h(\ell \mp \mu) \\ 0 & \textit{otherwise} \end{cases}$$

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Proposition (continued)

(ii) *If* a **is odd** *then*

$$d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h} = (-1)^{q+r}\left(\left(\tfrac{2(\ell-\mu)}{p}\right) - \left(\tfrac{2(\ell+\mu)}{p}\right)\right) p^{\frac{a-1}{2}}$$

In particular,

$$d^+_{0,\mu,h} - d^-_{0,\mu,h} = \begin{cases} 0 & p \equiv 1\,(4) \\ (-1)^r\, 2\left(\tfrac{2\mu}{p}\right) p^{\frac{a-1}{2}} & p \equiv 3\,(4) \end{cases}$$

The differences $d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}$

Sketch of proof

- Rewrite $d^+_{0,\mu,h} - d^-_{0,\mu,h}$ in terms of "character Gauß sums"

$$d^+_{0,\mu,h} - d^-_{0,\mu,h} = \begin{cases} -i^{m+1}\, 2\, p^{\frac{a}{2}-1} F_h^{\chi_0}(\ell, c_\mu) & a \text{ even} \\ -i^{m+1}\, 2\, p^{\frac{a}{2}-1} (-1)^{\left(\frac{p^2-1}{8}\right)} F_h^{\chi_p}(\ell, c_\mu) & a \text{ odd} \end{cases}$$

where
χ_0 = trivial character mod p
χ_p = quadratic character mod p

- Compute the blue Gauß sums

The eta series $\eta_{\ell,h}(s)$

$\eta_{\ell,h}(s)$ can be computed in terms of **Hurwitz zeta functions**

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$$

where

$$\alpha \in (0,1] \qquad Re\,(s) > 1$$

The eta series $\eta_{\ell,h}(s)$

Theorem

Let (M, ε_h) *be an exceptional* $\mathbb{Z}_p$*-manifold. Put* $r = [\frac{n}{4}]$, $t = [\frac{p}{4}]$.

(i) *If* a **is even** *then* $\eta_{0,1}(s) = \eta_{0,2}(s) = 0$ *and for* $\ell \neq 0$

$$\eta_{\ell,1}(s) = \frac{(-1)^r}{(2\pi p)^s}\, p^{\frac{a}{2}} \left(\zeta(s, \tfrac{\ell}{p}) - \zeta(s, \tfrac{p-\ell}{p})\right)$$

$$\eta_{\ell,2}(s) = \begin{cases} \frac{(-1)^r}{(2\pi p)^s}\; p^{\frac{a}{2}} \left(\zeta(s, \frac{1}{2} + \frac{\ell}{p}) - \zeta(s, \frac{1}{2} - \frac{\ell}{p})\right) & 1 \leq \ell \leq q \\ \frac{(-1)^r}{(2\pi p)^s}\, p^{\frac{a}{2}} \left(\zeta(s, \frac{1}{2} - \frac{p-\ell}{p}) - \zeta(s, \frac{1}{2} + \frac{p-\ell}{p})\right) & q < \ell < p \end{cases}$$

The eta series $\eta_{\ell,h}(s)$

Theorem (continued)

(ii) *If* a **is odd** *then*

$$\eta_{\ell,1}(s) = \frac{(-1)^{t+r}}{(2\pi p)^s}\, p^{\frac{a-1}{2}} \sum_{j=1}^{p-1} \Big(\big(\tfrac{\ell-j}{p}\big) - \big(\tfrac{\ell+j}{p}\big) \Big)\, \zeta(s, \tfrac{j}{p})$$

$$\eta_{\ell,2}(s) = \frac{(-1)^{q+r}}{(\pi p)^s}\, p^{\frac{a-1}{2}} \sum_{j=0}^{p-1} \Big(\big(\tfrac{2\ell-(2j+1)}{p}\big) - \big(\tfrac{2\ell+(2j+1)}{p}\big) \Big)\, \zeta(s, \tfrac{2j+1}{2p})$$

In particular, $\eta_{0,h}(s) = 0$ *for* $p \equiv 1\,(4)$

Papers on eta-invariants

Incomplete list of authors

M. Atiyah, V. Patodi, I. Singer
P. Gilkey
W. Müller
N. Hitchin
H. Donelly
U. Bunke

and others

S. Goette
J. Park
R. Mazzeo, R. Melrose, P. Piazza
X. Dai, D. Freed
J. Brüning, M. Lesch
W. Zhang

Computation of eta invariants

We will now compute, for $0 \le \ell \le p-1$,

- the eta invariants

$$\eta_\ell = \eta_\ell(0)$$

- the reduced eta invariants

$$\bar{\eta}_\ell = \frac{\eta_\ell + \dim \ker D_\ell}{2} \mod \mathbb{Z}$$

- the relative eta invariants

$$\bar{\eta}_\ell - \bar{\eta}_0$$

Eta invariants $\eta_{\ell,h}$

Theorem

Let (M, ε_h) be an exceptional $\mathbb{Z}_p$-manifold. Put $r = [\frac{n}{4}]$, $t = [\frac{p}{4}]$.

(i) *If* a **is even** *then*

$$\eta_{0,h} = 0$$

and for $\ell \neq 0$

$$\eta_{\ell,1} = (-1)^r \, p^{\frac{a}{2}-1} \, (p - 2\ell)$$

$$\eta_{\ell,2} = (-1)^r \, p^{\frac{a}{2}-1} \, 2\big([\tfrac{2\ell}{p}]p - \ell\big)$$

Eta invariants $\eta_{\ell,h}$

Theorem (continued)

(ii) *If* a **is odd** *then*

$$\eta_{\ell,1} = \begin{cases} (-1)^{t+r+1} p^{\frac{a-1}{2}}\, S_1^-(\ell,p) & p \equiv 1\,(4) \\ (-1)^{t+r} p^{\frac{a-1}{2}} \big(S_1^+(\ell,p) + \frac{2}{p}\sum_{j=1}^{p-1} \big(\frac{j}{p}\big) j\big) & p \equiv 3\,(4) \end{cases}$$

$$\eta_{\ell,2} = \begin{cases} (-1)^{q+r+1} p^{\frac{a-1}{2}} \Big(S_2^-(\ell,p) - \big(\frac{2}{p}\big) S_1^-(\ell,p)\Big) & p \equiv 1\,(4) \\ (-1)^{q+r} p^{\frac{a-1}{2}} \Big\{ S_2^+(\ell,p) + \big(\frac{2}{p}\big)\, S_1^+(\ell,p) + \\ \qquad + \big(1 - \big(\frac{2}{p}\big)\big)\frac{2}{p}\sum_{j=1}^{p-1}\big(\frac{j}{p}\big) j \Big\} & p \equiv 3\,(4) \end{cases}$$

Eta invariants $\eta_{\ell,h}$

where

Notation

$$S_1^{\pm}(\ell,p) := \sum_{j=1}^{p-\ell-1} \left(\tfrac{j}{p}\right) \quad \pm \quad \sum_{j=1}^{\ell-1} \left(\tfrac{j}{p}\right)$$

$$S_2^{\pm}(\ell,p) := \sum_{j=1}^{p+\left[\frac{2\ell}{p}\right]p-2\ell-1} \left(\tfrac{j}{p}\right) \quad \pm \quad \sum_{j=1}^{2\ell-\left[\frac{2\ell}{p}\right]p-1} \left(\tfrac{j}{p}\right)$$

Eta invariants $\eta_{\ell,h}$

Sketch of proof

- Evaluate $\eta_{\ell,h}(s)$ in $s = 0$, using that $\zeta(0,\alpha) = \frac{1}{2} - \alpha$
- a even trivial, a odd:

$$\eta_{\ell,1}(0) = (-1)^{t+r}\, p^{\frac{a-1}{2}} \sum_{j=1}^{p-1} \Big(\big(\tfrac{\ell-j}{p}\big) - \big(\tfrac{\ell+j}{p}\big) \Big) \big(\tfrac{1}{2} - \tfrac{j}{p}\big)$$

$$\eta_{\ell,2}(0) = (-1)^{q+r}\, p^{\frac{a-1}{2}} \sum_{j=0}^{p-1} \Big(\big(\tfrac{2\ell-(2j+1)}{p}\big) - \big(\tfrac{2\ell+(2j+1)}{p}\big) \Big) \big(\tfrac{p-1}{2p} - \tfrac{j}{p}\big)$$

- Study the violet sums!

Eta invariants $\eta_{\ell,h}$: integrality, parity

Corollary

(i) *If* $(p, a) \neq (3, 1)$ *then*

$$\eta_{\ell,h} \in \mathbb{Z}$$

Furthermore, $\eta_{0,h}$ *is even,* $\eta_{\ell,1}$ *is odd and* $\eta_{\ell,2}$ *is even* $(\ell \neq 0)$

(ii) *If* $(p, a) = (3, 1)$ *then*

$$\eta_{\ell,1} = \begin{cases} -2/3 & \ell = 0 \\ 1/3 & \ell = 1, 2 \end{cases} \qquad \eta_{\ell,2} = 4/3 \quad \ell = 0, 1, 2$$

dim ker D_ℓ

It is known that

$$\begin{aligned} \dim \ker D &= \text{multiplicity of the 0-eigenvalue} \\ &= \#\ \text{independent harmonic spinors} \end{aligned}$$

So, we will compute

$$d_{\ell,0,h} = \dim \ker D_{\ell,h}$$

dim ker D_ℓ

Proposition

Let (M, ε_h) be **any** *$\mathbb{Z}_p$-manifold, $1 \leq h \leq 2^{b+c}$.*
Then $d_{\ell,0}(\varepsilon_h) = 0$ for $h \neq 1$ and

$$d_{\ell,0}(\varepsilon_1) = \frac{2^{\frac{b+c-1}{2}}}{p}\Big(2^{(a+b)q} + (-1)^{(\frac{p^2-1}{8})(a+b)}\,(p\delta_{\ell,0} - 1)\Big)$$

In particular, if $b + c > 1$ then $d_{\ell,0,1}$ is even for any $0 \leq \ell \leq p-1$ while if $b + c = 1$ then $d_{0,0,1}$ is even and $d_{\ell,0,1}$ is odd for $\ell \neq 0$.

dim ker D_ℓ

sketch of proof:

- We have

$$d_{\ell,0}(\varepsilon_1) = \frac{1}{p}\sum_{k=0}^{p-1} e^{\frac{2\pi i k\ell}{p}}\ \chi_{L_n}(\varepsilon_1(\gamma^k))$$

and

$$\varepsilon_1(\gamma^k) = (-1)^{k[\frac{q+1}{2}](a+b)}\ x_{a+b}\left(\frac{k\pi}{p}, \frac{2k\pi}{p}, \ldots, \frac{qk\pi}{p}\right)$$

- Thus

$$d_{\ell,0,1} = \frac{2^m}{p}\sum_{k=0}^{p-1}(-1)^{k[\frac{q+1}{2}](a+b)}\left(\prod_{j=1}^{q}\cos\left(\frac{jk\pi}{p}\right)\right)^{a+b} e^{\frac{2\pi i k\ell}{p}}$$

The reduced eta invariant of $\mathbb{Z}_p$-manifolds

Recall that $\bar{\eta}_{\ell,h} = \frac{1}{2}(\eta_{\ell,h} + d_{\ell,0,h}) \mod \mathbb{Z}$

Studying the parities of $\eta_{\ell,h}$ and $d_{0,\ell,h}$ we get our **main result**

Theorem

Let p be an odd prime and $0 \leq \ell \leq p-1$. Let M be a $\mathbb{Z}_p$-manifold with spin structure ε_h, $1 \leq h \leq 2^{b+c}$. Then

$$\bar{\eta}_{\ell,h} = \begin{cases} \frac{2}{3} & \mathit{mod}\ \mathbb{Z} \qquad p = n = 3 \\ 0 & \mathit{mod}\ \mathbb{Z} \qquad \mathit{otherwise} \end{cases}$$

Moreover, the relative eta invariants are

$$\bar{\eta}_{\ell,h} - \bar{\eta}_{0,h} = 0$$

The exception: the tricosm

- There is only one $\mathbb{Z}_p$-manifold with non-trivial reduced eta invariant
- The tricosm: the only 3-dimensional $\mathbb{Z}_3$-manifold $M = M_{3,1}$

Case $\ell = 0$

- In the untwisted case $\ell = 0$ we have a better insight
- and there is a close relation with number theory

We can put

- $\eta(s)$ is in terms of the *L*-**function**

$$L(s, \chi_p) = \sum_{n=1}^{\infty} \frac{\left(\frac{n}{p}\right)}{n^s}$$

- η is in terms of **class numbers** h_{-p} of **imaginary quadratic fields** $\mathbb{Q}(\sqrt{-p}) = \mathbb{Q}(i\sqrt{p})$

Case $\ell = 0$, eta series

Theorem ([Miatello-P, PAMQ '08])

Let (M, ε_h) be a $\mathbb{Z}_p$-manifold of dimension n.

If M is exceptional and $n \equiv p \equiv 3\,(4)$, $a \equiv 1\,(4)$ then

$$\eta_{0,1}(s) = \frac{-2}{(2\pi p)^s}\, p^{\frac{a-1}{2}}\, L(s, \chi_p)$$

$$\eta_{0,2}(s) = \frac{2}{(2\pi p)^s}\, p^{\frac{a-1}{2}}\, \left(1 - \left(\tfrac{2}{p}\right) 2^s\right) L(s, \chi_p)$$

In particular,

$$\eta_{0,2}(s) = \left(\left(\tfrac{2}{p}\right) 2^s - 1\right) \eta_{0,1}(s)$$

Otherwise we have $\eta_{0,1}(s) = \eta_{0,2}(s) \equiv 0$

Case $\ell = 0$, eta invariants

Theorem ([Miatello-P, PAMQ '08])

In the non-trivial case before, we have

(i) *If* $p = 3$ *then* $\eta_{0,1} = -2 \cdot 3^{\frac{a-3}{2}}$ *and* $\eta_{\varepsilon_2} = 4 \cdot 3^{\frac{a-3}{2}}$

(ii) *If* $p \geq 7$ *then*

$$\eta_{0,1} = -2\, p^{\frac{a-1}{2}}\, h_{-p}$$

$$\eta_{0,2} = \left(\left(\tfrac{2}{p}\right) - 1\right) \eta_{\varepsilon_1} = \begin{cases} 0 & p \equiv 7\ (8) \\ 4\, p^{\frac{a-1}{2}}\, h_{-p} & p \equiv 3\ (8) \end{cases}$$

where h_{-p} = *the class number of* $\mathbb{Q}(\sqrt{-p})$

Case $\ell = 0$, trigonometric expressions

Proposition ([Miatello-P, PAMQ '08])

The eta invariants of an exceptional $\mathbb{Z}_p$-manifold (M, ε_h) can be expressed in the following ways

$$\eta_{0,1} = -p^{\frac{a-2}{2}} \sum_{k=1}^{p-1} \left(\tfrac{k}{p}\right) \cot\left(\tfrac{\pi k}{p}\right) = -p^{\frac{a-2}{2}} \sum_{k=1}^{p-1} \cot\left(\tfrac{\pi k^2}{p}\right)$$

$$\eta_{0,2} = p^{\frac{a-1}{2}} \sum_{k=1}^{p-1} (-1)^k \left(\tfrac{k}{p}\right) \csc\left(\tfrac{\pi k}{p}\right)$$

Legendre symbol

Definition

For p an odd prime, the **Legendre symbol** of k mod p is

$$\left(\frac{k}{p}\right) := \begin{cases} 1 & \text{if } x^2 \equiv k\,(p) \text{ has a solution} \\ -1 & \text{if } x^2 \equiv k\,(p) \text{ does not have a solution} \end{cases}$$

if $(k,p) = 1$ and $(\frac{k}{p}) = 0$ otherwise

We have

$$(\tfrac{2}{p}) = (-1)^{\frac{p^2-1}{8}} \qquad (\tfrac{-1}{p}) = (-1)^{\frac{p-1}{2}}$$

Trigonometric products

Lemma

Let $p = 2q + 1$ be an odd prime, $k \in \mathbb{N}$ with $(k, p) = 1$. Then

$$(i) \qquad \prod_{j=1}^{q} \sin\left(\frac{jk\pi}{p}\right) = (-1)^{(k-1)\left(\frac{p^2-1}{8}\right)} \left(\frac{k}{p}\right) 2^{-q} \sqrt{p}$$

$$(ii) \qquad \prod_{j=1}^{q} \cos\left(\frac{jk\pi}{p}\right) = (-1)^{(k-1)\left(\frac{p^2-1}{8}\right)} 2^{-q}$$

Sketch of proof

(i) use

- identities of $\Gamma(z)$

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

$$(2\pi)^{\frac{d-1}{2}}\,\Gamma(z) = d^{z-\frac{1}{2}}\,\Gamma(\tfrac{z}{d})\Gamma(\tfrac{z+1}{d})\cdots\Gamma(\tfrac{z+(d-1)}{d})$$

- Gauß Lemma

$$(-1)^{\sum_{j=1}^{(p-1)/2}[\frac{jk}{p}]} = (-1)^{(k-1)(\frac{p^2-1}{8})}\left(\frac{k}{p}\right)$$

(ii) follows from (i)

Classical character Gauß sums

Definition

For $\ell \in \mathbb{N}_0$ the **character Gauß sum** is

$$G(\ell, p) := G(\chi_p, \ell) = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right) e^{\frac{2\pi i \ell k}{p}}$$

We have

$$G(\ell, p) = \begin{cases} \left(\frac{\ell}{p}\right) \sqrt{p} & p \equiv 1\ (4) \\ i \left(\frac{\ell}{p}\right) \sqrt{p} & p \equiv 3\ (4) \end{cases}$$

Modified character Gauß sums

Definition

For $p \in \mathbb{P}$, $\ell \in \mathbb{N}_0$, $c \in \mathbb{N}$, $1 \leq h \leq 2$, χ a character mod p we define

$$G_h^\chi(\ell) := \sum_{k=1}^{p-1} (-1)^{k(h+1)}\, \chi(k)\, e^{\frac{\pi i k\,(2\ell+\delta_{h,2})}{p}}$$

$$F_h^\chi(\ell, c) := \sum_{k=1}^{p-1} (-1)^{k(h+1)}\, \chi(k)\, e^{\frac{2\pi i \ell k}{p}} \sin\left(\frac{\pi k\,(2c+\delta_{h,2})}{p}\right)$$

We want to compute $G_h^\chi(\ell)$ and $F_h^\chi(\ell, c)$ for

- $\chi = \chi_0 =$ **trivial character** mod p
- $\chi = \chi_p =$ **quadratic character** mod p given by $\left(\frac{\cdot}{p}\right)$

The sums $G_h^{\chi}(\ell)$

$$G_1^{\chi_0}(\ell) = \sum_{k=1}^{p-1} e^{\frac{2\ell\pi ik}{p}}$$

$$G_2^{\chi_0}(\ell) = \sum_{k=1}^{p-1} (-1)^k \, e^{\frac{(2\ell+1)\pi ik}{p}}$$

$$G_1^{\chi_p}(\ell) = \sum_{k=1}^{p-1} \left(\tfrac{k}{p}\right) e^{\frac{2\ell\pi ik}{p}}$$

$$G_2^{\chi_p}(\ell) = \sum_{k=1}^{p-1} (-1)^k \left(\tfrac{k}{p}\right) e^{\frac{(2\ell+1)\pi ik}{p}}$$

The sums $G_h^{\chi_0}(\ell)$

Proposition

We have

$$G_1^{\chi_0}(\ell) = \begin{cases} p-1 & p \mid \ell \\ -1 & p \nmid \ell \end{cases}$$

$$G_2^{\chi_0}(\ell) = \begin{cases} p-1 & p \mid 2\ell+1 \\ -1 & p \nmid 2\ell+1 \end{cases}$$

In particular,

$$G_1^{\chi_0}(\ell) \equiv G_2^{\chi_0}(\ell) \equiv p-1 \quad mod\ p$$

The sums $G_h^{\chi_p}(\ell)$

Proposition

We have

$$G_1^{\chi_p}(\ell) = \delta(p)\left(\frac{\ell}{p}\right)\sqrt{p}$$
$$G_2^{\chi_p}(\ell) = \delta(p)\left(\frac{2}{p}\right)\left(\frac{2\ell+1}{p}\right)\sqrt{p}$$

where

$$\delta(p) := \begin{cases} 1 & p \equiv 1\ (4) \\ i & p \equiv 3\ (4) \end{cases}$$

In particular, $G_1^{\chi_p}(\ell) = 0$ *if* $p \mid \ell$ *and* $G_2^{\chi_p}(\ell) = 0$ *if* $p \mid 2\ell + 1$

The sums $F_h^{\chi}(\ell, c)$

$$F_1^{\chi_0}(\ell, c) = \sum_{k=1}^{p-1} e^{\frac{2\ell\pi i k}{p}} \sin\left(\tfrac{2c\pi k}{p}\right)$$

$$F_2^{\chi_0}(\ell, c) = \sum_{k=1}^{p-1} (-1)^k e^{\frac{2\ell\pi i k}{p}} \sin\left(\tfrac{(2c+1)\pi k}{p}\right)$$

$$F_1^{\chi_p}(\ell, c) = \sum_{k=1}^{p-1} \left(\tfrac{k}{p}\right) e^{\frac{2\ell\pi i k}{p}} \sin\left(\tfrac{2c\pi k}{p}\right)$$

$$F_2^{\chi_p}(\ell, c) = \sum_{k=1}^{p-1} (-1)^k \left(\tfrac{k}{p}\right) e^{\frac{2\ell\pi i k}{p}} \sin\left(\tfrac{(2c+1)\pi k}{p}\right)$$

The sums $F_h^{\chi_0}(\ell, c)$

Proposition

We have

1. *If* $p \mid \ell$ *then* $F_h^{\chi_0}(\ell, c) = 0$ *for* $h = 1, 2$
2. *If* $p \nmid \ell$ *then*

$$F_1^{\chi_0}(\ell, c) = \begin{cases} \pm i\, \frac{p}{2} & \text{if } p \mid \ell \mp c \\ 0 & \text{otherwise} \end{cases}$$

$$F_2^{\chi_0}(\ell, c) = \begin{cases} \pm i\, \frac{p}{2} & \text{if } p \mid 2(\ell \mp c) \mp 1 \\ 0 & \text{otherwise} \end{cases}$$

The sums $F_h^{\chi_p}(\ell, c)$

Proposition

We have

$$F_1^{\chi_p}(\ell,c) = i\,\delta(p)\,\Big(\big(\tfrac{\ell-c}{p}\big) - \big(\tfrac{\ell+c}{p}\big)\Big)\,\tfrac{\sqrt{p}}{2}$$

$$F_2^{\chi_p}(\ell,c) = i\,\delta(p)\,\big(\tfrac{2}{p}\big)\Big(\big(\tfrac{2(\ell-c)-1}{p}\big) - \big(\tfrac{2(\ell+c)+1}{p}\big)\Big)\,\tfrac{\sqrt{p}}{2}$$

In particular, if $p \mid \ell$ *then*

$$F_1^{\chi_p}(\ell,c) = \begin{cases} 0 & p \equiv 1\,(4) \\ \big(\tfrac{c}{p}\big)\sqrt{p} & p \equiv 3\,(4) \end{cases}$$

$$F_2^{\chi_p}(\ell,c) = \begin{cases} 0 & p \equiv 1\,(4) \\ \big(\tfrac{2}{p}\big)\big(\tfrac{2c+1}{p}\big)\,\sqrt{p} & p \equiv 3\,(4) \end{cases}$$

Sums involving Legendre symbols

For $0 \leq \ell \leq p-1$, we want to compute the sums

Definition

$$S_1(\ell, p) := \sum_{j=1}^{p-1} \left(\left(\tfrac{\ell-j}{p}\right) - \left(\tfrac{\ell+j}{p}\right) \right) j$$

$$S_2(\ell, p) := \sum_{j=0}^{p-1} \left(\left(\tfrac{2\ell-(2j+1)}{p}\right) - \left(\tfrac{2\ell+(2j+1)}{p}\right) \right) j$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=1}^{p-1} \left(\frac{k\ell \pm j}{p}\right) = -\left(\frac{k\ell}{p}\right) \qquad k \in \mathbb{Z}$$

$$\sum_{j=0}^{p-1} \left(\frac{2\ell \pm (2j+1)}{p}\right) = 0$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=1}^{p-1} \left(\tfrac{\ell+j}{p}\right) j = p \sum_{j=1}^{\ell-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j$$

$$\sum_{j=1}^{p-1} \left(\tfrac{\ell-j}{p}\right) j = \left(\tfrac{-1}{p}\right) \left(p \sum_{j=1}^{p-\ell-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j \right)$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=1}^{p-1} \left(\tfrac{2\ell+j}{p}\right) j = p \sum_{j=1}^{2\ell-\left[\frac{2\ell}{p}\right] p-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j$$

$$\sum_{j=1}^{p-1} \left(\tfrac{2\ell-j}{p}\right) j = \left(\tfrac{-1}{p}\right) \left(p \sum_{j=1}^{p+\left[\frac{2\ell}{p}\right] p-2\ell-1} \left(\tfrac{j}{p}\right) + \sum_{j=1}^{p-1} \left(\tfrac{j}{p}\right) j\right)$$

Sums involving Legendre symbols

Lemma

$$\sum_{j=0}^{p-1} \left(\frac{2\ell \pm (2j+1)}{p}\right) j = \sum_{j=1}^{p-1} \left(\frac{2\ell \pm j}{p}\right) j - \left(\frac{2}{p}\right) \sum_{j=1}^{p-1} \left(\frac{\ell \pm j}{p}\right) j$$

Sums involving Legendre symbols

Proposition

$$S_1(\ell,p)=\begin{cases} p\,S_1^-(\ell,p) & p\equiv 1\,(4)\\ -p\,S_1^+(\ell,p)-2\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)j & p\equiv 3\,(4)\end{cases}$$

$$S_2(\ell,p)=\begin{cases} p\Big(S_2^-(\ell,p)-\left(\frac{2}{p}\right)S_1^-(\ell,p)\Big) & p\equiv 1\,(4)\\ -p\Big(S_2^+(\ell,p)-\left(\frac{2}{p}\right)S_1^+(\ell,p)\Big)+ & \\ \qquad +2\big(\left(\frac{2}{p}\right)-1\big)\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)j & p\equiv 3\,(4)\end{cases}$$

Sums involving Legendre symbols

where we have used the notations

$$S_1^{\pm}(\ell, p) := \sum_{j=1}^{p-\ell-1} \left(\tfrac{j}{p}\right) \pm \sum_{j=1}^{\ell-1} \left(\tfrac{j}{p}\right)$$

$$S_2^{\pm}(\ell, p) := \sum_{j=1}^{p+\left[\frac{2\ell}{p}\right] p-2\ell-1} \left(\tfrac{j}{p}\right) \pm \sum_{j=1}^{2\ell-\left[\frac{2\ell}{p}\right] p-1} \left(\tfrac{j}{p}\right)$$

Note that

$$S_1^{\pm}(0, p) = S_1^{\pm}(0, p) = 0$$

since $\sum_{1\le j\le p-1} \left(\frac{j}{p}\right) = 0$

Sums involving Legendre symbols

Dirichlet's class number formula

We recall

$$\frac{1}{p}\sum_{j=0}^{p-1}\left(\frac{j}{p}\right)j = -2\frac{h_{-p}}{\omega_{-p}} = \begin{cases} -h_{-p} & p \geq 5, \\ -2/3 & p = 3, \end{cases}$$

where

- h_{-p} = class number of $\mathbb{Q}(\sqrt{-p}) \subset \mathbb{Q}(\xi_p)$,
- ω_{-p} = the number of p^{th}-roots of unity of $\mathbb{Q}(\sqrt{-p})$.

In fact, and $h_{-3} = 1$, $\omega_{-3} = 6$ and $\omega_{-p} = 2$ for $p \geq 5$.

Sums involving Legendre symbols

Corollary

For $p \geq 5$,

$$S_1(0,p) = \begin{cases} 0 & p \equiv 1\,(4) \\ -2h_{-p} & p \equiv 3\,(4) \end{cases}$$

$$S_2(\ell,p) = \begin{cases} 0 & p \equiv 1\,(4) \\ 2\big(\big(\tfrac{2}{p}\big) - 1\big)\,h_{-p} & p \equiv 3\,(4) \end{cases}$$

Bordism groups

The integrality of $\eta_\ell - \eta_0$ implies

Theorem

Let $(M, \varepsilon, \sigma_p)$ and $(M, \varepsilon, \sigma_0)$ denote a $\mathbb{Z}_p$-manifold M equipped with a spin structure ε and with the natural and the trivial $\mathbb{Z}_p$-structures

$$\sigma_p : \mathbb{Z}_p \to T_\Lambda \to M$$
$$\sigma_0 : \mathbb{Z}_p \to M \times \mathbb{Z}_p \to M$$

Then

$$[(M, \varepsilon, \sigma_p)] - [(M, \varepsilon, \sigma_0)] = 0$$

in the reduced equivariant spin bordism group $\widetilde{\mathrm{M}}\,\mathrm{Spin}_n(B\mathbb{Z}_p)$

Summary of results

We have

1. considered the "models" $M_{p,a}^{b,c}(\mathfrak{a})$ of $\mathbb{Z}_p$-manifolds
2. given an explicit description of the **spin strucures** of $M_{p,a}^{b,c}(\mathfrak{a})$
3. explicitly computed, for twisted Dirac operators D_ℓ acting on an arbitrary $\mathbb{Z}_p$-manifold $(M_\Gamma, \varepsilon_h)$, the following
    - the **eta series** $\eta_{\ell,h}(s)$
    - the **eta invariants** $\eta_{\ell,h}$
    - the **number of independent harmonic spinors** $d_{\ell,0,h}$
    - the **reduced eta invariants** $\bar{\eta}_{\ell,h} = 0$ (except for $M_{3,1}$)
    - the **relative eta invariants** $\bar{\eta}_{\ell,h} - \bar{\eta}_{0,h} = 0$

Note on methodology

★ There are indirect methods to compute η-invariants (representation techniques, computing $Ind(D)_{geo} - Ind(D)_{top}$)

★ However, we have performed the **direct approach**, that is, we have explicitly computed

1. the spectrum $\lambda = \pm 2\pi\mu$, $d_\lambda = d^{\pm}_{\ell,\mu,h}$
2. the eta series $\eta_\ell(s) = \frac{1}{(2\pi)^s} \sum_{\mu \neq 0} \frac{d^+_{\ell,\mu,h} - d^-_{\ell,\mu,h}}{|\mu|^s}$
3. the different eta invariants

$$\eta_\ell, \qquad \bar\eta_\ell = \tfrac{1}{2}(\eta_\ell + \dim\ker D_\ell) \mod \mathbb{Z}, \qquad \bar\eta_\ell - \bar\eta_0$$

References

R. Miatello - R. Podestá

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Thanks