Exceptional holonomy based on Hitchin's flow equations

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$SU(3)$-, G_2- and Spin(7)-structures I

An $SU(3)$-structure on a 6-manifold is determined by a 2-form ω and a 3-form ρ. There is in fact a $U(1)$-family of 3-forms ρ^θ, $\theta \in \mathbb{R}$, which determine the same $SU(3)$-structure. We nevertheless denote $SU(3)$-structures mostly by (ω, ρ).

$SU(3)$-, G_2- and Spin(7)-structures I

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A G_2-structure on a 7-manifold is determined by a 3-form ϕ.

A Spin(7)-structure on an 8-manifold is determined by a 4-form Φ.

The above forms have to satisfy certain constraints.

$SU(3)$-, G_2- and Spin(7)-structures II

(ω, ρ), ϕ and Φ yield symmetric bilinear forms denoted by g^6, g^7 and g^8. In order to define an $SU(3)$-, G_2- or Spin(7)-structure, we need $g^6, g^7, g^8 > 0$.

ω, ρ, ϕ should be stable, i.e. the $GL(6)$- or $GL(7)$-orbit is open.

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ω, ρ, ϕ should be stable, i.e. the $GL(6)$- or $GL(7)$-orbit is open.

Moreover:

- $\omega \wedge \rho = 0$, $\left(J_\rho^* \rho\right) \wedge \rho = \frac{2}{3}\omega^3$.
- There exists a frame such that Φ has certain coefficients.

Exceptional Holonomies

$d\phi = d * \phi = 0 \Leftrightarrow$ Holonomy of g^7 contained in G_2.

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We call such manifolds G_2- or Spin(7)-manifolds. First examples by Bryant, Salamon and Joyce.

Aim: Find further examples and understand the structure of G_2- and Spin(7)-manifolds.

Hitchin's flow equations I

On an oriented hypersurface in a

- G_2-manifold, there exists a canonical $SU(3)$-structure with $d\rho = 0$ and $d\omega \wedge \omega = 0$ (half-flat).
- Spin(7)-manifold, there exists a canonical G_2-structure with $d * \phi = 0$ (cocalibrated).

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On an equidistant one-parameter family of oriented hypersurfaces we have:

- $\frac{\partial}{\partial t}\rho = d\omega$ and $\left(\frac{\partial}{\partial t}\omega\right) \wedge \omega = dJ_\rho^* \rho$,
- $\frac{\partial}{\partial t} * \phi = d\phi$.

These are Hitchin's flow equations.

Hitchin's flow equations II

Theorem(Hitchin) Let (ω_0, ρ_0) be a half-flat $SU(3)$-structure on a compact N^6. The initial value problem

$$\begin{aligned} &\frac{\partial}{\partial t}\rho = d\omega \,, \\ &\left(\frac{\partial}{\partial t}\omega\right) \wedge \omega = dJ_\rho^* \rho \,, \\ &\omega(t_0) = \omega_0 \,, \\ &\rho(t_0) = \rho_0 \end{aligned} \tag{1}$$

has a unique solution on $N^6 \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $(\omega(t), \rho(t))$ always is a half-flat $SU(3)$-structure, too.

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has a unique solution on $N^6 \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $(\omega(t), \rho(t))$ always is a half-flat $SU(3)$-structure, too. In this situation,

$$\phi := dt \wedge \omega + \rho \tag{2}$$

is parallel ($d\phi = 0$, $d * \phi = 0$).

Hitchin's flow equations III

Theorem(Hitchin) Let ϕ_0 be a cocalibrated G_2-structure on a compact N^7. The initial value problem

$$\begin{aligned} \frac{\partial}{\partial t} * \phi &= d\phi\,, \\ \phi(t_0) &= \phi_0 \end{aligned} \tag{3}$$

has a unique solution on $N^7 \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\phi(t)$ always is a cocalibrated G_2-structure, too.

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$$\Phi := dt \wedge \phi + *\phi \tag{4}$$

is parallel ($d\Phi = 0$).

Why study degenerations?

A solution on $N^6 \times I$ or $N^7 \times I$ with $I \subsetneq \mathbb{R}$ is not complete.

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Nevertheless, a complete (or compact) G_2- or Spin(7)-manifold may be foliated by equidistant hypersurfaces and finitely many lower-dimensional submanifolds.

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Nevertheless, a complete (or compact) G_2- or Spin(7)-manifold may be foliated by equidistant hypersurfaces and finitely many lower-dimensional submanifolds.

In this situation, the existence and uniqueness of the Hitchin flow is not always granted.

Examples I

Let N^7 be a nearly parallel G_2-manifold ($d\phi_0 = \lambda * \phi_0$ with $\lambda \neq 0$).

$\phi(t) = \frac{\lambda^3}{64} t^3 \phi_0$ solves Hitchin's flow equation.

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Degeneration of N^7 into a point $\{p\}$.

Since there are many (non-homeomorphic) nearly parallel G_2-manifolds (cf. Friedrich et al.), the Hitchin flow near $\{p\}$ is far from unique.

Examples II

Cohomogeneity-one examples (N^6 or N^7 is homogeneous)

- Bryant and Salamon: $N^7 = S^7$, degeneration of S^7 into a sphere S^4 for small t.

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- Bryant and Salamon: $N^7 = S^7$, degeneration of S^7 into a sphere S^4 for small t.
- Cvetič et al., R.: An Aloff-Wallach space $N^{k,l} := SU(3)/U(1)_{k,l}$ degenerates into $\mathbb{CP}^2$. The third derivative of a certain coefficient w.r.t. t can be chosen freely. $\Rightarrow$ No uniqueness.

Examples III

- N^7 is a generic Aloff-Wallach space which degenerates into $SU(3)/U(1)^2$. No solution of Hitchin's flow equation (if we assume $SU(3)$-invariance).

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- N^7 is a generic Aloff-Wallach space which degenerates into $SU(3)/U(1)^2$. No solution of Hitchin's flow equation (if we assume $SU(3)$-invariance).
- Same situation, but now $k = l = 1$. (M^8, g) has a singularity, which can be repaired by replacing $N^{1,1}$ by $N^{1,1}/\mathbb{Z}_2$ (Bazaikin, R.). $\Rightarrow$ Smoothness of the flow is another non-trivial problem.

Shape of M^8 I

From now on, we restrict ourselves to Spin(7)-manifolds (M^8, Φ).

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The fixed point set shall be a six-dimensional connected submanifold N^6.

$\{p \in M^8 | \mathrm{dist}(p, N^6) = c\} =: N^7$ is $U(1)$-invariant. Moreover, it is a $U(1)$-bundle over N^6 if c is small. (Project p to the nearest point on N^6.)

Shape of M^8 II

Local picture of M^8: See flip chart.

- t becomes the radial coordinate r.
- Define $e_r := \frac{\partial}{\partial r}$. The integral curves of e_r are geodesics.

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- t becomes the radial coordinate r.
- Define $e_r := \frac{\partial}{\partial r}$. The integral curves of e_r are geodesics.
- Let e_φ be the infinitesimal action of $U(1)$ such that the flow of e_φ at the time 2π is the identity map.
- $g(e_r, e_r) = 1$, $g(e_r, e_\varphi) = 0$, $g(e_\varphi, e_\varphi) = f(r)^2$.

Shape of M^8 III

On N^6, there exists a canonical $SU(3)$-structure (ω_0, ρ_0).

On $M^8 \setminus N^6$, the Spin(7)-structure can be written as:

$$\Phi = \frac{1}{2}\omega \wedge \omega + e_r^* \wedge J_\rho^* \rho + f \cdot e_\varphi^* \wedge \rho + f \cdot e_r^* \wedge e_\varphi^* \wedge \omega \,. \tag{5}$$

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Relations:

- $df(e_\varphi) = 0$,
- $de_r^* = 0$, $[e_r, e_\varphi] = 0$,
- $de_\varphi^*(e_\varphi, \ldots) = 0$, $de_\varphi^*(e_r, \ldots) = 2f'f\, e_\varphi^*$,
- $\mathcal{L}_{e_\varphi}\omega = 0$, $\mathcal{L}_{e_\varphi}\rho = k \cdot J_\rho^* \rho$ with $k \in \mathbb{Z}$.

Hitchin flow

Flow equations outside N^6 are "nice". (f' does not depend on terms containing "$\frac{1}{f}$".)

Same theory as in the non-degenerate case. We merely extend the solution from $(0, \epsilon)$ to $[0, \epsilon)$.

In particular, we have existence and uniqueness.

Smoothness conditions I

Consider $\mathbb{R}^2$ with the canonical action of $SO(2)$:

$$g(x,y) := \sum_{i,j=0}^{\infty} c_{ij} x^i y^j \tag{6}$$

is $SO(2)$-invariant iff

$$g(x,y) = \sum_{i=0}^{\infty} c_i (x^2 + y^2)^i =: h(r) \,. \tag{7}$$

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$\Rightarrow g$ determined by $h(r) = g(r,0)$. Smoothness condition: $h(r) = h(-r)$.

Smoothness consitions II

This translates into conditions on the objects on M^8 as follows:

- ω, ρ invariant under $-1 \in U(1)$,
- f^2 even.

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- ω, ρ invariant under $-1 \in U(1)$,
- f^2 even.

Moreover,

- $f(0) = 0$,
- $\sqrt{g(e_\varphi, e_\varphi)} = t + O(t^2)$
  $\Leftrightarrow |f'(0)| = 1 \quad (\Rightarrow f \text{ odd})$
  $\Leftrightarrow \pm\rho = -k \cdot \rho - \underbrace{df \wedge \omega}_{=0}$ on N^6.

Smoothness conditions III

Remarks:

- $k = -f'(0)$.
- f may be non-constant on N^7.
- k cannot always be chosen freely and is not always ± 1. If $N^7 = G/H$ is homogeneous, k is determined by G and H.

Main Theorem

Theorem: Let M^8 and N^6 be as above and let $k \in \mathbb{Z}$ be arbitrary. N^6 shall carry an $SU(3)$-structure (ω_0, ρ_0) satisfying $(d\omega_0) \wedge \omega_0 = 0$.

Then, there exists a unique $U(1)$-invariant Spin(7)-structure Φ on a neighbourhood of $N^6 \subseteq M^8$ such that its restriction to N^6 induces (ω_0, ρ_0) and $\mathcal{L}_{e_\varphi}\rho = k \cdot J_\rho^* \rho$.

Φ is smooth near N^6 iff $k = \pm 1$.

Examples I

- Let N^7 be a generic Aloff-Wallach space and N^6 be $SU(3)/U(1)^2$. We assume that everything is $SU(3)$-invariant. $\mathcal{L}_{e_\varphi}\rho = 0 \Rightarrow f'(0) = 0 \Rightarrow$ No meaningful examples.

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- Let N^7 be a generic Aloff-Wallach space and N^6 be $SU(3)/U(1)^2$. We assume that everything is $SU(3)$-invariant. $\mathcal{L}_{e_\varphi}\rho = 0 \Rightarrow f'(0) = 0 \Rightarrow$ No meaningful examples.
- Let N^7 be one of the following homogeneous spaces: $N^{1,1}$, $Q^{1,1,1}$ or $M^{1,1,0}$. We have $k = \pm 2 \Rightarrow$ There is a singularity along N^6.

Examples II

- Let N^7 be a product of a homogeneous N^6 and a circle. We always have $f'(0) = 0 \Rightarrow$ No metrics with holonomy Spin(7).

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- Let N^7 be a product of a homogeneous N^6 and a circle. We always have $f'(0) = 0 \Rightarrow$ No metrics with holonomy Spin(7).
- By our methods we can construct many further examples of Spin(7)-manifolds (of cohomogeneity one).

Outlook I

- Similar results for the G_2-case to be expected.

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- Similar results for the G_2-case to be expected.
- Degeneration into a four-dimensional calibrated submanifold $N^4 \subseteq M^8$. Fiber: homogeneous $S^3 \cong SU(2)$. More insights into fibrations of M^8 by calibrated submanifolds?

Outlook II

In superstring theory (M-theory) space-time is sometimes modelled as:

$$(M^8, \Phi) \times \mathbb{R}^{2,1} \,. \tag{8}$$

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Let $U(1)$ act on M^8 such that $M^8/U(1)$ is smooth and the fixed point set N^4 is four-dimensional. Duality:

M-theory on $M^8 \times \mathbb{R}^{2,1}$
$\Leftrightarrow$
IIA theory on $M^8/U(1) \times \mathbb{R}^{2,1}$ with D6-branes on $N^4 \times \mathbb{R}^{2,1}$.

(Cf. Acharya, Gukov)