Transformations of surfaces and their applications to spectral theory

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The Euler–Poisson–Darboux equation

The equation

$$z_{xy} - \frac{n}{x-y} z_x + \frac{m}{x-y} z_y - \frac{p}{(x-y)^2} z = 0$$

after a substitution

$$z = (x-y)^\alpha w$$

takes the form

$$w_{xy} - \frac{n'}{x-y} w_x + \frac{m'}{x-y} w_y - \frac{p'}{(x-y)^2} w = 0,$$

where $n' - n = m' - m = \alpha, \quad p' = p + (m+n)\alpha + \alpha(\alpha - 1)$.

The Euler exact solution

Let $m' = n' = k$ are integers and $p' = 0$. The equation is reduced to the form

$$w_{xy} - \frac{k}{x-y} w_x + \frac{k}{x-y} w_y = 0$$

and after the substitution $w = (x-y)^{-k} u$ we derive

$$u_{xy} = \frac{k(1-k)}{(x-y)^2} u.$$

A general solution of this equation is as follows

$$u(x,y) = (x-y)^k \frac{\partial^{2k-2}}{\partial x^{k-1} \partial y^{k-1}} \left(\frac{f(x)+g(y)}{x-y} \right).$$

The Laplace transformation

$$\psi_{xy} + A\psi_x + B\psi_y + C\psi = 0.$$

Replace ψ by

$$\widetilde{\psi} = \left(\frac{\partial}{\partial y} + A\right)\psi.$$

The equation on $\widetilde{\psi}$ has another coefficients:

$$A \to A - (\log h)_y,$$

$$B \to B,$$

$$C \to C - A_x + B_y - (\log h)_y B,$$

where

$$h = AB + A_x - C.$$

The analogous transformation is obtained after swapping $x \leftrightarrow y$, therewith h is replaced by

$$k = AB + B_y - C.$$

The Laplace integration method

Under the first transformation

$$h \to 2h - k - (\log h)_{xy}, \quad k \to h;$$

after the transformation

$$\psi \to \widetilde{\psi} = f(x, y)\psi$$

the values of h and k are preserved (*the Darboux invariants*). Note that

$$\widetilde{\psi}_x = -B\widetilde{\psi} + h\psi,$$

hence $h = 0$ implies the integrability.

Exactly solvable operators with a magnetic field

Consider a two-dimensional Schrödinger operator

$$L = \partial\bar{\partial} + A\bar{\partial} + B\partial + C$$

with an electric potential $V = -\frac{h}{2} = -\frac{1}{2}(AB + A_{\bar{z}} - C)$ and in a magnetic field $H = \frac{1}{2}(B_z - A_{\bar{z}})$. It is represented as follows

$$L = (\bar{\partial} + B)(\partial + A) + 2V = (\partial + A)(\bar{\partial} + B) + 2U,$$

where $U = V + H = -k$. The Laplace transformation takes the form

$$\widetilde{H} = H + \frac{1}{2}\partial\bar{\partial}\log V, \quad \widetilde{V} = V + \widetilde{H}.$$

By exploiting that Novikov and Veselov constructed integrable on two energy levels periodic Schrödinger operators with nonvanishing magnetic flux.

The Darboux transformation I

Given the conjugate coordinates x, y on a surface $\mathbf{r}(x, y) \subset \mathbb{R}^3$, we have

$$\mathbf{r}_{xy} + a\mathbf{r}_x + b\mathbf{r}_y = 0.$$

For surfaces in $\mathbb{R}P^3$ we have

$$\mathbf{r}_{xy} + a\mathbf{r}_x + b\mathbf{r}_y + c\mathbf{r} = 0.$$

A generic congruence C (2-dimensional family) of lines in RP^3 has two focal surfaces $\mathbf{r}$ and $\widetilde{\mathbf{r}}$ to which every line is tangent. Then the Laplace transformation

$$\widetilde{\mathbf{r}} = \mathbf{r}_y + a\mathbf{r}$$

defines a mapping from $\mathbf{r}$ to $\widetilde{\mathbf{r}}$ (here we assume that lines from C are tangent along y-directions) [Darboux].
A general congruence (line, spherical and etc) which relates two enveloping surfaces is called *the Darboux transformation*.

The Darboux transformation II

$$H = -\frac{d^2}{dx^2} + u(x)$$

— a one-dimensional Schrödinger operator. Every solution ω to

$$H\omega = 0.$$

defines a factorization of H:

$$H = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega'}{\omega}.$$

The Darboux transformation of H consists in swapping $A^\top$ and A:

$$H = A^\top A \longrightarrow \widetilde{H} = AA^\top = -\frac{d^2}{dx^2} + \widetilde{u}(x),$$

$$u = v^2 + v' \longrightarrow \widetilde{u} = v^2 - v'$$

and it acts on eigenfunctions as follows:

$$\psi \longrightarrow \widetilde{\psi} = A\psi.$$

The harmonic oscillator

Let $v = ax, a > 0$, then

$$v' = \mathrm{const} = a$$

and

$$AA^\top = 2H - a, \quad A^\top A = 2H + a,$$

where

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + a^2x^2\right)$$

is the harmonic oscillator operator. It follows from the commutation relation $[A^\top, A] = 2a$ that if

$$H\psi = E\psi,$$

then

$$H(A\psi) = (E + a)(A\psi), \quad H(A^\top\psi) = (E - a)(A^\top\psi).$$

Note that

$$(2E - a)(\psi, \psi) = (AA^\top \psi, \psi) = (A^\top \psi, A^\top \psi) \geq 0,$$

which implies

$$E \geq \frac{a}{2}.$$

The equality is attained on a solution of the equation

$$A^\top \psi = \left(\frac{d}{dx} + ax \right) \psi = 0,$$

which is up to a constant multiple equals

$$\psi_1(x) = e^{-\frac{ax^2}{2}}.$$

The basis of eigenfunctions has the form

$$\psi_N = A^{N-1} \psi_1, \quad N = 1, 2, 3, \dots$$

with eigenvalues

$$\frac{a}{2} + (N - 1)a.$$

The Crum method

Consider the problem

$$-\varphi'' + u\varphi = \lambda\varphi, \quad 0 < x < 1,$$

$$\varphi(0) = a\varphi'(0), \quad \varphi(1) = b\varphi'(1),$$

where $u(x)$ is continuous on $[0, 1]$. Denote by

$$\lambda_0 < \lambda_1 < \dots$$

the spectrum of this problem, and by $\varphi_0, \varphi_1, \dots$ — the corresponding eigenfunctions.
Let W_n be the Wronskian of φ_0, ..., φ_{n-1} and W_{ns} be the Wronskian of $\varphi_0, \dots, \varphi_{n-1}, \varphi_s$ $(s \geq n)$.

The Crum theorem:

- *the problem*

$$-\varphi'' + u_n\varphi = \lambda\varphi, 0 < x < 1, \lim_{x\to 0}\varphi(x) = 0, \qquad \lim_{x\to 1}\varphi(x) = 0,$$

where $u_n = u - 2\frac{d^2}{dx^2}\log W_n$ *has the spectrum*

$$\lambda_n < \lambda_{n+1} < \dots$$

and a complete family of corresponding eigenfunctions

$$\varphi_{ns} = \frac{W_{ns}}{W_n}, \quad s \geq n.$$

For $n \geq 2$ *the problem is not regular and*

$$u_n \sim \frac{n(n-1)}{x^2},\ x \to 0; \qquad u_n \sim \frac{n(n-1)}{(1-x)^2},\ x \to 1.$$

The Moutard equation

If x and y are the asymptotic coordinates on a surface $\mathbf{r}(x,y) \subset \mathbb{R}^3$, then

$$(\mathbf{r}_x, \mathbf{n}_u) = (\mathbf{r}_y, \mathbf{n}_v) = 0$$

with $\mathbf{n}$ the normal field. This implies

$$\mathbf{r}_x = \lambda \mathbf{n}_x \times \mathbf{n}, \quad \mathbf{r}_y = \mu \mathbf{n} \times \mathbf{n}_y.$$

Put $\Psi = \sqrt{\lambda}\mathbf{n}$ and derive

$$\Psi_{xy} \times \Psi = 0$$

which is equivalent to *the Moutard equation*:

$$\Psi_{xy} = Q(x,y)\Psi, \quad \text{or} \quad (\partial_x \partial_y - Q)\Psi = 0.$$

Every (vector-valued) solution to this equation defines a surface with the asymptotic coordinates x, y and vise versa.

The Moutard transformation

Let H be a two-dimensional potential Schrödinger operator and ω be a solution of the equation

$$H\omega = (-\Delta + u)\omega = 0,$$

where Δ is the two-dimensional Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The Moutard transformation of H is defined as

$$\widetilde{H} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

If ψ satisfies $H\psi = 0$, then the function θ, defined via the system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\psi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\psi}{\omega}\right)_x,$$

satisfies $\widetilde{H}\theta = 0$.

REMARKS:
1) the Moutard transformation describes deformations only of "eigenfunctions" with zero "eigenvalue";
2) the action of the Moutard transformation on "eigenfunctions" ψ is multi-valued and is defined modulo multiples of $\frac{1}{\omega}$;
3) if $u = u(x)$ and $\omega = f(x)e^{\kappa y}$, the the Moutard transformation reduces to the Darboux transformation defined by f.
Introduce the following notation:

$$M_\omega(u) = \widetilde{u} = u - 2\Delta \log \omega, \quad M_\omega(\varphi) = \{\theta + \frac{C}{\omega},\ C \in \mathbb{C}\}.$$

The double iteration

Let

$$H = -\Delta + u_0$$

— an operator with potential $u_0(x, y)$ and ω_1 and ω_2 satisfy the equation

$$H\omega_1 = H\omega_2 = 0.$$

Let $\theta_1 \in M_{\omega_1}(\omega_2)$ и $\theta_2 = -\frac{\omega_1}{\omega_2}\theta_1 \in M_{\omega_2}(\omega_1)$. Then

- The diagram

$$\begin{array}{ccccc} & u_0 & \xrightarrow{\omega_1} & u_1 & \\ \omega_2 & \downarrow & & \downarrow & \theta_1 \\ & u_2 & \xrightarrow{\theta_2} & u_{12} = u_{21} & \end{array},$$

  where $\theta_1 \in M_{\omega_1}(\omega_2),\ \theta_2 = -\frac{\omega_1}{\omega_2}\theta_1 \in M_{\omega_2}(\omega_1)$, is commutative, i.e.

$$u_{12} = u_{21} = u.$$

- For $\psi_1 = \frac{1}{\theta_1}$ and $\psi_2 = \frac{1}{\theta_2}$ we have

$$H\psi_1 = H\psi_2 = 0, \quad \text{where } H = -\Delta + u.$$

A remark on the kernel of a two-dimensional rational Schrödinger operator

If a one-dimensional potential meets the condition

$$\int_{-\infty}^{\infty} |u(x)|(1+|x|)\,dx < \infty,$$

then there are finitely many eigenvalues and all of the m are negative (Faddeev, Marchenko).
For $n \geq 5$ it is easy by regularizing the Green function $G(x) = \frac{c_n}{r^{n-2}}$ to obtain a positive function ψ which lies in the kernel of the Schrödinger operator with finite potential

$$u = \frac{\Delta\psi}{\psi}.$$

The question do there exist two-dimensional Schrödinger operators with smooth fast decaying potential and nontrivial kernel was opened until recently.

A two-dimensional Schrödinger operator with nontrivial kernel (T.–Tsarev)

Let

$$\omega_1 = x + 2(x^2 - y^2) + xy, \quad \omega_2 = x + y + \frac{3}{2}(x^2 - y^2) + 5xy.$$

Then the double iteration of the Moutard transformation gives the potential

$$u^* = -\frac{5120(1 + 8x + 2y + 17x^2 + 17y^2)}{(160 + (4 + 16x + 4y)(x^2 + y^2) + 17(x^2 + y^2)^2)^2}$$

and the eigenfunctions with $E = 0$:

$$\psi_1 = \frac{x + 2x^2 + xy - 2y^2}{160 + (4 + 16x + 4y)(x^2 + y^2) + 17(x^2 + y^2)^2}$$

$$\psi_2 = \frac{2x + 2y + 3x^2 + 10xy - 3y^2}{160 + (4 + 16x + 4y)(x^2 + y^2) + 17(x^2 + y^2)^2}.$$

u^*, ψ_1, ψ_2 are smooth rational functions.
u^* decays as r^{-6} as $r \to \infty$.
ψ_1 and ψ_2 decay as r^{-2} as $r \to \infty$

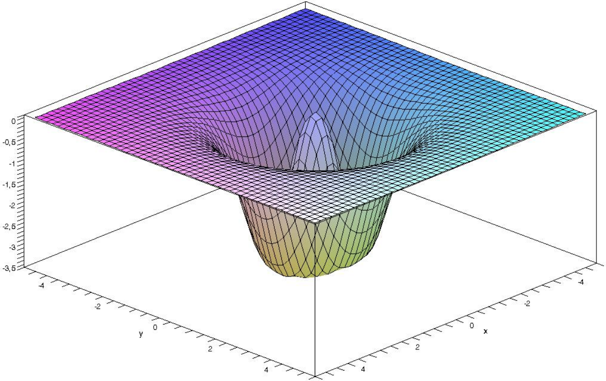

Fig. 1. The potential u^*.

Some problems

1. There are examples of potentials $u \sim O(r^{-8})$ and eigenfunctions $\psi_1, \psi_2 \sim O(r^{-3})$ as $r \to \infty$. We think that for any $N > 0$ by using this construction one may construct smooth rational potentials u and their eigenfunctions ψ_1 and ψ_2 which decay faster than $\frac{1}{r^N}$.
2. It looks that by using a k-th iteration it is possible to construct operators H with $\dim \mathrm{Ker}\, H \geq k$.
3. The potential u is nonpositive and $H = -\Delta + u^*$ has to have negative discrete eigenvalues. How looks the discrete spectrum of H and other Schrödinger operators with two-dimensional rational solitons as potentials?

The Novikov–Veselov equation

The Novikov–Veselov (NV) equation:

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(UV) + 3\bar{\partial}(\bar{V}U) = 0, \quad \bar{\partial}V = \partial U.$$

The one-dimensional reduction $U = U(x), U = V = \bar{V}$ leads to the Korteweg–de Vries equation $U_t = \frac{1}{4}U_{xxx} + 6UU_x$.
The NV equation is the compatibility condition for the system

$$\begin{aligned} H\psi = (\partial\bar{\partial} + U)\psi = 0, \\ \partial_t\psi = -A\psi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial})\psi \end{aligned} \qquad (1)$$

and is represented by a "Manakov triple" of the form $H_t = [H, A] + BH$. Equations represented by such triples preserve the "spectrum on the zero energy level" deforming "eigenfunctions" via

$$(\partial_t + A)\psi = 0.$$

The extended Moutard transformation

The system (1) is invariant under the transformation

$$\varphi \to \theta = \frac{i}{\omega} \int (\varphi \partial \omega - \omega \partial \varphi) dz - (\varphi \bar{\partial} \omega - \omega \bar{\partial} \varphi) d\bar{z} +$$

$$+ [\varphi \partial^3 \omega - \omega \partial^3 \varphi + \omega \bar{\partial}^3 \varphi - \varphi \bar{\partial}^3 \omega + 2(\partial^2 \varphi \partial \omega - \partial \varphi \partial^2 \omega) -$$

$$-2(\bar{\partial}^2 \varphi \bar{\partial} \omega - \bar{\partial} \varphi \bar{\partial}^2 \omega) + 3V(\varphi \partial \omega - \omega \partial \varphi) + 3\bar{V}(\omega \bar{\partial} \varphi - \varphi \bar{\partial} \omega)] dt,$$

$$U \to U + 2\partial \bar{\partial} \log \omega, \quad V \to V + 2\partial^2 \log \omega.$$

Therefore if two holomorphic in z functions $p_1(z,t)$ and $p_2(z,t)$ satisfy the equation

$$\frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial z^3},$$

then the double iteration of the extended Moutard transformation defined by them and applied to $U = 0$ gives a solution of the Novikov–Veselov equation.

Blowing up solution of the Novikov–Veselov equation (T.–Tsarev)

Apply this construction to a pair of polynomials $p_k = p_k(z,0)$:

$$p_1 = i\,z^2, \qquad p_2 = z^2 + (1+i)z$$

and obtain a solution

$$U = \frac{H_1}{H_2},$$

where

$$H_1 = -12\Big(12t(2(x^2+y^2)+x+y) + x^5 - 3x^4y + 2x^4 - 2x^3y^2 -$$

$$-4x^3y - 2x^2y^3 - 60x^2 - 3xy^4 - 4xy^3 - 30x + y^5 + 2y^4 - 60y^2 - 30y\Big),$$

$$H_2 = (3x^4 + 4x^3 + 6x^2y^2 + 3y^4 + 4y^3 + 30 - 12t)^2.$$

It decays as r^{-3}, is nonsingular for $0 \le t < T_* = \frac{29}{12}$ and is singular for $t \ge T_* = \frac{29}{12}$.

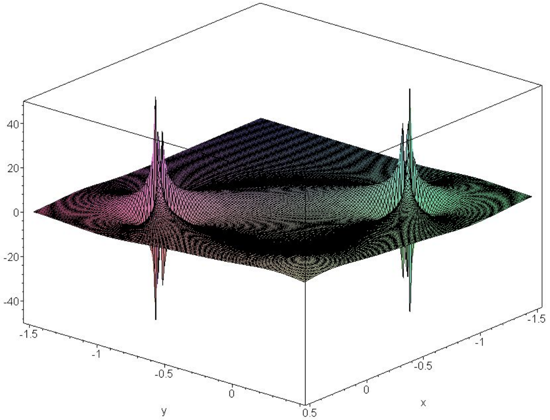

Fig. 2. The potential U as $t \longrightarrow \frac{29}{12}$.

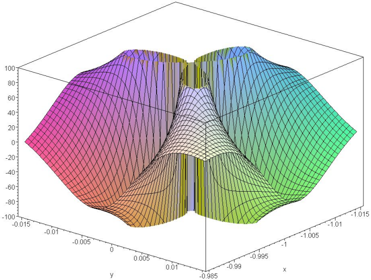

Fig. 3. The potential U at $t = \frac{29}{12}$ near $(-1, 0)$.