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Clifford Modules, Dirac Operators and the STM Action.

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Clifford Modules, Dirac Operators and the STM action

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Content

1. The Einstein-Hilbert vs. the Yang-Mills action
2. Clifford modules and the universal Dirac action
3. Simple type Dirac operators and the STM action

1 Brief motivation

– Einstein-Hilbert action:

$$\begin{aligned} \mathcal{I}_{\rm EH} &:= \int_M *scal(g_{\rm M}) \\ &\sim \int_M *\mathrm{tr}_\gamma(curv(\not{\partial}_{\rm A}))\,; \end{aligned} \tag{1}$$

– Yang-Mills action:

$$\begin{aligned} \mathcal{I}_{\rm YM} &:= \int_M *\mathrm{tr}\left(g_{\Lambda\rm M}(F_{\rm A}, F_{\rm A})\right) \\ &\sim \int_M *\mathrm{tr}_\gamma(\not{F}_{\rm A}^{\,2})\,; \end{aligned} \tag{2}$$

$$\Rightarrow \qquad \not{\partial}_{\rm A} F_{\rm A} = 0\,, \qquad \left\{ \begin{array}{rcl} \not{\partial}_{\rm A} & = & d_{\rm A} + \varepsilon\delta_{\rm A}\,, \\ \not{\partial}_{\rm A}^2 & = & \varepsilon \mathrm{ev}_g(\partial_{\rm A}^2) - \frac{\varepsilon}{4} scal(g_{\rm M}) + \not{F}_{\rm A}\,. \end{array} \right. \tag{3}$$

2 Clifford modules and the universal Dirac action

Let M be a smooth, connected (simply connected) and orientable (spin-) manifold of even dimension $n = p + q$. Also, let g_{M} be a (semi-)Riemannian metric of index $s = p - q$.

2.1 Some generalities:

Let $(\mathcal{E}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\mathrm{M}})$ be a **general** $\mathbb{Z}_2-$ **graded Clifford module** with an **odd Clifford mapping**:

$$\gamma_{\mathcal{E}}(\alpha)^2 = \varepsilon\, g_{\mathrm{M}}(\alpha, \alpha)\, \mathrm{id}_{\mathcal{E}}\ , \tag{4}$$

for all $\alpha \in T^*M$.

A first order differential operator: $\not{D} : \mathfrak{S}ec(M,\mathcal{E}) \longrightarrow \mathfrak{S}ec(M,\mathcal{E})$, satisfying

$$[\not{D}, f] = \gamma_{\mathcal{E}}(df) \tag{5}$$

for all $f \in \mathcal{C}^\infty(M)$ is called of **Dirac type**. An odd Dirac type operator is called a **Dirac operator**.

- Canonical one-form: $\Theta \overset{\text{loc.}}{=} \frac{\varepsilon}{n}\, e^k \otimes \gamma_{\mathcal{E}}(e_k^\flat) \in \Omega^1(M, \mathrm{End}^-(\mathcal{E}))\,;$

$$\Rightarrow \qquad \begin{array}{rcl} ext_\Theta : \Omega^0(M,\mathrm{End}^\pm(\mathcal{E})) & \longrightarrow & \Omega^1(M,\mathrm{End}^\mp(\mathcal{E})) \\ \Phi & \mapsto & \Theta \wedge \Phi\,, \end{array} \tag{6}$$

right-inverse of the “quantization map”:

$$\begin{array}{rcl} \delta_\gamma : \Omega^*(M,\mathrm{End}(\mathcal{E})) & \longrightarrow & \mathfrak{S}ec(M,\mathrm{End}(\mathcal{E})) \\ \omega \otimes \chi & \mapsto & \gamma_{\mathcal{E}}(\sigma_{\mathrm{Ch}}^{-1}(\omega)) \circ \chi\,. \end{array} \tag{7}$$

- Clifford connections:

$$\mathcal{A}_{\mathrm{Cl}}(\mathcal{E}) := \{\partial_{\mathrm{A}} \in \mathcal{A}(\mathcal{E}) \,|\, \partial_{\mathrm{A}}\Theta \equiv 0\}\,. \tag{8}$$

- First and second order decomposition:

$$\not{D} = \not{\partial}_{\mathrm{B}} + \Phi_{\mathrm{D}}\,, \tag{9}$$

$$\not{D}^2 = \triangle_{\mathrm{B}} + V_{\mathrm{D}}\,. \tag{10}$$

Here, respectively, $\not{\partial}_{\mathrm{B}} \equiv \delta_\gamma \circ \partial_{\mathrm{B}}$ is the quantization of the Bochner connection $\partial_{\mathrm{B}} \in \mathcal{A}(\mathcal{E})$:

$$2\,\mathrm{ev}_g(df, \partial_{\mathrm{B}}\psi) := \varepsilon\left([\not{D}^2, f] - \delta_{\mathrm{g}} df\right)\psi\,, \tag{11}$$

for all $f \in \mathcal{C}^\infty(M)$ and $\psi \in \mathfrak{S}ec(M, \mathcal{E})$; $\triangle_{\mathrm{B}} := \varepsilon \mathrm{ev}_g(\partial_{\mathrm{B}} \circ \partial_{\mathrm{B}})$ is the corresponding Bochner-Laplacian (or “trace/connection Laplacian”).

- **Dirac connections**:

$$\partial_{\mathrm{D}} := \partial_{\mathrm{B}} + \omega_{\mathrm{D}}\,, \tag{12}$$

$$\omega_{\mathrm{D}} \equiv ext_{\Theta}(\Phi_{\mathrm{D}}) \qquad \textbf{Dirac form}\,, \tag{13}$$

whereby $\not{\partial}_{\mathrm{D}} \equiv \delta_\gamma \circ \partial_{\mathrm{D}} = \not{D}$.

- **Universal Dirac-Lagrangian:**

$$\begin{aligned} *\mathcal{L}_{\mathrm{D}} &:= \mathrm{tr}_{\mathcal{E}} V_{\mathrm{D}} \\ &= \mathrm{tr}_\gamma\big(curv(\not{D}) - \varepsilon ev_g(\omega^2_{\mathrm{D}})\big) + div\xi_{\mathrm{D}}\,. \end{aligned} \tag{14}$$

Here, $\xi_{\mathrm{D}} := -\varepsilon\,\mathrm{tr}_{\mathcal{E}}\,\omega^\sharp_{\mathrm{D}} \in \mathfrak{S}ec(M, TM)$ is the **Dirac vector field** and

$$curv(\not{D}) := \partial_{\mathrm{D}} \wedge \partial_{\mathrm{D}} \in \Omega^2(M, \mathrm{End}(\mathcal{E})) \tag{15}$$

is the **curvature of the Dirac operator** $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ and $\mathrm{tr}_\gamma \equiv \mathrm{tr}_{\mathcal{E}} \circ \delta_\gamma$ is the “quantized trace”.

- **Dirac operators of simple type:**

$$\not{D} = \not{\partial}_{\rm A} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} \,, \tag{16}$$

where $\phi_{\mathcal{E}} \in \mathrm{Sec}(M, \mathrm{End}^-_\gamma(\mathcal{E}))$.

- **Dirac operators of Yang-Mills-Higgs type:**

$$\not{D} \;=\; \not{\partial}_{\rm A} + \varphi_{\rm H} \,, \tag{17}$$

where $\varphi_{\rm H} \in \mathrm{Sec}(M, \mathrm{End}_\gamma(\mathcal{E}))$. They have the property that the corresponding Dirac connections read:

$$\begin{aligned} \partial_{\rm D} &= \partial_{\rm B} + \Theta \wedge \Phi_{\rm D} \\ &= \partial_{\rm A} + \varphi_{\rm H}\Theta \\ &\equiv \partial_{\rm YMH} \,. \end{aligned} \tag{18}$$

Higgs gauge potential: $\quad H := \varphi_{\rm H}\Theta \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$

2.2 The universal Dirac action:

$$\mathcal{I}_{\mathrm{D}} := \int_M *\mathrm{tr}_\gamma\big(curv(\not{D}) - \varepsilon ev_g(\omega_{\mathrm{D}}^2)\big)\ . \tag{19}$$

It generalizes "Einstein's biggest blunder":

$$\mathcal{I}_{\mathrm{EH},\Lambda} := \int_M *(scal(g_{\mathrm{M}}) + \Lambda)\,. \tag{20}$$

2.2.1 Geometrical picture:

Let $\mathcal{E} := \Lambda_{\mathrm{M}} \otimes_{\mathbb{C}} E \twoheadrightarrow M$ be a twisted Grassmann bundle. Also, let $\mathcal{D}_{\mathrm{S}}(\mathcal{E})$ be the subset of "S-reducible" Dirac operators on $\mathfrak{S}ec(M,\mathcal{E})$ and $\mathcal{T}_{\mathrm{D}} = \Omega^1(M, \mathrm{End}(E))$.

For

$$\mathcal{E}_{\mathrm{EH}} := \mathcal{F}_{\mathrm{M}} \times_{SO(p,q)} GL(p+q)/SO(p,q) \twoheadrightarrow M \tag{21}$$

and $\mathcal{E}_{\rm D} := \mathcal{E}_{\rm EH} \times_M \mathrm{End}(\mathcal{E})$, we put

$$\Gamma_{\rm D} := \mathfrak{S}ec(M, \mathcal{E}_{\rm D})/\mathcal{T}_{\rm D} \,. \tag{22}$$

$$(g_{\rm M}, \Phi) \sim (g'_{\rm M}, \Phi') \quad :\Leftrightarrow \quad \begin{cases} g'_{\rm M} & = & g_{\rm M} \,, \\ \Phi' & = & \Phi + \not{\alpha} \,, \end{cases} \tag{23}$$

$\alpha \in \mathcal{T}_{\rm D}$.

It follows that $\Gamma_{\rm D} \simeq \mathcal{D}_{\rm S}(\mathcal{E})/\mathcal{T}_{\rm D}$ and

$$\begin{array}{rcl} \mathcal{D}_{\rm S}(\mathcal{E}) & \twoheadrightarrow & \Gamma_{\rm D} \\ \not{D} = \not{\partial}_{\rm A} + \Phi_{\rm A} & \mapsto & [(g_{\rm M}, \Phi_{\rm A})] \,. \end{array} \tag{24}$$

Each connection $\nabla^{\rm E} \in \mathcal{A}(E)$ yields a (global) section:

$$\begin{array}{rcl} \sigma_{\rm A} : \Gamma_{\rm D} & \longrightarrow & \mathcal{D}_{\rm S}(\mathcal{S}) \\ [(g_{\rm M}, \Phi)] & \mapsto & \not{\partial}_{\rm A} + \Phi \,. \end{array} \tag{25}$$

It follows that $\sigma_{\rm A}^*\mathcal{I}_{\rm D} : \Gamma_{\rm D} \to \mathbb{C}$ is independent of $\sigma_{\rm A}$.

In particular, with respect to the identification:

$$\begin{aligned} \Gamma_{\rm EH} &:= \{ [(g_{\rm M}, \Phi)] \in \Gamma_{\rm D} \,|\, \Phi \sim \not{d} \} \\ &\simeq \mathfrak{S}ec(M, \mathcal{E}_{\rm EH}) \,, \end{aligned} \tag{26}$$

it follows that

$$\begin{aligned} \sigma_{\rm A}^*\mathcal{I}_{\rm D} : \mathfrak{S}ec(M, \mathcal{E}_{\rm EH}) &\longrightarrow \mathbb{C} \\ g_{\rm M} &\mapsto \mathcal{I}_{\rm EH}(g_{\rm M}) \,. \end{aligned} \tag{27}$$

Locally, $\sigma_{\rm A}(g_{\rm M}) \overset{loc.}{=} d + \varepsilon\delta_g + \not{A}$.

More general:

$$\sigma_{\rm A}([(g_{\rm M}, \Phi)]) \overset{\rm loc.}{=} d + \varepsilon\delta_g + \Phi_{\rm A} \,, \tag{28}$$

where locally: $\Phi_{\rm A} := \Phi + \not{A} \in \mathfrak{S}ec(M, {\rm End}(\mathcal{E})) \simeq \mathfrak{S}ec(M, \Lambda T^*M \otimes {\rm End}_\gamma(\mathcal{E}))$.

3 Simple type Dirac operators and the STM action

Definition 3.1 *A Dirac operator $\not{D}$ is of simple type if its Bochner connection is given by a Clifford connection.*

Lemma 3.1 *A Dirac operator $\not{D}$ on an arbitrary Clifford module $(\mathcal{E}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{M})$ is of simple type, iff it fulfills:*

$$\{\not{D} - \not{\partial}_{B}, \gamma_{\mathcal{E}}(\alpha)\} = 0\,, \tag{29}$$

for all $\alpha \in T^{*}M \subset Cl_{M}$.

Likewise, a Dirac operator $\not{D}$ is of simple type, iff it reads:

$$\not{D} = \not{\partial}_{A} + \tau_{\mathcal{E}} \circ \phi_{D}\,, \tag{30}$$

where $\phi_{D} \in \mathfrak{S}ec(M, \mathrm{End}^{-}_{\gamma}(\mathcal{E}))$

When restricted to simple type Dirac operators, the universal Dirac action explicitly reads:

$$\begin{aligned}\mathcal{I}_{\rm D} &= \int_M * \left(\mathrm{tr}_\gamma(curv(\partial\!\!\!/_{\rm A})) - \lambda \mathrm{tr}_{\mathcal{E}}\phi_{\rm D}^2\right) \\ &\sim \mathcal{I}_{\rm EH,\Lambda}\,, \end{aligned} \tag{31}$$

with $\lambda \in \mathbb{R}$ being a positive constant that is determined by the dimension of M.

Remark:

The "cosmological constant" $\Lambda \equiv -\lambda \mathrm{tr}_{\mathcal{E}}\phi_{\rm D}^2$ reduces the gauge symmetry similar to the Higgs potential in the Standard Model. Whence, in terms of simple type Dirac operators one gets spontaneous symmetry breaking without a Higgs potential but makes use of gravity.

QUESTION:

How to obtain a non-trivial dynamics for the sections $\phi_{\mathrm{D}} \in \mathfrak{S}ec(M, \mathrm{End}_\gamma(\mathcal{E}))$ in a manner that is compatible with such a gauge symmetry reduction?

The answer is provided by the action functional of the Standard Model.

Proposition 3.1 *There is a class of (real) Clifford modules $(\mathcal{P}, \tau_{\mathcal{P}}, \gamma_{\mathcal{P}}, J_{\mathcal{P}})$ and a class of (real) Dirac operators of simple type: $\not{P}_{YMH} = \not{\partial}_A + \tau_{\mathcal{P}} \circ \phi_{YMH}$, determined by Yang-Mills-Higgs connections $\partial_{YMH} = \partial_A + H$, such that*

$$\begin{aligned} \mathcal{I}_D &= \int_M * \left(\mathrm{tr}_\gamma(curv(\not{P}_{YMH}) - \varepsilon \mathrm{ev}_g(\omega_D^2))\right) \\ &= \int_M \left[\mathrm{tr}_\gamma curv(\not{\partial}_A) + \mathrm{tr}_{\mathcal{P}} \phi_{YMH}^2\right] dvol_M \,. \end{aligned} \tag{32}$$

Here,

$$\mathrm{tr}_{\mathcal{P}} \phi_{YMH}^2 = -\mathrm{tr}_g F_A^2 + 2\varepsilon(\tfrac{n-1}{n})^2 \,\mathrm{tr}_g(\partial_A \varphi_H)^2 + 2(\tfrac{n-1}{n})^2 \,\mathrm{tr}\varphi_H^4 - 2\mathrm{tr}\varphi_H^2 \,. \tag{33}$$

A consequence on the physics side:

The geometrical description of the STM in terms of the universal Dirac action and (real) Dirac operators of simple type allows to make a verifiable prediction of the **mass of the Higgs boson**:

$$m_{\mathrm{H}} = 182 \pm 20\,\mathrm{GeV}\,, \tag{34}$$

which will be checked within the next few years in the LHC experiments made at CERN/Switzerland.