Quaternionic contact Einstein structures

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Quaternionic Contact Structures

Definition

M^{4n+3}-quaternionic contact if we have

i) *codimension three distribution H, locally,* $H = \bigcap_{s=1}^{3} Ker\, \eta_s$, $\eta_s \in T^*M$.

ii) *a 2-sphere bundle of "almost complex structures" locally generated by* $I_s : H \to H, \quad I_s^2 = -1,$ *satisfying* $I_1 I_2 = -I_2 I_1 = I_3$;

iii) *a metric tensor g on H, s.t.,*

$$2g(I_s X, Y) = d\eta_s(X, Y),$$
$$g(I_s X, I_s Y) = g(X, Y), \quad X, Y \in H.$$

Quaternionic Contact Structures

- Given η (and *H*) there exists at most one triple of a.c.str. and metric *g* that are compatible.
- Rotating η we obtain the same qc-structure.

The Biquard connection

Theorem (O. Biquard)

Under the above conditions and $n > 1$, there exists a unique supplementary distribution V of H in TM and a linear connection ∇ on M, s.t.,

1. *V and H are parallel*
2. *g and $\Omega = \sum_{j=1}^{3}(d\eta_j|_H)^2$ are parallel*
3. *torsion $T_{A,B} = \nabla_A B - \nabla_B A - [A, B]$ satisfies*
   - $\forall X, Y \in H, \quad T_{X,Y} = -[X, Y]|_V \in V$
   - $\forall \xi \in V$, *and* $\forall X \in H$, $T_{\xi,X} \in H$ *and* $T_\xi := (X \mapsto T_{\xi,X}) \in (sp(n)+sp(1))^\perp$

Reeb vector fields

- Note: V is generated by the Reeb vector fields $\{\xi_1, \xi_2, \xi_3\}$

  $\eta_s(\xi_k) = \delta_{sk}, \quad (\xi_s \lrcorner d\eta_s)_{|H} = 0$

  $(\xi_s \lrcorner d\eta_k)_{|H} = -(\xi_k \lrcorner d\eta_s)_{|H}.$

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- D. Duchemin showed that if we assume their existence, then there is a connection as before.
- Henceforth, by a qc structure in dimension 7 we mean a qc structure satisfying the Reeb conditions

- curvature: $\mathcal{R}(A, B)C = [\nabla_A, \nabla_B]C - \nabla_{[A,B]}C$;
- (horizontal) Ricci tensor: $Ric(X, Y) = Ric^{\nabla}|_H = tr_H\{Z \mapsto \mathcal{R}(Z, X)Y\}$ for $X, Y \in H$
- scalar curvature: $\mathrm{Scal} = tr_H\, Ric$.
- Kähler forms
  $2\omega_{i|H} \; = \; d\eta_{i|H}, \qquad \xi \lrcorner \omega_i = 0, \quad \xi \in V.$

- Sp(1)= {unit quaternions} $\subset$ SO(4n), $\lambda q = q \cdot \lambda^{-1}$.
- Sp(n)-quaternionic unitary $\subset SO(4n)$.
- Sp(n)Sp(1)-product in $SO(4n)$.
- Let $\Psi \in$End(H). The $Sp(n)$-invariant parts are follows

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

- The two $Sp(n)Sp(1)$-invariant components are given by

$$\Psi_{[3]} = \Psi^{+++}, \qquad \Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}.$$

Using End(H) $\overset{g}{\cong} \Lambda^{1,1}$ the $Sp(n)Sp(1)$-invariant components are the projections on the eigenspaces of $\Upsilon = I_1 \otimes I_1 + I_2 \otimes I_2 + I_3 \otimes I_3$.

The Torsion Tensor. $T_{\xi_j} = T^0_{\xi_j} + I_j U$, $U \in \Psi_{[3]}$.

$T^0_{\xi_j}$-symmetric, $I_j U$-skew-symmetric.

Theorem (w/ St. Ivanov, I. Minchev)

Define $T^0 = T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3 \in \Psi_{[-1]}$. *We have* $Ric = (2n+2)T^0 + (4n+10)U + \frac{Scal}{4n} g$.

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Definition

M is called qc-Einstein if $T^0 = 0$ and $U = 0$. M is called qc-pseudo-Einstein if $U = 0$.

Theorem (w/ St. Ivanov, I. Minchev)

Let $(M^{4n+3}, g, \mathbb{Q})$ *be a QC manifold. TFAE*

i) *The torsion of the Biquard connection is identically zero,* $T_\xi, \xi \in V$.

ii) $(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold.*

iii) *Each Reeb vector field is a qc vector field,* $\mathcal{L}_Q \eta = (\nu I + O) \cdot \eta$.

iv) *Each Reeb vector field preserves the horizontal metric and the quaternionic structure simultaneously, i.e.* $\mathcal{L}_Q g = 0$ *and* $\mathcal{L}_Q I = O \cdot I$,

where

$$\nu \in \mathcal{C}^\infty(M), \quad O \in \mathcal{C}^\infty(M, so(3)), \quad I = (I_1, I_2, I_3)^t.$$

Vanishing horizontal connection 1-forms

Lemma (w/ St. Ivanov, I. Minchev)

If a qc structure has zero connection one forms restricted to the horizontal space H then the qc structure is qc-Einstein.

The connection one forms are $\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j$.

It is also useful to note $R(A, B, \xi_i, \xi_j) = 2\rho_k(A, B) = (d\alpha_k + \alpha_i \wedge \alpha_j)(A, B)$.

Proposition (w/ St. Ivanov)

Let $(M^{4n+3}, \eta, \mathbb{Q})$ *be a (4n+3)- dimensional qc manifold. Let* $s = \frac{Scal}{8n(n+2)}$*, so that a 3-Sasakian manifold has* $s = 2$*. The following equations hold*

$$2\omega_i = d\eta_i + \eta_j \wedge \alpha_k - \eta_k \wedge \alpha_j + s\eta_j \wedge \eta_k,$$

$$d\omega_i = \omega_j \wedge (\alpha_k + s\eta_k) - \omega_k \wedge (\alpha_j + s\eta_j) - \rho_k \wedge \eta_j + \rho_j \wedge \eta$$

$$d\Omega = \sum_{(ijk)} \Big[2\eta_i \wedge (\rho_k \wedge \omega_j - \rho_j \wedge \omega_k) + ds \wedge \omega_i \wedge \eta_j \wedge \eta_k \Big],$$

where $\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$*.*
In particular, the structure equations of a 3-Sasaki manifold have the form $d\eta_i = 2\omega_i + 2\eta_j \wedge \eta_k$*.*

Properties of qc-Einstein manifolds

Theorem (w/ St. Ivanov, I. Minchev)

If M is qc-Einstein then Scal=const., V is integrable, and for $X \in H$, $s, t = 1, 2, 3$ we have $\rho_{t|_H} = \tau_{t|_H} = -2\zeta_{t|_H} = -s\omega_t, \quad \rho_i(\xi_i, \xi_j) + \rho_k(\xi_k, \xi_j) = 0, \quad Ric(\xi_s, X) = \rho_s(X, \xi_t) = \zeta_s(X, \xi_t) = 0.$

Here, $\rho_s(A, B) = \frac{1}{4n} R(A, B, e_\alpha, I_s e_\alpha), \quad \zeta_s(A, B) = \frac{1}{4n} R(e_\alpha, A, B, I_s e_\alpha), \quad \tau_s(A, B) = \frac{1}{4n} R(e_\alpha, I_s e_\alpha, A, B), \quad \omega_s = \frac{1}{2} d\eta_{s|_H}$.

The Proof uses the Bianchi's identities.

Theorem (w/ St. Ivanov, I. Minchev)

The divergences of the curvature tensors satisfy the system $B\,b = 0$, *where*

$$\mathbf{B} = \begin{pmatrix} -1 & 6 & 4n-1 & \frac{3}{16n(n+2)} & 0 \\ -1 & 0 & n+2 & \frac{3}{16n(n+2)} & 0 \\ 1 & -3 & 4 & 0 & -1 \end{pmatrix},$$

$\mathbf{b} = (\nabla^* T^o,\ \nabla^* U,\ A,\ dScal,\ Ric(\xi_j, I_j\,.)\,)^t,$
$A = I_1[\xi_2, \xi_3] + I_2[\xi_3, \xi_1] + I_3[\xi_1, \xi_2].$

Vanishing tor using the str eqs

Using qc-Einstein $\Rightarrow$ *Scal*=const., $[V, V] \subseteq V$, and

Lemma (w/ St. Ivanov)

On a qc manifold of dimension $(4n+3) > 7$ *we have*

$$U(X, Y) = -\frac{1}{16n}\Big[d\Omega(\xi_i, X, I_k Y, e_a, I_j e_a) + d\Omega(\xi_i, I_i X, I_j Y, e_a, I_j e_a)\Big]$$

$$T^0(X, Y) = \frac{1}{8(1-n)} \sum_{(ijk)} \Big[d\Omega(\xi_i, X, I_k Y, e_a, I_j e_a) - d\Omega(\xi_i, I_i X, I_j Y, e_a, I_j e_a)\Big].$$

we prove

Theorem (w/ St. Ivanov)

Let $(M^{4n+3}, \eta, \mathbb{Q})$ *be a qc manifold,* $n > 1$. *The following conditions are equivalent*

a) $(M^{4n+3}, \eta, \mathbb{Q})$ *has closed fundamental four form,* $d\Omega = 0$;

b) $(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold;*

c) *Each Reeb vector* ξ_s *field preserves the horizontal metric and the quaternionic structure simultaneously,* $\mathbb{L}_{\xi_s} g = 0, \quad \mathbb{L}_{\xi_s}\mathbb{Q} \subset \mathbb{Q}$.

d) *Each Reeb vector field* ξ_s *preserves the fundamental four form,* $\mathbb{L}_{\xi_s}\Omega = 0$.

The ”positive” qc-Einstein case

Proposition (w/ St. Ivanov)

The structure equations characterizing a 3-Sasaki manifold among all qc structures are $d\eta_i = 2\omega_i + 2\eta_j \wedge \eta_k$.

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Proof.

Recall, $2\omega_i = d\eta_i + \eta_j \wedge \alpha_k - \eta_k \wedge \alpha_j + s\eta_j \wedge \eta_k$.
For a 3-Sasakian manifold we have $s = 2, d\eta_i(\xi_j, \xi_k) = 2, \alpha_s = -2\eta_s$. Conversely, the Kähler forms $F_i = t^2(\omega_i + \eta_j \wedge \eta_k) + tdt \wedge \eta_i$ on the cone $N = M \times \mathbb{R}^+$ are closed and therefore $g_N = t^2(g + \sum_{s=1}^3 \eta_s \otimes \eta_s) + dt \otimes dt$ is hyper Kähler due to Hitchin's theorem

Main thrm for "positive" Einstein qc

Theorem (w/ St. Ivanov, I. Minchev)

Suppose Scal > 0. The next conditions are equivalent:

i) $(M^{4n+3}, g, \mathbb{Q})$ *is qc-Einstein manifold.*

ii) *M is locally 3-Sasakian: locally there exists a matrix* $\Psi \in \mathcal{C}^{\infty}(M : SO(3))$, *s.t.,* $(\frac{2}{s}\Psi \cdot \eta, Q)$ *is 3-Sasakian;*

Proof of the main theorem characterizing 3-Sasaki

Proof in the case $n > 1$ using the fundamental 4-form. The original proof works when $n = 1$ as well.

$d\Omega = 0 \ \Rightarrow M$ is qc-Einstein $\Rightarrow Scal = const$ and $[V, V] \subseteq V$. The qc structure $\eta' = \frac{16n(n+2)}{Scal}\eta$ has normalized qc scalar curvature $s' = 2$ and $d\Omega' = 0$ provided $Scal \neq 0$.
Drop the $'$ hereafter.

Claim: The Riemannian cone $N = M \times \mathbb{R}^+$, $g_N = t^2(g + \sum_{s=1}^{3} \eta_s \otimes \eta_s) + dt \otimes dt$ is hyper-Kähler.

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  $F = F_1 \wedge F_1 + F_2 \wedge F_2 + F_3 \wedge F_3$.
- $dF_i = tdt \wedge (2\omega_i + 2\eta_j \wedge \eta_k - d\eta_i) + t^2 d(\omega_i + \eta_j \wedge \eta_k)$.

Thus $(M, \Psi \cdot \eta)$ is locally a 3-Sasakian manifold.

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- $dF = t^4 d\Omega - 2t^3 \sum_{(ijk)} dt \wedge (\rho_i + 2\omega_i) \wedge \eta_j \wedge \eta_k = 0$.

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- N is Einstein and (warped metric) is Ricci flat.

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- N is quaternionic Kähler manifold if $n > 1$.
- N is Einstein and (warped metric) is Ricci flat.
- N is locally hyper-Kähler. Locally, there exists a $SO(3)$-matrix Ψ with smooth entries, possibly depending on t, such that the triple of two forms $(\tilde{F}_1, \tilde{F}_2, \tilde{F}_3) = \Psi \cdot (F_1, F_2, F_3)^t$ are closed.

Thus $(M, \Psi \cdot \eta)$ is locally a 3-Sasakian manifold.

Zero torsion examples

- For some constant τ the following structure equations hold $d\eta_i = 2\omega_i + 2\tau\eta_j \wedge \eta_k$, for any cyclic permutation (i, j, k) of $(1, 2, 3)$.

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- Examples of such qc manifolds are:
  (i) the quaternionic Heisenberg group, where $\tau = 0$;
  (ii) any 3-Sasakian manifold, where $\tau = 1$;
  (iii) the zero torsion qc-flat group $G_{-1/4}$ described next, where $\tau = -1/4$.

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  (ii) any 3-Sasakian manifold, where $\tau = 1$;
  (iii) the zero torsion qc-flat group $G_{-1/4}$ described next, where $\tau = -1/4$.
- For $\tau < 0$ ($\tau > 0$), the qc homothety $\eta_i \mapsto -2\tau\eta_i$ ($\eta_i \mapsto \tau\eta_i$) brings the qc-structure $G_{-1/4}$ (a 3-Sasakain structure) to one satisfying the above structure equations.

Example of a "negative" qc-Einstein

This is the only Lie group s.t. $d\eta_i = 2\omega_i + 2\tau\eta_j \wedge \eta_k$, $\tau \neq 0$, for some (necessarily) negative constant τ.
Consider the Lie algebra $\mathfrak{g}_{-1/4}$

$$de^1 = 0, \quad de^2 = -e^{12} - 2e^{34} - \tfrac{1}{2}e^{37} + \tfrac{1}{2}e^{46}$$
$$de^3 = -e^{13} + 2e^{24} + \tfrac{1}{2}e^{27} - \tfrac{1}{2}e^{45}$$
$$de^4 = -e^{14} - 2e^{23} - \tfrac{1}{2}e^{26} + \tfrac{1}{2}e^{35}$$
$$de^5 = 2e^{12} + 2e^{34} - \tfrac{1}{2}e^{67}$$
$$de^6 = 2e^{13} + 2e^{42} + \tfrac{1}{2}e^{57}, \quad de^7 = 2e^{14} + 2e^{23} - \tfrac{1}{2}e^{56}$$

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$$H = span\{e^1, \ldots, e^4\}, \ \eta_1 = e^5, \eta_2 = e^6, \eta_3 = e^7,$$
$$\omega_1 = e^{12} + e^{34}, \ \omega_2 = e^{13} + e^{42}, \ \omega_3 = e^{14} + e^{23}$$

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Theorem (w/ L. de Andres, M. Fernandez, St. Ivanov, J. Santisteban and L. Ugarte)

Let $(G_{-1/4}, \eta, \mathbb{Q})$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_{-1/4}$ equipped with the left invariant qc structure $(\eta, \mathbb{Q})$ defined above. Then

a) *$G_{-1/4}$ is qc-Einstein and the normalized qc scalar curvature is a negative constant, $S = -\frac{1}{2}$.*

b) *The qc conformal curvature is zero, $W^{qc} = 0$, i.e., $(G_{-1/4}, \eta, \mathbb{Q})$ is locally qc conformally flat.*