Scattering on manifolds with cylindrical ends and stable systoles

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- joint work with Alexander Strohmaier

1) Scattering theory

Scattering theory seeks to understand large-time asymptotic behavior of solutions to evolution equations such as the Schrödinger equation

$$i\frac{\partial u}{\partial t} = Hu$$

and the wave equation

$$\frac{\partial^2 u}{\partial t^2} = -\Delta u.$$

Solution operators are

$$e^{itH}, \quad \text{and} \quad \cos(t\sqrt{\Delta}) \quad \text{or} \quad \sin(t\sqrt{\Delta})/\sqrt{\Delta},$$

depending in the latter case on initial conditions.
Main features: Open systems, particle may escape to infinity,

1.1. The time-dependent approach

- $\mathcal{H}$ separable Hilbert space, H_0 and H self-adjoint operators in $\mathcal{H}$.
- H_0 free Hamiltonian, H Hamiltonian with interaction

Example. Scattering by a potential $V \in C_c^\infty(\mathbb{R}^n)$.

$$\mathcal{H} = L^2(\mathbb{R}^n), \quad \Delta = d^* d, \quad H_0 = \overline{\Delta}, \quad H = \overline{\Delta + V}.$$

We say that

$$u(t) = e^{-itH}\varphi$$

is **asymptotically free** as $t \to \pm\infty$, if there exist $\varphi_\pm \in \mathcal{H}$:

$$\lim_{t\to\pm\infty} \| e^{-itH}\varphi - e^{-itH_0}\varphi_\pm \| = 0.$$

Equivalent to

$$\lim_{t\to\pm\infty} \| e^{itH} e^{-itH_0}\varphi_\pm - \varphi \| = 0.$$

Problem. Existence of the limit.

Wave operators

- $\mathcal{H}_{\mathrm{ac}}$ and $\mathcal{H}_{0,\mathrm{ac}}$ absolutely continuous subspaces of H and H_0, respectively.
- P_{ac} and $P_{0,\mathrm{ac}}$ projections onto $\mathcal{H}_{\mathrm{ac}}$ and $\mathcal{H}_{0,\mathrm{ac}}$, respectively.
- H_{ac} and $H_{0,\mathrm{ac}}$ absolutely continuous parts of H and H_0, respectively.

Put

$$W_{\pm}(H, H_0) := \text{s-lim}_{t\to\pm\infty}\, e^{itH} e^{-itH_0} P_{0,\mathrm{ac}}$$

Problem. Existence and completeness of wave operators.

- If $W_\pm(H, H_0)$ exist and are complete, then

$$W_\pm \colon \mathcal{H}_{0,\mathrm{ac}} \cong \mathcal{H}_{\mathrm{ac}}, \quad W_\pm H_{0,\mathrm{ac}} = H_{\mathrm{ac}} W_\pm.$$

- Absolutely continuous parts of H_0 and H are unitarily equivalent.

Birman-Kato invariance principle:
1) Let $\lambda \in \mathbb{C} - \mathbb{R}$ and $k \in \mathbb{N}$ such that

$$(H + \lambda\,\mathrm{Id})^{-k} - (H_0 + \lambda\,\mathrm{Id})^{-k}$$

is trace class. Then the wave operators exist and are complete.

2) Suppose that $H, H_0 \geq 0$ and

$$e^{-tH} - e^{-tH_0}$$

is trace class for all $t > 0$. Then the wave operators exist and are complete.

Scattering operator:

$$S = W_+^* \circ W_-.$$

- S unitary operator in $\mathcal{H}_{0,\mathrm{ac}}$, commutes with $H_{0,\mathrm{ac}}$.
- Let σ_0 be the absolutely continuous spectrum of H_0, and $\{E_0(\lambda)\}_{\lambda\in\sigma_0}$ the spectral family of $H_{0,\mathrm{ac}}$.

Then

$$S = \int_{\sigma_0} S(\lambda) dE_0(\lambda).$$

- $S(\lambda)$ "on-shell scattering matrix".

1.2. Stationary (time-independent) approach

Example. $H = \overline{\Delta + V}$, $V \in C_c^\infty(\mathbb{R}^n)$.

The stationary approach is related to the spectral decomposition of H.
Let $\omega \in S^{n-1}$, $\lambda > 0$, $\hat{x} = x/|x|$.

$$e^{i\lambda\langle\omega,x\rangle}, \quad \text{plane wave.}$$

Sommerfeld radiation condition: There exists a unique solution of

$$(\Delta + V - \lambda^2)\psi = 0$$

such that

$$\psi(x;\omega,\lambda) = e^{i\lambda\langle\omega,x\rangle} + a(\hat{x},\omega,\lambda)|x|^{-(n-1)/2}e^{-i\lambda|x|} + o\left(|x|^{-(n-1)/2}\right), \quad |x| \to \infty.$$

distorted plane wave.

- $a(\theta, \omega, \lambda)$ scattering amplitude.
- $|a(\theta, \omega, \lambda)| d\theta$ scattering cross section with respect to angle $d\theta$.
- fundamental quantity, measured in scattering experiments

$$S(\lambda) \colon L^2(S^{n-1}) \to L^2(S^{n-1})$$

on-shell scattering matrix,

- $S(\lambda) - \mathrm{Id}$ integral operator with kernel

$$e^{\frac{\pi}{4}(n-1)i}(2\pi)^{-\frac{1}{2}(n-1)}\lambda^{(n-1)/2}a(\theta, \omega, \lambda).$$

Time delay operator.

Example. $H_0 = \overline{\Delta}$, $H = \overline{H + V}$.

Let $\Omega \subset \mathbb{R}^n$, $\phi \in L^2(\mathbb{R}^n)$.

Quantum mechanics: Probability to find particle with wave function ϕ at time t in Ω is given by

$$\int_\Omega \left| e^{-itH} \phi(x) \right|^2 dx = \| \chi_\Omega e^{-itH} \phi \|^2 .$$

Total time:

$$\int_{\mathbb{R}} \| \chi_\Omega e^{-itH} \phi \|^2 \; dt.$$

- By definition, $e^{-itH} W_- \phi$ and $e^{-itH_0} \phi$ are asymptotically equivalent as $t \to \infty$.

Time excess due to interaction:

$$\int_{\Omega} \left(\| \chi_{\Omega} e^{-itH} W_{-} \phi \|^2 - \| \chi_{\Omega} e^{-itH_0} \phi \|^2 \right).$$

Eisenbud-Wigner time delay operator:

$$\langle \phi, \mathcal{T} \phi \rangle = \lim_{R \to \infty} \int_{\mathbb{R}} \left(\| \chi_{B_R} e^{-itH} W_{-} \phi \|^2 - \| \chi_{B_R} e^{-itH_0} \phi \|^2 \right).$$

defines closable quadratic form, $\mathcal{T}$ self-adjoint, commutes with H_0. Let

$$\mathcal{T} = \int_{\sigma_{ac}(H_0)} \mathcal{T}(\lambda)\, dE_0(\lambda).$$

Eisenbud-Wigner formula: Let $S(\lambda)$ be the on-shell scattering matrix. Then

$$\mathcal{T}(\lambda) = -iS(\lambda)^{-1} \frac{dS}{d\lambda}(\lambda).$$

movie

Resonances

• In physics a resonance $E - i\gamma$ is related to a dissipative metastable state with energy E and decay rate γ.

• Mathematically resonances can be defined as poles of the meromorphic continuation of the resolvent.

• Resonances describe the longtime behaviour of solutions of the wave equation.

• In many settings a meromorphic extension of the scattering matrix exists and resonances can be defined as poles of the scattering matrix.

• Resonances replace bound states in any system in which particles have the possibility to escape to infinity.

2) Geometric scattering theory

• Basic tool to study of continuous spectrum of geometric differential operators on noncompact manifolds.

Examples: Manifolds with cylindrical ends, wave guides, locally symmetric spaces $\Gamma\backslash G/K$ of finite volume, moduli spaces, etc.

• Common feature: Special geometric structure at infinity.

Manifolds with cylindrical ends:

- M compact Riemannian manifold with boundary Y,
- $g^M|_{(-\varepsilon,0]\times Y} = du^2 + g^Y$,

$$X = M \cup_Y Z, \quad Z = \mathbb{R}^+ \times Y,\ g^Z = du^2 + g^Y.$$

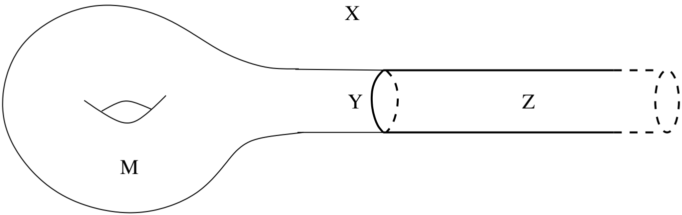

Figure: Elongation X of M.

- $E \to X$ Hermitian vector bundle, $E|_Z = pr_Y^*(E^Y)$, $E^Y \to Y$.

$$D\colon C^\infty(X,E) \to C^\infty(X,E)$$

Dirac type operator.

Assumption: $D|_Z = \gamma\left(\frac{\partial}{\partial u} + D_Y\right)$,

- $\gamma\colon E|_Y \to E|_Y$ bundle isomorphism,
- $D_Y\colon C^\infty(Y, E|_Y) \to C^\infty(Y, E|_Y)$ symmetric, 1st order, elliptic, s.th.

$$\gamma^2 = -\operatorname{Id}, \quad \gamma^* = -\gamma, \quad D_Y\gamma = -\gamma D_Y.$$

Put

$$D_0 = \gamma\left(\frac{\partial}{\partial u} + D_Y\right) : C_c^\infty(\mathbb{R}^+ \times Y, E) \to L^2(\mathbb{R}^+ \times Y, E).$$

- $\mathcal{D} = \overline{D}$, $\mathcal{D}_0$ closure of D_0 w.r.t. Atiyah-Patodi-Singer boundary conditions.

Let $J\colon L^2(Z, E|_Z) \subset L^2(X, E)$ be the inclusion.

Theorem. For all $t > 0$,

$$e^{-t\mathcal{D}^2} - Je^{-t\mathcal{D}_0^2}, \quad \mathcal{D}e^{-t\mathcal{D}^2} - J\mathcal{D}_0 e^{-t\mathcal{D}_0^2}$$

are trace class operators.
By the Kato-Birman invariance principle, the wave operators

$$W_\pm(\mathcal{D}, \mathcal{D}_0) = \text{s-lim}_{t\to\pm\infty} e^{it\mathcal{D}} J e^{-it\mathcal{D}_0}$$

exist and are complete.

- $S(\lambda)$, $\lambda \in \mathbb{R}$, on-shell scattering matrix.

Let $0 \leq \mu_1 < \mu_2 < \cdots$ be the nonnegative eigenvalues of D_Y, $\mathcal{E}(\mu_k)$ eigenspace of μ_k.

$$S(\lambda)\colon \bigoplus_{\mu_k \leq |\lambda|} (\mathcal{E}(\mu_k) \oplus \mathcal{E}(-\mu_k)) \to \bigoplus_{\mu_k \leq |\lambda|} (\mathcal{E}(\mu_k) \oplus \mathcal{E}(-\mu_k)) .$$

Let $\Sigma \to \mathbb{C}$ be the ramified covering associated to the functions $\sqrt{\lambda \pm \mu_k}$, $k \in \mathbb{N}$. Then $S(\lambda)$ extends to a meromorphic function on Σ.

Question: What information about X can be extracted from $S(\lambda)$? – inverse scattering theory.

Example. $S(0)\colon \ker(D_Y) \to \ker(D_Y)$ satisfies

$$S(0)^2 = \mathrm{Id}, \quad S(0)\gamma = -\gamma S(0).$$

Let $E_Y = E_Y^+ \oplus E_Y^-$ be the decomposition of $E_Y \to Y$ into the ± 1 eigenspaces of γ. Let $D_Y^\pm$ be the restriction of D_Y to $C^\infty(Y, E_Y^\pm)$. Then

$$D_Y^\pm \colon C^\infty(Y, E_Y^\pm) \to C^\infty(Y, E_Y^\mp).$$

Then

$$S(0)\colon \ \ker(D_Y^+) \cong \ker(D_Y^-).$$

Cobordism invariance of the index for Dirac type operators:

$$\operatorname{Ind}(D_Y^+) = 0.$$

3) Harmonic forms and scattering length

From now on we restrict attention to

$$d + d^* \colon \Lambda^*(X) \to \Lambda^*(X).$$

Let $\omega \in \Lambda^p(\mathbb{R}^+ \times Y)$. Then

$$\omega = \phi + du \wedge \psi,$$

where $\phi \in C^\infty(\mathbb{R}^+, \Lambda^p(Y))$ and $\psi \in C^\infty(\mathbb{R}^+, \Lambda^{p-1}(Y))$.

- $S(0)$ preserves this decomposition.
- $0 \leq \mu_1 < \mu_2 < \cdots$ eigenvalues of $\Delta_{Y,p}$

Then for $\lambda \geq 0$:

$$S_p(\lambda) \colon \bigoplus_{\mu_k \leq \lambda} \mathcal{E}(\mu_k) \to \bigoplus_{\mu_k \leq \lambda} \mathcal{E}(\mu_k).$$

Let $\mu > 0$ be the first positive eigenvalue of $\Delta_{Y,p}$. Then

$$S_p(\lambda)\colon \mathcal{H}^p(Y) \to \mathcal{H}^p(Y)$$

is analytic in $\lambda \in B(0,\mu)$.

Generalized eigenforms: Let $\phi \in \mathcal{H}^p(Y)$. For $|\lambda| < \mu$ there is a unique $E(\phi,\lambda) \in \Lambda^p(X)$ which is a solution of

$$\Delta_p E(\phi,\lambda) = \lambda^2 E(\phi,\lambda),$$

such that on $\mathbb{R}^+ \times Y$ we have

$$E(\phi,\lambda,(u,y)) = e^{-i\lambda u}\phi(y) + e^{i\lambda u}(S_p(\lambda)\phi)(y) + R(\phi,\lambda,(u,y))$$

with $R(\phi,\lambda) \in L^2$.

- Analog of Sommerfeld radiation condition.

Relation with cohomology

Let $\phi \in \mathcal{H}^p(Y)$ and assume that $S_p(0)\phi = \phi$. Then

$$E(\phi, 0)|_Z = 2\phi + \psi, \quad \psi \in L^2.$$

- $\frac{1}{2}E(\phi, 0)$ extended harmonic form on X (in the sense of A-P-S) with limiting value ϕ.

$X = M \cup_Y Z$, M compact manifold with boundary Y.

Thoerem. The $+1$-eigenspace of $S(0)$ on $H^p(Y, \mathbb{R})$ coincides with $\mathrm{Im}(H^p(M, \mathbb{R}) \to H^p(Y, \mathbb{R}))$.

Let

$$\mathcal{H}^p_{\mathrm{ext}}(X) = \{\psi \in \Lambda^p(X)\colon \Delta_p\psi = 0,\ \exists\phi_1 \in \mathcal{H}^p(Y),\ \phi_2 \in \mathcal{H}^{p-1}(Y)$$
$$\psi|_Z = \phi_1 + du\wedge\phi_2 + \theta, \quad \theta \in L^2\Lambda^p(Z)\}.$$

- ϕ_1 and ϕ_2 are uniquely determined by ψ.
- Put $\psi_t = \phi_1$ and $\psi_n = \phi_2$.

$$\mathcal{H}^p_{ext,abs}(X) := \{\psi \in \mathcal{H}^p_{ext}(X) \mid \psi_n = 0\},$$
$$\mathcal{H}^p_{ext,rel}(X) := \{\psi \in \mathcal{H}^p_{ext}(X) \mid \psi_t = 0\}.$$

Let $\psi \in \mathcal{H}^p_{\mathrm{ext}}(X)$. Since ψ is harmonic, it follows that

$$(\psi - \psi_t - du\wedge\psi_n)(u,y) \ll e^{-cu}, \quad (u,y) \in Z.$$

Applying Green's formula to $M_a = M \cup_Y ([0,a]\times Y)$, we get

$$0 = \langle\Delta\psi,\psi\rangle_{M_a} = \| d\psi \|^2_{M_a} + \| \delta\psi \|^2_{M_a} + O(e^{-ca}),$$

which implies that

$$d\psi = 0,\ \delta\psi = 0 \quad \text{for all } \psi \in \mathcal{H}^p_{ext}(X).$$

Thus we get a canonical map

$$R\colon \mathcal{H}^p_{ext,abs}(X) \to H^p(X,\mathbb{R}).$$

Let $\psi \in \mathcal{H}^p_{ext,rel}(X)$. Then $d\psi = 0$,

$$\psi|_Z = du \wedge \psi_n + d\theta,$$

and θ is exponentially decaying.

- χ cut-off function, support on the cylinder Z equal to 1 outside a compact set.

Define

$$R_c\colon \mathcal{H}^p_{ext,rel}(X) \to H^p_c(X,\mathbb{R}), \quad \psi \mapsto [\psi - d(\chi(u\psi_n + \theta))].$$

This map is well defined and independent of the choice of χ. There are maps

$$\hat{F} : \mathcal{H}^p(Y) \to \mathcal{H}^p_{ext,abs}(X), \quad \phi \mapsto \frac{1}{2}E(\phi,0),$$

$$\hat{G} : \mathcal{H}^p(Y) \to \mathcal{H}^p_{ext,rel}(X), \quad \phi \mapsto \frac{i}{2}dE'(\phi,0).$$

Lemma. $S'(0)$ is invertible on $\mathcal{H}^*(Y)$.

Put

$$\tilde{\partial} = \hat{G} \circ \left(\frac{i}{2}S'(0)\right)^{-1}.$$

Then

$$\tilde{\partial}\colon \mathcal{H}^p(Y) \to \mathcal{H}^{p+1}_{\mathrm{ext},rel}(X)$$

corresponds to the boundary operator

$$\partial\colon H^p(Y,\mathbb{R}) \to H^{p+1}_c(X,\mathbb{R}).$$

There is a long exact sequence

$$\cdots \to \mathcal{H}^p_{\mathrm{ext},rel}(X) \to \mathcal{H}^p_{\mathrm{ext},abs}(X) \to \mathcal{H}^p(Y) \to \mathcal{H}^{p+1}_{\mathrm{ext},rel}(X) \to \cdots$$

which is equivalent to

$$\cdots \to H^p_c(X,\mathbb{R}) \to H^p(X,\mathbb{R}) \to H^p(Y,\mathbb{R}) \to H^{p+1}_c(X,\mathbb{R}) \to \cdots$$

4) Stable norm and comass norm

Federer, Gromov

- V n-dimensional Euclidean vector space

For $\omega \in \Lambda^p V^*$ define the comass norm by

$$\|\omega\|_\infty = \sup\{\omega(e_1, \dots, e_p) \mid e_k \in V, \|e_k\| = 1\} \qquad (1)$$

Since the norms are equivalent there is a constant C such that

$$\|\omega\|^2 \leq C\|\omega\|_\infty^2. \qquad (2)$$

- $C(n,p)$ the optimal such constant.
- $C(n,0) = C(n,1) = 1,\ C(n,p) \leq \binom{n}{p}$.

• B a compact manifold with boundary ∂B, $\omega \in \Lambda^p(B)$. Define comass by

$$\begin{aligned}\|\omega\|_\infty &= \sup\{\omega_x(e_1,\dots,e_p) \mid x \in B,\ e_i \in T_xB,\ g(e_i,e_i)=1\}\\ &= \sup\{\|\omega_x\|_\infty \mid x \in B\}.\end{aligned}$$

Get norm on $H^p(B,\partial B,\mathbb{R})$ by

$$\|\phi\|_\infty = \inf\{\|\omega\|_\infty \mid \phi = [\omega],\ \omega \in \Lambda^p(B,\partial B),\ d\omega = 0\}.$$

For $z \in H_p(B,\partial B,\mathbb{R})$ the stable norm $\|z\|_{st}$ is defined as

$$\|z\|_{st} = \inf\{\sum_i |\alpha_i| \mathrm{Vol}(c_i) \mid z = \sum_i \alpha_i [c_i],\ \alpha_i \in \mathbb{R}\},$$

where the infimum is taken over all Lipschitz continuous simplices c_i.

Geometric measure theory: Federer, Gromov

$$\|z\|_{st} = \sup\{|\phi(z)| \mid \phi \in H^p(B, \partial B, \mathbb{R}), \|\phi\|_\infty \leq 1\}.$$

5) Scattering length:

$$T_p(0) = -iS_p(0)^{-1}\left(\frac{d}{ds}S_p\right)(0).$$

Put

$$\mathrm{Vol}_*(M) = \mathrm{Vol}(M) + \frac{1}{\sqrt{\mu}}\mathrm{Vol}(Y),$$

where μ is the samllest postive eigenvalue of Δ_Y.

Theorem

Let $0 \leq p \leq n$. For every $\phi \in \mathcal{H}^p(Y)$ in the -1-eigenspace of $S_p(0)$ we have

$$\frac{1}{2}C(n,p+1)^{-1}\mathrm{Vol}_*(M)^{-1}\|[M]\cap\partial\phi\|_{st}^2$$
$$\leq \langle\phi, T(0)^{-1}\phi\rangle \leq \frac{1}{2}C(n,p+1)\mathrm{Vol}(M)\|\partial\phi\|_\infty^2.$$

Example. Y has two components Y_1 and Y_2, $p=0$.
There is a canonical basis in $H^0(Y,\mathbb{R})$ s.th.

$$T_0(0) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix},$$

so that

$$t_1 = 2\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(Y)}, \quad C_2 \leq t_2 \leq C_1,$$

where

$$C_1 = 2\mathrm{Vol}_*(M)\frac{\mathrm{Vol}(Y_1)\mathrm{Vol}(Y_2)}{\|\iota_*([Y_1])\|_{st}^2(\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2))},$$

$$C_2 = 2\mathrm{Vol}(M)^{-1}\frac{\mathrm{dist}(Y_1, Y_2)^2\mathrm{Vol}(Y_1)\mathrm{Vol}(Y_2)}{\mathrm{Vol}(Y_1)+\mathrm{Vol}(Y_2)}.$$

and ι is the inclusion of Y into M.