1. Abstract

We present a notion of Einstein manifolds with skew torsion on Riemannian manifolds for all dimensions, initiated by the doctoral work of the second author in dimension four.

2. Overview

- Torsion, and in particular skew torsion, has been a topic of interest to both mathematicians and physicists in recent decades.
- The first attempts to introduce torsion in general relativity go back to the 1920s with the work of H. Cartan. More recently, torsion makes its appearance in string theory, where the basic model for type II consists of a Riemannian manifold, a connection with skew torsion, a spinorial field and a dilaton function.
- From the mathematical point of view, skew torsion has a significant role in the work of Bismut and his local index theorem for non-Kähler manifolds. Skew torsion is also an important feature in generalised geometry, where there are two natural connections with skew torsion that come from the exterior derivative of the B-field.
- Torsion is also ubiquitous in the theory of non-integrable geometries. The idea is to choose a G-structure so that the G-connection with torsion admits desired parallel objects, in particular spinors, interpreted as supermetry transformations. As a first step in this investigation, T. Friedrich and S. Ivanov proved that many non-integrable geometric structures admit a unique invariant connection with parallel totally antisymmetric torsion, thus being a natural replacement for the Levi-Civita connection.

3. Metric connections with skew torsion

Definition, existence and uniqueness

- Let \((M, g, H)\) be a Riemannian manifold. Suppose that \(\nabla\) is a connection on a \(TM\) and let \(T\) be the \((1,2)\) tensor field. If we contract \(T\) with the metric we get a \((0,3)\) tensor which we still call the torsion of \(\nabla\). If \(T\) is a three-form then we say that \(\nabla\) is a connection with skew-symmetric torsion.
- Given any three-form \(H\) on \(M\) there exists a unique metric connection with skew torsion \(\nabla\) which is defined explicitly by \(g(\nabla_X Y, Z) = g(\nabla^\nabla_X Y, Z) + \frac{1}{2}H(X, Y, Z)\) where \(\nabla^\nabla\) is the Levi-Civita connection.

4. Motivation and definition

The obvious way to define the Einstein equations with skew torsion is simply to set \(Ric^\nabla = \lambda g\) for some function \(\lambda\). This, however, presents two immediate problems: the first is that the function \(\lambda\) might not be constant, and also that \(Ric^\nabla\) might not be symmetric since \(Ric^\nabla(X, Y) = Ric^\nabla(\theta(X), \theta(Y)) + \frac{1}{2}H(X, Y)\), where \(\theta\) is an orthogonal frame of the tangent bundle and requiring \(H\) to be co-closed, in a priori too restrictive.

The standard Einstein equations of Riemannian geometry can be obtained by a variational argument. They are the critical points of the Hilbert functional \(\int_M \frac{1}{2} |R|^2 \, dv_g\), where \(\lambda\) is the cosmological constant.

So one way of obtaining Einstein equations with skew torsion is to look for the critical points of the functional \(\int_M \frac{1}{2} |\nabla^\nabla|^2 \, dv_g\) given by \(g(\nabla^\nabla_X Y, Z) = g(\nabla^\nabla_X Y, Z) + \frac{1}{2}H(X, Y, Z)\) for any connection \(\nabla\) on \(TM\) and \(H\) a three-form on \(M\).

Proposition 1: The critical points of the functional \(L(g, H) = \int_M \frac{1}{2} |\nabla^\nabla|^2 \, dv_g\)

are given by pairs \((g, H)\) such that the Ricci tensor \(Ric^\nabla\) is symmetric and satisfies the equation \(-Ric^\nabla = \frac{1}{2}|H|^2 g + \lambda g\).

In the case of non-integrable geometries, requiring that \(H\) is \(\nabla\) parallel is a natural condition and this will then imply that \(H\) is co-closed. Also, under the assumption that \(\nabla^\nabla = 0\), the \(\nabla\)-curvature tensor simplifies to \(\nabla^\nabla(X, Y, Z) = \nabla(X, Y, Z) + \omega(X) \wedge Y + \omega(Y) \wedge X + \omega(Z) \wedge [X, Y]\)

Following the classical theory of decomposition of algebraic curvature tensors, if \(\varpi\) denotes the Kilikian-Nomizu product, we have the following:

Proposition 2: Under the action of the orthogonal group, the \(\nabla\)-curvature tensor decomposes as

\[ R^\nabla = W^\nabla + \frac{1}{n-2} (\nabla^\nabla \cdot \nabla^\nabla) \cdot g + \frac{1}{n} \nabla g + \frac{1}{2} \nabla H \]

where \(\nabla^\nabla\) is the metric connection with skew torsion \(\nabla^\nabla\).

5. \(\nabla\)-Einstein and \(\nabla\)-Einstein

An interesting question is what is the relation between the standard Einstein condition and the Einstein condition with skew torsion. By looking at the formulas for the Ricci tensor, we see that if \((M, g, H)\) is Einstein then \((M, g)\) being Einstein depends only on the algebraic type of the three-form \(H\). Up to dimension 6, we have a classification provided by Schouten which can be easily illustrated by the following graphs.

Theorem 1: If \((M, g, H)\) is Einstein with parallel skew torsion \(\nabla\), then the scalar curvature \(\kappa^\nabla\) is constant.

Theorem 2: Let \(G\) be a Lie group equipped with a bi-invariant metric. Consider the \(1\)-parameter family of connections with skew torsion \(\nabla^G = g[X, Y]\).

Then for \(t = 0\) and \(t = 1\), \(\nabla^G\) is Einstein (it is in flat) for any \(t \neq 0\) and \(t \neq 1\). \(\nabla^G\) is Einstein if and only if \(\nabla^G = \nabla\).

6. Special geometries

Lie groups

Classical examples of manifolds where skew torsion arises naturally are those of Lie groups equipped with a bi-invariant inner product on the corresponding Lie algebra. They provide many examples of Einstein manifolds with skew torsion. Erie Thị kobon

References: