Half-flat structures on $S^3 \times S^3$ and G_2 holonomy

Thomas Madsen King's College London

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joint with **Simon Salamon**

based on contributions of Brandhuber, Chiossi, Hitchin, Schulte-Hengesbach, . . .

Outline

Aim: study explicit metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{G}_2$ on non-compact 7-manifolds:

1. $M^6 = S^3 \times S^3 \qquad \mathrm{SU}(3)$
2. $N^7 = (s, t) \times M \qquad \mathrm{G}_2$

“Die sechs Schwäne”

(Anne Anderson illustration)

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Non-degeneracy of forms

V^6 vector space over $\mathbb{R}$

$\alpha \in \Lambda^k V^*$

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For $k = 2$ notions happen to coincide...

...but situation differs for 3-forms:

If $\phi \in \Lambda^3 V^*$ is non-degenerate then, using the $\mathrm{GL(V)}$-action, it can be normalised to one of the following:

1. $f^{123} + f^{456}$
2. $f^{135} - f^{146} - f^{236} - f^{245}$
3. $f^{156} + f^{264} + f^{345}$

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1. $f^{123} + f^{456}$ stable
2. $f^{135} - f^{146} - f^{236} - f^{245}$ stable
3. $f^{156} + f^{264} + f^{345}$ not stable

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$K_\phi \in \mathrm{End}(V) \otimes \Lambda^6 V^*$ via

$$K_\phi(v) = (v \lrcorner \phi) \wedge \phi \in \Lambda^5 V^* \cong V \otimes \Lambda^6 V^*,$$

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If $\phi \in \Lambda^3 V^*$ is non-degenerate then, using the $\mathrm{GL(V)}$-action, it can be normalised to one of the following:

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$$K_{\phi_0} = \sum_{i=1}^{3} (f_i \otimes f^i - f_{2i} \otimes f^{2i}) \otimes f^{123456}$$

$$\lambda_{\phi_0} = \mathrm{tr}(K_{\phi_0}^2) = 6(f^{123456})^2$$

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If $\phi \in \Lambda^3 V^*$ is non-degenerate then, using the $\mathrm{GL(V)}$-action, it can be normalised to one of the following:

1. $f^{123} + f^{456}$ $\lambda_\phi > 0$
2. $f^{135} - f^{146} - f^{236} - f^{245}$ $\lambda_\phi < 0$
3. $f^{156} + f^{264} + f^{345}$ $\lambda_\phi = 0$

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$$\lambda_\phi = \mathrm{tr}(K_\phi^2)$$

Pairs of "compatible" stable forms

If $(\omega, \phi) \in \Lambda^2 V^* \times \Lambda^3_0 V^*$ is a pair of stable forms then, using the $\mathrm{GL(V)}$-action, this pair can be normalised to one of the following:

1. $\omega = f^{12} + f^{34} + f^{56}$, $\phi = c(f^{135} - f^{146} - f^{236} - f^{245})$
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3. $\omega = f^{14} + f^{25} + f^{36}$, $\phi = c(f^{123} + f^{456})$

where $c > 0$.

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   $\mathrm{SU(3)}$
2. $\omega = f^{12} + f^{34} + f^{56}$, $\phi = c(f^{135} - f^{146} + f^{236} + f^{245})$
   $\mathrm{SU(1,2)}$
3. $\omega = f^{14} + f^{25} + f^{36}$, $\phi = c(f^{123} + f^{456})$
   $\mathrm{SL(3,\mathbb{R})}$

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where $c > 0$.

Interested in pairs (ω, ϕ) with stabiliser $\mathrm{SU}(3)$.
Note: ϕ determines almost complex structure $J_\phi = \frac{K_\phi}{\sqrt{-\lambda_\phi}}$, and we have additional 3-form $\psi = J_\phi(\phi)$.

SU(3) structures on M^6

On smooth M^6 consider (positive) (ω, ϕ) "modelled" pointwise on

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Then we also have J_ϕ, $\psi = J_\phi(\phi)$, and Riemannian metric $h = \omega(\cdot, J_\phi \cdot)$.

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$\mathrm{SU}(3)$ compatibility:

$$\omega \wedge \phi = 0 = \omega \wedge \psi \qquad 3\phi \wedge \psi = 2\omega^3$$

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$\mathrm{SU}(3)$ compatibility:

$$\omega \wedge \phi = 0 = \omega \wedge \psi \qquad 3\phi \wedge \psi = 2\omega^3$$

Also impose “1/2 integrability”, meaning

$$d(\omega^2) = 0 \qquad d\phi = 0 \tag{1}$$

CY: $\mathrm{Hol}(h) \subseteq \mathrm{SU}(3)$ iff $d\phi = 0 = d\psi$ and $d\omega = 0$

(1) means “$21/42 = 1/2$” CY (in terms of intrinsic torsion).

Invariant half-flat structures on $S^3 \times S^3$

Fix $M = S^3 \times S^3$ ($T := T_e M$) and consider

$$T^* = A \oplus B$$

$$A = \langle e^1, e^3, e^5 \rangle \qquad\qquad B = \langle e^2, e^4, e^6 \rangle$$

$de^1 = e^{35}$, $de^2 = e^{46}$ and so forth; with d induced via $[\cdot, \cdot]$ on $\mathfrak{su}(2) \cong \mathfrak{so}(3)$.

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Look for pairs $(\omega, \phi) \in \Lambda^2 T^* \times \Lambda^3 T^*$, ω non-degenerate, such that

$$d(\omega^2) = 0 \qquad d\phi = 0$$
$$\omega \wedge \phi = 0$$

Simplifications

Use

$$\Lambda^2 T^* \cong \Lambda^2 A \oplus (A \otimes B) \oplus \Lambda^2 B \cong \Lambda^4 T^*$$
$$\Lambda^3 T^* \cong \Lambda^3 A \oplus (\Lambda^2 A \otimes B) \oplus (A \otimes \Lambda^2 B) \oplus \Lambda^3 B$$

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So (ϕ, ω) determines $(a, b, Q, P) \in \mathbb{R}^2 \times \mathbb{R}^{3,3} \times \mathbb{R}^{3,3}$. Find

$$\omega \wedge \phi = 0 \quad \Leftrightarrow \quad QP^T,\ P^T Q \text{ symmetric}$$

"...und der Weg war so schwer zu finden."

From $\mathbb{R}^{3,3}$ to $S^2_0(\mathbb{R}^4)$

Local isomorphisms

$$\mathrm{SU}(2)^2 \xrightarrow{2:1} \mathrm{SO}(4) \xrightarrow{2:1} \mathrm{SO}(3)^2$$

reflected in usual splitting of $\Lambda^2 T^*$ on Riemannian 4-manifold

$$\begin{array}{ccccccccc} \Lambda^2(\mathbb{R}^4)^* & = & \Lambda^2_+ & \oplus & \Lambda^2_- & \cong & A & \oplus & B \\ & & f^{12}+f^{34} & & f^{12}-f^{34} & & e^1 & & e^2 \\ & & \cdots & & \cdots & & \cdots & & \cdots \end{array}$$

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From trace-free Ricci tensor $\mathrm{Ric}_0 \in \Lambda^2_+ \otimes \Lambda^2_- \cong A \otimes B$,
recall we have isomorphism $\mathbb{R}^{3,3} \cong S^2_0(\mathbb{R}^4)$

$$P = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \mapsto \begin{pmatrix} -c_{11}-c_{22}-c_{33} & c_{23}-c_{32} & -c_{13}+c_{31} & c_{12}-c_{21} \\ c_{23}-c_{32} & -c_{11}+c_{22}+c_{33} & -c_{12}-c_{21} & -c_{13}-c_{31} \\ -c_{13}+c_{31} & -c_{12}-c_{21} & c_{11}-c_{22}+c_{33} & -c_{23}-c_{32} \\ c_{12}-c_{21} & -c_{13}-c_{31} & -c_{23}-c_{32} & c_{11}+c_{22}-c_{33} \end{pmatrix} = \mathcal{P}$$

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Is this any better?...

...I'd say YES:

$$\underline{\omega \wedge \phi = 0} \quad \Leftrightarrow \quad QP^T,\ P^TQ \text{ symmetric} \Leftrightarrow \underline{[\mathcal{Q}, \mathcal{P}] = 0}.$$

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So (ϕ, ω) determines $(\mathcal{Q}, \mathcal{P}) \in (S^2(\mathbb{R}^4))^2$ with $\mathcal{Q}$, $\mathcal{P}$ commuting matrices.

(Two options: incorporated $a, b \in \mathbb{R}$ as the traces of $\mathcal{Q}$,$\mathcal{P}$, or fix (a, b) and consider $(S^2_0(\mathbb{R}^4))^2$)

Side remark: a dictionary

Both pictures ($\mathbb{R}^{3,3}$ and $S^2_0(\mathbb{R}^4)$) are useful, so we should be able to use them interchangeably.

$\mathbb{R}^{3,3}$	$S^2_0(\mathbb{R}^4)$
C	$\mathcal{C}$
$4\,\mathrm{tr}(CC^T)$	$\mathrm{tr}(\mathcal{C}^2)$
$-2\ \mathrm{Adj}(C^T)$	$(\mathcal{C}^2)_0$
$-24\ \det(C)$	$\mathrm{tr}(\mathcal{C}^3)$
$4\ \mathrm{tr}(CC^T)C$	$\mathrm{tr}(\mathcal{C}^2)\mathcal{C}$
$2\,CC^TC$	$\frac{3}{4}\ \mathrm{tr}(\mathcal{C}^2)\mathcal{C} - (\mathcal{C}^3)_0$
$4\ \mathrm{tr}((CC^T)^2)$	$3\det(\mathcal{C}) + \frac{1}{4}\,\mathrm{tr}(\mathcal{C}^4)$
$2\,\mathrm{tr}(CC^T)^2$	$\det(\mathcal{C}) + \frac{1}{4}\,\mathrm{tr}(\mathcal{C}^4)$
$-24\ \det(C)C$	$\mathrm{tr}(\mathcal{C}^3)\mathcal{C}$
$4\ \mathrm{tr}(CC^T)\,\mathrm{Adj}(C)$	$\frac{1}{3}\,\mathrm{tr}(\mathcal{C}^3)\mathcal{C} - (\mathcal{C}^4)_0$

Conclusion: invariant 1/2 structures on $S^3 \times S^3$

Fix cohomology class $c = (a, b) \in H^3(M, \mathbb{R}) \cong \mathbb{R}^2$, and let $V = S_0^2(\mathbb{R}^4)$.

Theorem

The set $\mathcal{H}_c$ of invariant half-flat structures on M with $[\phi] = c$ can be regarded as a subset of the **commuting variety**

$$\{(\mathcal{Q}, \mathcal{P}) \in V \oplus V \colon [\mathcal{Q}, \mathcal{P}] = 0\}$$

Hamiltonian rewriting

$\mathrm{SO}(4)$ acts Hamiltonian on $V \oplus V$ with moment map

$$\mu = [\cdot,\cdot]\colon\ V \oplus V \to \Lambda^2(\mathbb{R}^4) \cong \mathfrak{so}(4)^*$$
$$(\mathcal{Q},\mathcal{P}) \mapsto [\mathcal{Q},\mathcal{P}]$$

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Corollary

For each fixed cohomology class, the invariant half-flat structures, modulo equivalence relations, form a subset of the symplectic quotient

$$\mu^{-1}(0)/\mathrm{SO}(4) \cong (\mathbb{R}^3 \times \mathbb{R}^3)/S_3$$

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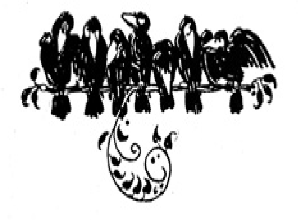
In particular, we may assume $\mathcal{Q},\mathcal{P}$ are diagonal matrices!

"Die sieben Raben"



(Ernst Kutzer illustration)

$$N^7 = (s, t) \times M \qquad \mathrm{G}_2$$

Stable forms in dimension 7

From $(\omega, \phi) \in \Lambda^2 V^* \times \Lambda^3_0 V^*$ with normal forms

$$\omega = f^{12} + f^{34} + f^{56}, \phi = f^{135} - f^{146} - f^{236} - f^{245},$$

we construct 3-form on $W = \mathbb{R} \oplus V$:

$$\begin{aligned}\Phi &= f \wedge \omega + \phi \\ &= f \wedge f^{12} + f(f^{34} + f^{56}) + f^1(f^{35} - f^{46}) - f^2(f^{36} + f^{45})\end{aligned}$$

$(V^0 = \langle f \rangle)$.

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$(V^0 = \langle f \rangle)$. Then

$$\mathrm{GL(W)} \cdot \Phi = \mathrm{G_2} \subset \mathrm{SO(7)}.$$

Induced pos. def. inner product g and orientation, so can define 4-form

$$*\Phi (= \psi \wedge f + \frac{\omega^2}{2})$$

G_2 structures on $N^7 = (s, t) \times M$

On smooth $N^7 = I \times M^6$ consider Φ "modelled" pointwise on

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Determines Riemannian metric g, orientation and then $*\Phi$.

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Torsion-free:

$$\mathrm{Hol}(g) \subseteq \mathrm{G}_2,$$

equivalently (Fernández-Gray),

$$d\Phi = 0 \qquad \text{and} \qquad d{*}\Phi = 0$$

G_2 structures on $N^7 = (s, t) \times M$

On smooth $N^7 = I \times M^6$ consider Φ "modelled" pointwise on

$$f \wedge f^{12} + f(f^{34} + f^{56}) + f^1(f^{35} - f^{46}) - f^2(f^{36} + f^{45}).$$

Determines Riemannian metric g, orientation and then $*\Phi$.

Torsion-free:

$$\mathrm{Hol}(g) \subseteq G_2,$$

equivalently (Fernández-Gray),

$$d\Phi = 0 \qquad \text{and} \qquad d{*}\Phi = 0$$

Now: turn to link btw $1/2$-flat $\mathrm{SU}(3)$ & torsion-free G_2 structures

Recall: Hitchin's description

Let $V = \Omega^3_{exact}(M)$ then $V^* \cong \Omega^4_{exact}(M)$ via pairing

$$\langle \alpha, \beta \rangle = \int_M A \wedge \beta = -\int_M \alpha \wedge B,$$

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On symplectic space $V \times V^*$ consider the functional

$$H = \left(\frac{1}{2} \int J_\alpha(\alpha) \wedge \alpha \right) - \left(\frac{1}{3} \int \omega^3 \right) \qquad \lambda_\alpha < 0,\ \beta = \frac{\omega^2}{2}$$

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Actually, H works more generally: fix $([\alpha], [\beta]) \in H^3(M) \times H^4(M)$
$\Rightarrow$ affine space modelled on $V \times V^*$.

Hitchin's Hamiltonian flow

- Let (ω, ϕ) be a (normalised) half-flat $\mathrm{SU}(3)$ structure; in particular, $([\phi], [\frac{1}{2}\omega^2]) \in H^3(M) \times H^4(M)$.

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Note: Remarkably normalisation and the condition $\omega \wedge \phi = 0$ are preserved! Evolution equations for Hamiltonian flow are of the form

$$\begin{cases} \phi' = \widehat{d}\omega \\ (\frac{\omega^2}{2})' = -\widehat{d}\psi \end{cases}$$

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Indeed, we are eventually let to investigate certain flow equations on

$$\mu^{-1}(0)/\mathrm{SO}(4) \cong (\mathbb{R}^3 \times \mathbb{R}^3)/S_3.$$

Flow equations: first class of solutions with $a + b = 0$

Take matrices of the form

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Focus on solutions of form:

$$\mathcal{Q} = q\,\mathrm{diag}(-3, 1, 1, 1) + a\mathrm{I} \qquad \mathcal{P} = p\,\mathrm{diag}(-3, 1, 1, 1)$$

so that flow equations reduce to

$$\begin{cases} q' = p \\ q'q'' = \frac{-q(q+a)^2}{\sqrt{(3q-a)(q+a)^3}} \end{cases}$$

Alternatively, consider the Hamiltonian

$$H = \frac{1}{3}\left(\sqrt{-\det(\mathcal{Q})} - \frac{1}{12}\operatorname{tr}(\mathcal{P}^3)\right)$$

in (q, r), $r = p^2$ and subject to $H = 0$. Hamilton's equations read

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Proposition

Solution is given by

$$\begin{cases} q(\tau) = \frac{1}{3}(4\tau^3 + a) \\ r(\tau) = \frac{4}{3}\tau^4(1 + a\tau^{-3}) \end{cases}$$

where $s = -\int \sqrt{\frac{12}{1+a\tau^{-3}}} d\tau$.

Associated holonomy G_2 metrics

If parameter

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2. $a \neq 0$ then we have Asymptotically Conical G_2-holonomy metric

$$g = \frac{12 d\tau^2}{1 + a\tau^{-3}} + \tau^2 \sum_{i=1}^{3} (e^{2i-1} - e^{2i})^2 + \frac{\tau^2(1 + a\tau^{-3})}{3} \sum_{i=1}^{3} (e^{2i-1} + e^{2i})^2$$

Flow equations: $a + b = 0$ but slightly more advanced

(Mark Haskins' terminology)

Consider $\mathcal{Q} = \mathrm{diag}(-2q_1 - q_2, q_2, q_2, 2q_1 - q_2) + aI$ and $\mathcal{P} = \mathrm{diag}(-2p_1 - p_2, p_2, p_2, 2p_1 - p_2)$ then system

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reads

$$\begin{cases} q_i' = p_i \\ (p_1^2)' = \frac{-2(q_2+a)(2q_1^2+aq_2-q_2^2)}{\sqrt{-\det(\mathcal{Q})}} \\ (p_1p_2)' = \frac{-2q_1(q_2+a)^2}{\sqrt{-\det(\mathcal{Q})}} \\ -\det(\mathcal{Q}) = (2q_1+q_2-a)(2q_1-q_2+a)(a+q_2)^2 \end{cases}$$

If, say, $a = 1$ one finds:

Proposition

Solution is given by

$$\begin{cases} q_1(\tau) = \frac{\tau^3 - 3\tau}{18},\ q_2(\tau) = 1 - \frac{2}{9}\tau^2 \\ p_1(\tau) = \frac{\tau^2 - 1}{6}\sqrt{\frac{\tau^2 - 9}{\tau^2 - 1}},\ p_2(\tau) = -\frac{4\tau}{9}\sqrt{\frac{\tau^2 - 9}{\tau^2 - 1}} \end{cases}$$

where $\tau = \tau(s)$ *satisfies* $s = \int \sqrt{\frac{\tau^2 - 1}{\tau^2 - 9}}\, d\tau$.

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Get Asymptotically circle Bundle over Cone G_2-holonomy metric

$$\begin{aligned} g = {} & \frac{\tau^2 - 1}{\tau^2 - 9} d\tau^2 + 4\frac{\tau^2 - 9}{\tau^2 - 1}(e^5 + e^6)^2 \\ & + \frac{(\tau+1)(\tau-3)}{12}((e^1 + e^2)^2 + (e^3 + e^4)^2) \\ & + \frac{\tau^2}{9}(e^5 - e^6)^2 + \frac{(\tau-1)(\tau+3)}{12}((e^1 - e^2)^2 + (e^3 - e^4)^2) \end{aligned}$$

“Nun ging es immerzu, weit weit, bis an der Welt Ende...”

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