Gradient Ricci Solitons of Cohomogeneity One

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Rauischholzhausen Workshop, July 4, 2012
1. Basic Definitions and Facts

*Ricci soliton:* special solution of

Ricci flow equation $\frac{\partial g}{\partial t} = -2 \text{Ric}(g)$

of form $g(t) = \lambda(t) \phi_t^*(g_0)$ where

$\phi_t$ is a 1-parameter family of diffeomorphisms with $\phi(0) = \text{id}_M$

$\lambda(t)$ smooth function with $\lambda(0) = 1$ (scale change)

“Static” *Ricci soliton equation* for pair $(g, X)$ on manifold $M$:

$$\text{Ric}(g) + \frac{1}{2} \mathcal{L}_X g + \frac{\epsilon}{2} g = 0$$

where $g$ is a *complete metric,*

$X$ is a vector field

(necessarily complete, Z-H Zhang 2009)

$\epsilon = -\frac{\Lambda}{2}$ is a real constant
\( \epsilon > 0 \) expanding soliton \( (\Lambda < 0) \)

\( \epsilon = 0 \) steady soliton \( (\Lambda = 0) \)

\( \epsilon < 0 \) shrinking soliton \( (\Lambda > 0) \)

\( X \) Killing \( \implies \) \( g \) Einstein ("trivial" solitons)

In Einstein case, \( \Lambda = -\frac{\epsilon}{2} \approx \) Einstein constant.

gradient Ricci soliton: special solution where

\[ X^b = du \]

\( u : M \rightarrow \mathbb{R} \) (soliton potential)

static equation becomes

\[ \text{Ric}(g) + \text{Hess}_g(u) + \frac{\epsilon}{2} g = 0 \]

[Petersen-Wylie] \( g \) Einstein \( \implies \)

Gaussian or \( du \) parallel
3. Cohomogeneity One GRS Equations

Assume compact Lie group $G$ acts isometrically on manifold $M^{n+1}$ with

- orbit space an interval $I$ (closed or half-open)
- generic (principal) orbit type $G/K$
- singular orbits $G/H_i$ with $H_i/K \approx S^{k_i}$

Write metric as $\bar{g} = dt^2 + g_t$ where $g_t$ : a curve of $G$-invariant metrics on $P := G/K$

GRS equations become the system:

\[ - (\delta \nabla^t L_t)^b - d(\text{tr} L_t) = 0 \] (1)

\[ - \text{tr}(\dot{L}_t) - \text{tr}(L_t^2) + \ddot{u} + \frac{\epsilon}{2} = 0 \] (2)

\[ \text{ric}_t - \text{tr}(L_t) L_t - \dot{L}_t + \ddot{u} L_t + \frac{\epsilon}{2} I = 0 \] (3)
where

- $L_t$ is the shape operator of hypersurface $P_t$
- $\delta^{\nabla^t} : T^*(P) \otimes TP \to TP$ codifferential,
- $\text{ric}_t$ is the Ricci operator of $P_t$ defined by

$$\text{Ric}(g_t)(X, Y) = g_t(\text{ric}_t(X), Y)$$

Plus appropriate boundary conditions at endpoints of $I$ to guarantee smoothness and completeness.
**Conservation Law:** two formulations

\[ \ddot{u} + (-\dot{u} + \text{tr} L) \dot{u} = C + \epsilon u \quad (\text{R. Hamilton}) \quad (4) \]

\[ \Leftrightarrow S_t + \text{tr}(L^2) - (-\dot{u} + \text{tr}(L))^2 + (n-1) \frac{\epsilon}{2} = C + \epsilon u \]

**Useful Fact:** (A. Back for Einstein case)

smoothness (e.g. $C^3$) of $\bar{g}, u$ + Eq. (3) +

\[ \text{codim}(G/H) \geq 2 \implies \text{Eq. (1)} \]

above + conservation law $\implies$ Eq. (2)

**Hamiltonian Formulation:**

\[ \mathcal{C} = S^2_+ (p)^K \times \mathbb{R} \]

On $T^* \mathcal{C}$ (with canonical symplectic structure)

take Hamiltonian function
\[ \mathcal{H} = v(q) e^{-u} \left( 2\langle L, L \rangle + \dot{u}^2 - 2\dot{u} \text{tr}L \right) + E - \epsilon(n + 1 - u) - S(q) \]

(from Perelman's \( W \)-functional)

\( v(q) \) relative volume, \( E \) Lagrangian multiplier

KE has Lorentz signature

Then integral curves in \( \{ \mathcal{H} = 0 \} \) are equivalent to solutions of Eq. (2) and (3).

**Initial Value Problem at Singular Orbit:**

existence of a local solution (arbitrary \( \epsilon \)) in a \( G \)-invariant nbd of singular orbit \( G/H \) with *prescribed metric and shape operator* on \( G/H \)

M. Buzano (JGP 2011)

under *assumption*: at special orbit \( G/H \), the slice rep. and the isotropy rep. as \( K \)-reps, have no common irreducible summands
4. Non-existence Result ([DHW], after Böhm)

Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (Ad$_K$-invariant decomposition)

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r \quad (5)$$

where $\mathfrak{p}_i$ is the sum of all equivalent Ad$_K$-irreducible summands of a fixed type. This decomposition is unique up to permutation of summands.

Special orbits $G/H_i$: $h_i = s_i \oplus \mathfrak{k}$, $\mathfrak{p} = s_i \oplus q_i$

**Theorem 1.** Let $M$ be a closed cohomogeneity one $G$-manifold as described above. Assume that some summand $\mathfrak{p}_{i_0}$ in (5) is actually Ad$_K$-irreducible and that for any $G$-invariant metric on $G/K$, the restriction to $\mathfrak{p}_{i_0}$ of its traceless Ricci tensor is always negative definite. Assume further that $\mathfrak{p}_{i_0} \cap s_j = \{0\}$ for $j = 1, 2$.

Then there cannot be any $G$-invariant gradient Ricci soliton structure on $\overline{M}$.
**Sketch of Proof:**

Consider $\tilde{g} = v^{-2/n} g$, where $v := \sqrt{\det g_t}$

Set $F_i := \frac{1}{2} \text{tr}_i(\tilde{g}^2)$. Then one computes that

$$\ddot{F}_i + \dot{\xi} \dot{F}_i = \text{tr}_i(\tilde{g}^2) + \text{tr}_i(\tilde{g}\tilde{g}^{-1}\tilde{g}) + 2 \text{tr}_i(\tilde{g}^2 r(0))$$

Pick $i_0$.

At the singular orbits, $F_{i_0}$ tend to $+\infty$. So $F_{i_0}$ has an interior minimum.

There, $\dot{F}_{i_0} = 0$, while $\ddot{F}_{i_0} \geq 0$. $\square$

**Explicit example (C. Böhm):**

$$S^{k+1} \times (G'/K') \times M_3 \times \cdots \times M_r$$

with $M_i$ compact isotropy irreducible and

$$G'/K' = \text{SU}(\ell + m)/(\text{SO}(\ell) \cdot \text{U}(1) \cdot \text{U}(m)),$$

$$\ell \geq 32, \quad m = 1, 2, \quad k = 1, 2, \cdots, [\ell/3]$$
Complete, non-compact, non-trivial GRS

I. Steady Case ($\epsilon = 0$)

can apply and/or sharpen results of B. L. Chen ($\bar{R} > 0$), Munteanu-Sesum, Peng Wu, ...

**Proposition 2.** For a complete, non-compact, non-trivial steady GRS of cohomogeneity one:

(a) $u$ is strictly decreasing and concave (as function of $t$) with $\ddot{u}(0) = C/(k + 1) < 0$

(note: no curvature assumptions)

(b) $\text{tr}L$ is strictly decreasing; $0 < \text{tr}L \leq \frac{n}{t}$.

(c) generalized mean curvature $\xi := -\dot{u} + \text{tr}L$ is strictly decreasing with asymptotic limit $\sqrt{-C}$. Hence $C \xi^{-2}$ is a general Lyapunov function.

(d) ambient scalar curvature is strictly decreasing with asymptotic limit 0 (since $\bar{R} + \dot{u}^2 = -C$).

(e) quantity $\mathcal{F} := \frac{2}{v^n}(S + \text{tr}(L_0)^2)$ is non-increasing on any trajectory corresponding to a non-trivial soliton (Lyapunov function)
**Example 1.** [DW2009] $M = \mathbb{R}^{d_1+1} \times M_2 \times \cdots \times M_r$

$M_1 = S^{d_1}$, $d_1 > 1$, equipped with the constant curvature 1 metric $h_1$

$(M_i, h_i)$, $2 \leq i \leq r$ Einstein with Einstein constants $\lambda_i > 0$ and dimension $d_i$.

$\exists r-1$ parameter family of non-trivial steady GRS structures with $\bar{g} = dt^2 + g_1(t)^2 h_1 + \cdots + g_r(t)^2 h_r$, $\text{Ric}(\bar{g}) \geq 0$ (positive off the zero section )

**Remarks:**

(i) generally non-Kähler; generally not locally conformally flat if $r \geq 2$

(ii) $r = 1$: Bryant solitons on $\mathbb{R}^n$, $n \geq 3$

($n = 2$ is Hamilton's cigar, which is Kähler)

These have positive sectional curvature.
(iii) \( r = 2 \) Ivey’s generalization of Bryant solitons, PAMS (1994)

(iv) asymptotics: \( g_i \sim \sqrt{t}, \tr L \sim \frac{n}{2t} + O(t^{-2}) \) and \( u(t) \sim -\sqrt{-C}t + \frac{n}{4} \log t. \)

(iv) C. Böhm (1999): \( r - 2 \) parameter family of complete Ricci-flat metrics \((C=0)\); asymptotically Euclidean

**Example 2.** [DZ Chen 2010]

\( M = S^1 \times L_q \) where \( L_q \) is the complex line bundle over a Fano KE manifold with \( |q| \) the first Chern number.

\exists a 3-parameter family of “explicit” steady soliton solutions (modulo homothety)

Hypersurfaces are \( T^2 \) bundles over Fano with connection metric

Metric on \( T^2 = S^1 \times S^1 \) is not “diagonal”

Asymptotically, metric components \( \sim t \) (paraboloidal)
II. Expanding Case: \((\epsilon > 0)\) Set

\[
\xi := -\dot{u} + \text{tr}L \quad \text{(generalized mean curvature)}
\]

and \(\mathcal{E} = C + \epsilon u\).

Conservation law becomes: \(\ddot{\mathcal{E}} + \xi \dot{\mathcal{E}} - \epsilon \mathcal{E} = 0\).

Its derivative yields for \(y = \dot{u}\):

\[
\ddot{y} + \xi \dot{y} - \left(\frac{\epsilon}{2} + \text{tr}(L^2)\right)y = 0
\]

can apply and/or sharpen results of B.L. Chen \((\bar{R} + \frac{\epsilon}{2}(n+1) > 0)\), Shijin Zhang, Zhuhong Zhang, Carillo-Ni, Munteanu-Sesum, Pigola-Rimoldi-Setti,....

**Proposition 3.** For a non-trivial complete expanding GRS with \(u(0) = 0\)

(a) \(u\) is strictly decreasing and strictly concave; \(\ddot{u}(0) = C/(k+1) < 0\)

(b) volume grows at least logarithmically

(c) \(\exists t_1 > 0\) such that \(-\sqrt{\frac{\epsilon}{2}n} < \text{tr}L < \sqrt{\frac{\epsilon}{2}n}\) for \(t \geq t_1\)
Proposition 4. (gradient bound) \( \exists t_1 > 0 \) and \( a > 0 \) such that for \( t \geq t_1 \)

(a) \( \frac{9}{10} \left( \frac{-\dot{u}(t_1)}{\frac{\epsilon}{2} t + a} \right) \left( \frac{\epsilon}{2} t + a \right) < |\nabla u| < \frac{\epsilon}{2} t + \sqrt{-C} \)

Hence \( u \) is asymptotically bounded above and below by quadratics.

(b) \( \lim_{t \to +\infty} \xi = +\infty \)

(c) For \( t \) large, the quantity \( \mathcal{F} := \frac{\epsilon}{n} (S + \text{tr}(L_0)^2) \) is strictly decreasing on any trajectory in velocity phase space except when \( L_0 \) vanishes

(d) \( \ddot{u} + \frac{\epsilon}{2} \leq -\text{Ric}_g \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) \leq \frac{\epsilon}{2} \left( 1 + \frac{9}{10} \left( \frac{-\dot{u}(t_1)}{\frac{\epsilon}{2} t + a} \right) \left( \frac{\epsilon}{2} t + a \right) \right) \)

provided \( t \geq t_1 \).

(e) If \( \ddot{u} + \epsilon/2 \leq 0 \) for \( t \geq t_0 \), then \( \text{tr}L \) is strictly decreasing, \( 0 < \text{tr}L < n/t \) and \( \bar{R} > -\frac{\epsilon}{2} n \).
Example 3 [DW2009]

On the same manifolds as in Example 1, there is an $r$-parameter family of non-trivial (generally non-Kähler, non-locally conformally flat) expanding GRS structures

Remarks:

(i) $r = 1$ Ivey in [Chow et al] ;

(ii) $r = 2$ Gastel and Kronz (2004)

(iii) Böhm (1999): $r - 1$ parameter family of complete negative Einstein metrics on these manifolds

Einstein metric components grow exponentially with $t$, mean curvature asymptotically constant $\sim \sqrt{n\epsilon/2}$.

(iv) solitons are asymptotically conical and satisfy $\ddot{u} + \epsilon/2 \leq 0$ for $t \geq t_0$; also, ambient scalar curvature tends to 0 and $\xi \sim \frac{\epsilon}{2} t$. 
4. A General Winding Number for Shrinkers

Recall $\xi = -\dot{u} + \text{tr} L$ (generalized mean curvature for measure $e^{-u}d\mu_{\bar{g}}$)

Eq. (2) $\implies \xi$ is strictly decreasing from $+\infty$ to $-\infty$ in all cases (unique zero)

Let $\mathcal{E} := C + \epsilon u$ and $\mathcal{F} := \dot{u}$.

Recall Conservation law (4) in the form

$$\ddot{\mathcal{E}} + \xi \dot{\mathcal{E}} - \epsilon \mathcal{E} = 0$$

Let $ds := \xi dt$ and $'$ denote differentiation wrt $s$.

Note: insert $-1$ for change of variables after unique zero of $\xi$.

We now have (with $W := \xi^{-1}$)

$$\mathcal{E}' = \epsilon W \mathcal{F}$$

$$\mathcal{F}' = W \mathcal{E} - \mathcal{F}$$
Theorem 5. ([DHW 2011]) For trajectories of the flow of $(\mathcal{F}, \mathcal{E})$ starting from either the positive or negative $\mathcal{E}$ axis, the winding number about the origin up to the (unique) turning point is finite, non-positive and bounded from below by $-(6 + \frac{\pi}{4})$.

Remark: The origin corresponds to Einstein trajectories.
Some General Facts about Shrinking GRS:

(a) $\bar{R} \geq 0$. It is positive unless the soliton metric is flat. (B. L. Chen without sectional curvature bounds, Pigoli-Rimoldi-Setti for rigidity)

(b) Quadratic bound for soliton potential in complete, non-compact case

[H. D. Cao-D. Zhou 2010]

$$-rac{\epsilon}{2}(n + 1) + \frac{\epsilon}{4}(t\sqrt{-\epsilon} + c_2)^2 \leq \epsilon(t) = C + \epsilon u(t)$$

$$\leq -\frac{\epsilon}{2}(n + 1) + \frac{\epsilon}{4}(t\sqrt{-\epsilon} - c_1)^2$$

Note: $c_i$ depend only on $n + 1 = \dim M$. (Haslhofer-Müller 2011)

Ambient scalar curvature

$$\bar{R} = -2 \text{tr}(\dot{L}) - \text{tr}(L^2) - (\text{tr}L)^2 + S$$

$$= -\epsilon - \frac{\epsilon^2}{\epsilon^2} - \frac{\epsilon}{2}(n + 1)$$
So General Fact (a) implies in non-flat cases

$$\mathcal{E} < -\frac{\varepsilon}{2}(n + 1), \text{ and}$$

$$\ddot{u}(0) < -\frac{\varepsilon}{2}\left(\frac{n+1}{k+1}\right)$$

**Theorem 6.** [DHW 2011]

Let \((M, \bar{g}, u)\) be a non-trivial complete shrinking GRS of cohomogeneity one with invariant soliton potential and orbit space \(I\). Then, regarding \(\mathcal{E}\) as a function of \(t\):

(i) \(\mathcal{E} = C + \epsilon u\) must change sign and is a Morse-Bott function on \(M\).

(ii) If \(\bar{g}\) is nonflat, then \(\mathcal{E} < -\frac{\varepsilon}{2}(n + 1)\).

(iii) If \(M\) is compact, \(\mathcal{E}\) has at most 4 critical points in \(\text{int} I\). As a function of \(t\), \(\mathcal{E}\) is either a local max (where \(\mathcal{E} > 0\)) or a local min (where \(\mathcal{E} < 0\)).

(iv) If \(M\) is complete, noncompact, \(\mathcal{E}\) has at most 5 critical points in \(\text{int} I\).

*Remark:* In known examples, \(\mathcal{E}\) is monotone decreasing. But these are all Kähler.
Theorem rules out

\textbf{Example 1} smooth Gaussian (rigid in Petersen-Wylie sense)

\[ M = \mathbb{R}^{d_1+1} \times M_2 \times \cdots \times M_r \]

\( \mathbb{R}^{d_1+1} \) Euclidean, \( M_i \) positive Einstein \( i > 1 \)

\[ u(t) = -\frac{\epsilon}{4} t^2, \quad \text{tr} L = \frac{d_1}{t}, \quad \bar{R} = -\frac{\epsilon}{2} (n - d_1) \]
Example 2 [FIK 2003], [DW 2008]

\((V_i, J_i, h_i), 1 \leq i \leq r, r \geq 2\), Fano KE manifolds with complex dimension \(n_i\) and \(c_1(V_i) = p_i a_i\) where \(p_i > 0\) and \(a_i\) are indivisible classes in \(H^2(V_i, \mathbb{Z})\)

\(V_1 = \mathbb{CP}^{n_1}, n_1 \geq 0\), with normalised Fubini-Study metric

\(P_q : \) principal \(S^1\) bundle over \(V_1 \times \cdots \times V_r\) with Euler class \(-\pi_1^*(a_1) + \sum_{i=2}^{r} q_i \pi_i^*(a_i)\), i.e., \(q_1 = -1\).

Assume \(0 < - (n_1 + 1) q_i < p_i\) for all \(2 \leq i \leq r\).

Then there is a complete shrinking GKRS structure on the space \(\overline{M}\) obtained from the line bundle \(P_q \times_{S^1} \mathbb{C}\) by blowing the zero section down to \(V_2 \times \cdots \times V_r\).

soliton metric has an asymptotically conical end
Remarks: (a) Feldman-Illmanen-Knopf considered case with $r = 2, n_1 = 0$ and $V_2$ to be a complex projective space.

(b) The case $r = 2, n_2 = 0$ corresponds to flat $\mathbb{C}^{n_1+1}$ as a shrinking soliton.

(c) Also: Bo Yang (2008), A.Futaki-M.T.Wang (2010), Chi Li (2010)

(d) There is a version of theorem where the base is a coadjoint orbit and the principal orbits are suitable circle bundles over it.

(e) The condition $\ddot{u} + \frac{\epsilon}{2} > 0$ holds except in flat case.

**Proposition 7.** Assume $\ddot{u} \leq -\frac{\epsilon}{2}$ on some $[a, +\infty)$, $a > 0$.

- from some $t_0 \geq a$ on, $\text{tr}L$ is decreasing and $0 < \text{tr}L < (\frac{t}{n} + c(t_0))^{-1}$ and

- ambient scalar curvature $< -\frac{\epsilon}{2n}$. 

21
Numerical Search: [DHW]

negative search results in compact cases

(i) $S^5$ with $\text{SO}(3) \times \text{SO}(3)$ action

(ii) $S^2 \times S^3$ with $\text{SO}(3) \times \text{SO}(3)$ action

(iii) $S^{11}$ with $\text{SO}(6) \times \text{SO}(6)$ action

(iv) $\mathbb{HP}^{n+1} \# \mathbb{HP}^{n+1}$ with $\text{Sp}(1) \times \text{Sp}(n+1)$ action; connected sum of Cayley projective planes

(v) $\mathbb{R}^3$ bundle over $\mathbb{HP}^n$ with $G = \text{Sp}(n+1)$; principal orbit is twistor fibration over $\mathbb{HP}^n$

(vi) non-trivial sphere bundles over $S^2$ (Hashimoto-Sakaguchi-Yasui): principal orbit $S^3 \times S^{d-2}$