On the diagonalization of the Ricci flow on Lie groups

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1. Definitions
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Definitions

Space of Lie algebras of dimension $n \leftrightarrow V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \} \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.

Fix basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$) $\leftrightarrow \mu$. $\mu \leftrightarrow$ Lie group $N_\mu$: Simply connected Lie $(N_\mu) = (\mathbb{R}^n, \mu)$ endowed with left invariant metric defined by $\langle \cdot, \cdot \rangle$.

$GL_n(\mathbb{R})$ acts on $V$ by change of basis: $A \cdot \mu(X, Y) = A \mu(A^{-1}X, A^{-1}Y)$, $X, Y \in \mathbb{R}^n$, $A \in GL_n(\mathbb{R})$, $\mu \in V$.

Geometrically: $A \in GL_n(\mathbb{R}) \leftrightarrow a$ Riemannian isometry $(N_A \cdot \mu, \langle \cdot, \cdot \rangle) \rightarrow (N \cdot \mu, \langle A \cdot \mu, A \cdot \mu \rangle)$ by exponentiating $A^{-1}$: $(\mathbb{R}^n, A \cdot \mu) \rightarrow (\mathbb{R}^n, \mu)$. 


Definitions

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\[ V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}. \]
Definitions

Space of Lie algebras of dimension $n$ \( \iff \)

\[ V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}. \]

\[ \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \]
Definitions

Space of Lie algebras of dimension $n$ \( \sim \)

\[
V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}. \\
\subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n
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Fix basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$)
Definitions

Space of Lie algebras of dimension $n \iff$

$$V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}. \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

Fix basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$) $\iff$
Definitions

Space of Lie algebras of dimension $n \leftrightarrow \sim \\
\mathcal{V} = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}.
\subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \\\nFix \text{ basis of } \mathbb{R}^n \ (\text{and } \langle \cdot, \cdot \rangle) \sim n \leftrightarrow \mu.
Definitions

Space of Lie algebras of dimension $n \leftrightarrow \mathcal{V}$

$$\mathcal{V} = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}. $$

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Fix basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$) $\leftrightarrow n \leftrightarrow \mu$. 

$\mu$
Definitions

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Fix basis of $\mathbb{R}^n$ (and $\langle \cdot , \cdot \rangle$) $\leftrightarrow n \leftrightarrow \mu$. 

$\mu \leftrightarrow$
Define $V$ as the space of Lie algebras of dimension $n$:

$$V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \} \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

Fix a basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$) to get $n \leftrightarrow \mu$. Then, $\mu \sim \text{Lie group } N_\mu$.
Definitions

Space of Lie algebras of dimension $n \leftrightarrow$

$$V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}.$$

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Fix basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$) $\leadsto \mathfrak{n} \leftrightarrow \mu$.

$\mu \leadsto$ Lie group $N_\mu$: Simply connected $\text{Lie}(N_\mu) = (\mathbb{R}^n, \mu)$
Space of Lie algebras of dimension $n \mapsto V = \{ \mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n : \mu \text{ bilinear, skew-symmetric, Jacobi} \}. 
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Fix basis of $\mathbb{R}^n$ ( and $\langle \cdot, \cdot \rangle$ ) $\mapsto n \leftrightarrow \mu$.
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Fix basis of $\mathbb{R}^n$ (and $\langle \cdot, \cdot \rangle$) $\rightsquigarrow$ $n \leftrightarrow \mu$.

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Space of Lie algebras of dimension \( n \leftrightarrow \)

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$$(N_{A \cdot \mu}, \langle \cdot , \cdot \rangle) \rightarrow (N_\mu, \langle A \cdot, A \cdot \rangle)$$
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Space of Lie algebras of dimension $n$ \[\iff\]

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\[(N_{A \cdot \mu}, \langle \cdot , \cdot \rangle) \longrightarrow (N_\mu, \langle A \cdot , A \cdot \rangle)\]

by exponentiating $A^{-1} : (\mathbb{R}^n, A \cdot \mu) \longrightarrow (\mathbb{R}^n, \mu)$. 
The action of $\text{GL}_n(R)$ on $V$
The action of $GL_n(\mathbb{R})$ on $V$ ~→

$gl_n(\mathbb{R}) = so(n) \oplus \text{sym}(n)$ Cartan decomposition

$\Delta \subset a$, a system of roots, positive roots:

$$\Phi = \{E_{ll} - E_{mm} \in a, l > m\}.$$
The action of $\text{GL}_n(\mathbb{R})$ on $V \rightsquigarrow$ representation of $\text{gl}_n(\mathbb{R})$ on $V$:
The action of $GL_n(\mathbb{R})$ on $V \rightarrow$ representation of $gl_n(\mathbb{R})$ on $V$:

$$\pi(\alpha)\mu = \alpha \mu(\cdot, \cdot) - \mu(\alpha \cdot, \cdot) - \mu(\cdot, \alpha \cdot), \quad \alpha \in gl_n(\mathbb{R}), \, \mu \in V.$$
The action of $GL_n(\mathbb{R})$ on $V$ ~ representation of $gl_n(\mathbb{R})$ on $V :$

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$gl_n(\mathbb{R}) = so(n) \oplus sym(n) \text{ Cartan decomposition}$
The action of $GL_n(\mathbb{R})$ on $V \sim$ representation of $\mathfrak{gl}_n(\mathbb{R})$ on $V$:

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$\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \text{sym}(n)$ Cartan decomposition $\sim a = \text{diagonal } n \times n$
The action of $GL_n(\mathbb{R})$ on $V \rightsquigarrow$ representation of $gl_n(\mathbb{R})$ on $V :$

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$gl_n(\mathbb{R}) = so(n) \oplus \text{sym}(n)$ Cartan decomposition $\rightsquigarrow a =$ diagonal

$n \times n$ is maximal abelian subalgebra of sym$(n)$
The action of $GL_n(\mathbb{R})$ on $V \sim \pi$ representation of $gl_n(\mathbb{R})$ on $V$:

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$gl_n(\mathbb{R}) = so(n) \oplus \text{sym}(n)$ Cartan decomposition $\sim a = \text{diagonal}$

$n \times n$ is maximal abelian subalgebra of $\text{sym}(n) \sim \Delta \subset a$, a system of roots,
The action of $\text{GL}_n(\mathbb{R})$ on $V$ corresponds to a representation of $\mathfrak{gl}_n(\mathbb{R})$ on $V$:

$$\pi(\alpha)\mu = \alpha \mu(\cdot, \cdot) - \mu(\alpha \cdot, \cdot) - \mu(\cdot, \alpha \cdot), \quad \alpha \in \mathfrak{gl}_n(\mathbb{R}), \ \mu \in V.$$ 

$\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{sym}(n)$ Cartan decomposition corresponds to $\mathfrak{a} = \text{diagonal } n \times n$ is maximal abelian subalgebra of $\mathfrak{sym}(n)$, $\Delta \subset \mathfrak{a}$, a system of roots, positive roots:
The action of $GL_n(\mathbb{R})$ on $V \sim \pi \sim representation of \text{gl}_n(\mathbb{R})$ on $V$:

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$\text{gl}_n(\mathbb{R}) = \text{so}(n) \oplus \text{sym}(n)$ Cartan decomposition $\sim a = \text{diagonal}$ $n \times n$ is maximal abelian subalgebra of $\text{sym}(n) \sim \Delta \subset a$, a system of roots, positive roots:

$$\Phi = \{ E_{ll} - E_{mm} \in a, \, l > m \}.$$
Basis of weight vectors
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\[ \{ v_{ijk} = (e_i' \wedge e_j') \otimes e_k : 1 \leq i < j \leq n, \ 1 \leq k \leq n \} \]
Basis of weight vectors

\[ \{ v_{ijk} = (e'_i \wedge e'_j) \otimes e_k : 1 \leq i < j \leq n, 1 \leq k \leq n \} \]

\[ v_{ijk}(e_i, e_j) = -v_{ijk}(e_j, e_i) = e_k \]
Basis of weight vectors

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Corresponding weights
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Corresponding weights \( \alpha^k_{ij} := E_{kk} - E_{ii} - E_{jj} \)
Basis of weight vectors

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Corresponding weights \( \alpha^k_{ij} := E_{kk} - E_{ii} - E_{jj} : \)

if \( \alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathfrak{a}, \)
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\[ \pi(\alpha)v_{ijk} = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha^k_{ij} \rangle v_{ijk}, \]
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\( \mu \in V \)
Basis of weight vectors

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\[ v_{ijk}(e_i, e_j) = -v_{ijk}(e_j, e_i) = e_k \]

Corresponding weights \( \alpha_{ij}^k := E_{kk} - E_{ii} - E_{jj} : \)

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\[ \pi(\alpha)v_{ijk} = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha_{ij}^k \rangle v_{ijk}, \]

\( \mu \in V \) the structural constants \( c_{ij}^k \) are given by
Basis of weight vectors

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\[ \{ v_{ijk} = (e'_i \wedge e'_j) \otimes e_k : 1 \leq i < j \leq n, 1 \leq k \leq n \} \]

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if \( \alpha = \begin{bmatrix} a_1 \\ \cdot \cdot \\ a_n \end{bmatrix} \in a, \)

\[ \pi(\alpha)v_{ijk} = (a_k - a_i - a_j)v_{ijk} = \langle \alpha, \alpha^k_{ij} \rangle v_{ijk}, \]

\( \mu \in V \) the structural constants \( c^k_{ij} \) are given by

\[ [e_i, e_j] = \sum_k c^k_{ij} e_k, \quad \text{or} \quad [\cdot, \cdot] = \sum_{k; i<j} c^k_{ij} v_{ijk}. \]
Ricci Flow

Let $g(t)$ be a solution to the Ricci flow
$$\frac{\partial}{\partial t} g(t) = -2 \text{Rc}(g(t)).$$
Ricci Flow

\((N, g_0)\) a Lie group with a left-invariant metric
(N, g₀) a Lie group with a left-invariant metric
Ricci Flow

$(N, g_0)$ a Lie group with a left-invariant metric $\leftrightarrow$ metric Lie algebra $(n, \langle \cdot, \cdot \rangle_0)$. 
Ricci Flow

$(N, g_0)$ a Lie group with a left-invariant metric $\leftrightarrow$ metric Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle_0)$.

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Ricci Flow

$(N, g_0)$ a Lie group with a left-invariant metric $\leftrightarrow$ metric Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle_0)$.

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(N, g_0) a Lie group with a left-invariant metric \( \xmapsto{} \) metric Lie algebra \( (\mathfrak{n}, \langle \cdot, \cdot \rangle_0) \).
Let \( g(t) \) be a solution to the Ricci flow

\[
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\]

ODE for Lie groups.
Rc of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is given by
Rc of \((n, \langle \cdot, \cdot \rangle)\) is given by

\[
Rc = M - \frac{1}{2} B - S(\text{ad } H),
\]  

(1)
Rc of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is given by

$$
Rc = M - \frac{1}{2}B - S(\text{ad } H),
$$

(1)

Where

$B$ = Killing form,

$S(\text{ad } H) = \frac{1}{2}(\text{ad } H + (\text{ad } H)^t)$,

$H \in \mathfrak{n}$:

$\langle H, X \rangle = \text{tr } \text{ad } X$ for any $X \in \mathfrak{n}$,

$M(X, Y) = -\frac{1}{2} \sum \langle [X, X_i], X_j \rangle \langle [Y, X_i], X_j \rangle + \frac{1}{4} \sum \langle [X_i, X_j], X \rangle \langle [X_i, X_j], Y \rangle$. 

Rc of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is given by

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Rc of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is given by

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Rc of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is given by

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Rc of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is given by

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\[
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\(H \in \mathfrak{n} :\)
Rc of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is given by

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\[B = \text{Killing form}, \quad S(\text{ad} H) = \frac{1}{2}(\text{ad} H + (\text{ad} H)^t),\]

\[H \in \mathfrak{n}: \langle H, X \rangle = \text{tr ad} X \text{ for any } X \in \mathfrak{n},\]
Rc of \((\mathfrak{n}, \langle \cdot, \cdot \rangle)\) is given by

\[
\text{Rc} = M - \frac{1}{2}B - S(\text{ad } H),
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M(X, Y) = -\frac{1}{2} \sum \langle [X, X_i], X_j \rangle \langle [Y, X_i], X_j \rangle + \frac{1}{4} \sum \langle [X_i, X_j], X \rangle \langle [X_i, X_j], Y \rangle.
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If $\mathfrak{n}$ nilpotent $\leadsto \text{Rc} = M$.

$$\langle \text{Ric}_\mu, \alpha \rangle = 4\langle \pi(\alpha)\mu, \mu \rangle, \quad \forall \alpha \in \text{sym}(n).$$
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\[ \Rightarrow \] simplify the study.
Nice basis

Let $n$ be a nilpotent Lie algebra. A basis $\{X_1,...,X_n\}$ of $n$ is called nice if the structural constants given by $[X_i, X_j] = \sum c_{kj}^i X_k$ satisfy

- for all $i, j$ there exists at most one $k$ such that $c_{kj}^i \neq 0$, 
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Example: $n = (\mathbb{R}^4, [\cdot, \cdot])$ 
$\{X_1, ..., X_4\} \quad [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4$.

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Let $n = (\mathbb{R}^6, [\cdot, \cdot])$ where $[X_1, X_2] = X_4, \quad [X_1, X_4] = X_5, \quad [X_1, X_5] = [X_2, X_3] = [X_2, X_4] = X_6$.

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Nikolayevsky: simple criterium to decide whether a given nilpotent Lie algebra with a nice basis admits a nilsoliton or not.
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The free 3-step nilpotent Lie algebra in 3 generators
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Infinitely many 2-step nilpotent Lie algebras with type $(p, q)$ such that

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Theorem

A basis of a nilpotent Lie algebra is stably Ricci-diagonal if and only if it is nice.
Lemma

The canonical basis \(\{e_1, \ldots, e_n\}\) is nice for \(n\) if and only if

\[
\alpha^k_{ij} - \alpha^t_{rs} \notin \Phi, \quad \text{for any} \quad c^k_{ij}, c^t_{rs} \neq 0.
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\( \rightsquigarrow \) Generalization.
Theorem

For a nilpotent Lie algebra $\mathfrak{n}$, the following conditions are equivalent:

(i) The canonical basis $\{e_1, \ldots, e_n\}$ is nice for $\mathfrak{n}$.

(ii) $\langle \pi(X) v_{ijk}, \nu_{rst} \rangle = 0$, for all $X \in \mathfrak{g}$, $\lambda \in \Phi$, $c_{ij} \neq 0$.

(iii) $\text{Ric} A \cdot \mu(e_l, e_m) = 0$ for all $l \neq m$ and any diagonal $A \in \text{GL}_n(\mathbb{R})$.

(iv) The canonical basis $\{e_1, \ldots, e_n\}$ is stably Ricci-diagonal for $\mathfrak{n}$.
**Theorem**

*For a nilpotent Lie algebra \( \mathfrak{n} \), the following conditions are equivalent:*

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(i) ⇔ (ii) If \( \lambda \in \Phi, X \in g_{\lambda}, \alpha \in \mathfrak{a} \):

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(iii) ⇒ (ii) uses: $\pi$ is multiplicity free and special properties of the weights ($\alpha^k_{ij}$).
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(i) $\iff$ (ii) If $\lambda \in \Phi$, $X \in g_\lambda$, $\alpha \in a$:

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Nice but

\[ \text{Ric} = \begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}. \]

(for the metric which makes it orthonormal)
Stably Ricci diagonal \( \not\Rightarrow \) nice:

\[ X_1, X_3 = X_2 + X_3. \]

The basis \( \{X_1, X_2, X_3\} \) is not nice but it is stably-Ricci diagonal:

for every \( \langle \cdot, \cdot \rangle \)

\[
\begin{align*}
\langle X_i, X_i \rangle &= a_{2i}, \\
\langle X_i, X_j \rangle &= 0 \quad \forall i \neq j,
\end{align*}
\]

\[
\text{Ric}(X_r, X_s) = \frac{1}{4} \sum \langle [X_i, X_j], X_r \rangle \langle [X_i, X_j], X_s \rangle - \frac{1}{2} \langle [H, X_r], X_s \rangle - \frac{1}{2} \langle X_r, [H, X_s] \rangle.
\]

Vanishes if either \( r \) or \( s \) is 1, and

\[
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\]
Stably Ricci diagonal $\not\Rightarrow$ nice:

$\mathfrak{s}_3$ be the 3-dimensional Lie algebra defined by

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\]

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for every \[\langle \cdot, \cdot \rangle \] \[\langle \mathfrak{s}_i, \mathfrak{s}_i \rangle = a_{2i}, \quad a_i > 0, \]

\[\langle \mathfrak{s}_i, \mathfrak{s}_j \rangle = 0 \quad \forall \ i \neq j, \]

\[\text{Ric}(\mathfrak{s}_r, \mathfrak{s}_s) = \frac{1}{4} \sum \langle [\mathfrak{s}_1 a_i \mathfrak{s}_i, 1 a_j \mathfrak{s}_j], \mathfrak{s}_r \rangle \langle [\mathfrak{s}_1 a_i \mathfrak{s}_i, 1 a_j \mathfrak{s}_j], \mathfrak{s}_s \rangle - \frac{1}{2} \langle [\mathfrak{H}, \mathfrak{s}_r], \mathfrak{s}_s \rangle - \frac{1}{2} \langle \mathfrak{s}_r, [\mathfrak{H}, \mathfrak{s}_s] \rangle.\]

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**Diagonal solution of the Ricci Flow?**
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Let \((N, g_0)\) be a (non-necessarily nilpotent) Lie group endowed with a left-invariant metric
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\[
\text{Rc}(\langle \cdot, \cdot \rangle_t) := \text{Rc}(g(t))(e) : n \times n \rightarrow \mathbb{R}.
\]
If $P(t)$ is the positive definite operators of $(n, \langle \cdot, \cdot \rangle_0)$
If $P(t)$ is the positive definite operators of $(\mathbb{n}, \langle \cdot, \cdot \rangle_0)$

\[ \langle \cdot, \cdot \rangle_t = \langle P(t) \cdot, \cdot \rangle_0, \]
If $P(t)$ is the positive definite operators of $(n, \langle \cdot, \cdot \rangle_0)$

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If $P(t)$ is the positive definite operators of $(n, \langle \cdot, \cdot \rangle_0)$

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$$\frac{d}{dt} P(t) = -2P(t) \text{Ric}_t, \quad P(0) = I,$$

where $\text{Ric}_t := \text{Ric}(g(t))(e) : n \longrightarrow n$
If $P(t)$ is the positive definite operators of $(n, \langle \cdot, \cdot \rangle_0)$

$$\langle \cdot, \cdot \rangle_t = \langle P(t) \cdot, \cdot \rangle_0,$$

then the Ricci flow equation determines the following ODE for $P(t)$:

$$\frac{d}{dt} P(t) = -2P(t) \text{Ric}_t, \quad P(0) = I,$$

where $\text{Ric}_t := \text{Ric}(g(t))(e) : n \rightarrow n$ is the Ricci operator.
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Stably Ricci diagonal basis $\Rightarrow$ Ricci diagonal.
More examples:
More examples: Algebraic Ricci solitons.
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If \( \{X_1, \ldots, X_n\} \) is an orthonormal basis of eigenvectors of \( \text{Ric}_0 \),
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(Payne, Williams)
THANK YOU!