Symplectic Lie groups

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Based on the following joint work:

- Baues and C.–, *Symplectic Lie groups I-III*, arXiv:1307.1629

Definition

A symplectic Lie group is a Lie group endowed with a left-invariant symplectic structure.

Outline of the lecture:

I Structure results based on symplectic reduction

II Relation between flat Lie groups and symplectic Lie groups

III Lagrangian normal subgroups

I Structure results based on symplectic reduction

Simply connected symplectic Lie gps. $\longleftrightarrow$ Symplectic Lie algebras

Symplectic reduction

Let $(\mathfrak{g},\omega)$ be a symplectic LA and $\mathfrak{j}\subset\mathfrak{g}$ an isotropic ideal. Then

$$\bar{\mathfrak{g}} := \mathfrak{j}^{\perp}/\mathfrak{j}$$

inherits a symplectic form $\bar{\omega}$ and $(\bar{\mathfrak{g}},\bar{\omega})$ is called symplectic reduction.

Definition

$(\mathfrak{g},\omega)$ is called irreducible if $\nexists$ non-trivial $\mathfrak{j}\subset\mathfrak{g}$.

Uniqueness of the base

Definition

Given symplectic LAs $(\mathfrak{g}, \omega)$, $(\bar{\mathfrak{g}}, \bar{\omega})$, we say that $(\bar{\mathfrak{g}}, \bar{\omega})$ is an irreducible symplectic base for $(\mathfrak{g}, \omega)$ if

(i) $(\bar{\mathfrak{g}}, \bar{\omega})$ is irreducible and

(ii) there exist a sequence of reductions

$$(\mathfrak{g}, \omega), (\mathfrak{g}_1, \omega_1), \ldots, (\mathfrak{g}_\ell, \omega_\ell) = (\bar{\mathfrak{g}}, \bar{\omega}).$$

Theorem

Let $(\mathfrak{g}, \omega)$ be a symplectic LA. Then all its irreducible symplectic bases are isomorphic.

Proposition

A symplectic LA admits a Lagrangian subalgebra iff its irreducible symplectic base does.

Completely reducible symplectic LAs

Definition

A symplectic LA $(\mathfrak{g}, \omega)$ is called completely reducible if its irreducible symplectic base is trivial.

Proposition

Completely solvable symplectic LAs are completely reducible.

Corollary

Every completely reducible (in particular, every completely solvable) symplectic LA admits a Lagrangian subalgebra.

Classification of irreducible symplectic LAs

Extending results of Dardié and Medina (1996):

Theorem
A real symplectic LA $(\mathfrak{g}, \omega)$ is irreducible iff:

(i) $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$ is a maximal Abelian ideal and nondeg.,

(ii) $\mathfrak{h} = \mathfrak{a}^{\perp} \subset \mathfrak{g}$ is a complementary Abelian subalgebra and

(iii) the $\mathfrak{h}$-module $\mathfrak{a}$ is an orthogonal sum of 2-dimensional irreducible submodules which are pairwise non-isomorphic.

Remark
Such LAs are completely determined by $k = \frac{1}{2} \dim \mathfrak{h}$ and the characters $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$ of the $\mathfrak{h}$-module $\mathfrak{a}$ which are pairwise distinct and span $\mathfrak{h}^*$ ($\Rightarrow m \geq 2k$). The smallest possible dimension of a nontrivial such algebra is 6 ($k = 1, m = 2$).

Existence of Lagrangian subalgebras

Theorem

(i) Every real symplectic LA (whose base is) of dimension ≤ 6 admits a Lagrangian subalgebra.

(ii) There exists irreducible real symplectic LAs of dimension 8, which do not have any Lagrangian subalg.

II Relation between flat and symplectic Lie gps.

Cotangent Lie groups

Let G be a Lie group. Then the cotangent bundle T^*G is identified with $G \times \mathfrak{g}^*$ via

$$G \times \mathfrak{g}^* \to T^*G, \quad (g, \alpha) \mapsto \alpha_g.$$

Given a representation $\rho : G \to GL(\mathfrak{g}^*)$, we can form the semidirect product $G \ltimes_\rho \mathfrak{g}^*$ defining a Lie group structure on $G \times \mathfrak{g}^* \cong T^*G$:

$$(g, \alpha) \cdot (h, \beta) = (gh, \rho(h^{-1})\alpha + \beta). \tag{1}$$

Theorem

The canonical symplectic form Ω on T^*G is left-invariant wrt a Lie group structure of the form (1) iff G admits the structure of a flat Lie group with trivial linear holonomy.

Relation between flat and symplectic Lie gps.

Remarks concerning cotangent Lie groups

1. The case $\rho =$ coadjoint representation, G s.c. was considered by Chu (1974).
2. If Ω is left-invariant under (1) then the left-translates of G define a Lagrangian foliation and the above flat connection on G is precisely the Weinstein connection.
3. The fibre $T^*_e G \subset T^* G$ is a Lagrangian normal subgroup with the quotient

$$T^* G / T^*_e G = (G \ltimes_\rho \mathfrak{g}^*)/\mathfrak{g}^* = G.$$

Relation between flat and symplectic Lie gps.

Lagrangian extensions

- More generally, one can consider (as done by Boyom in 1993) symplectic LAs $(\mathfrak{g}, \omega)$, which admit a Lagrangian ideal $\mathfrak{a}$.
- Then
$$\mathfrak{h} := \mathfrak{g}/\mathfrak{a}$$
inherits a flat torsion free connection $\nabla : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$.
- Conversely, given a flat LA $(\mathfrak{h}, \nabla)$ we can ask for all possible symplectic LAs $(\mathfrak{g}, \omega)$ which admit $(\mathfrak{h}, \nabla)$ as a quotient by a Lagrangian ideal $\mathfrak{a}$.
- Such triples $(\mathfrak{g}, \omega, \mathfrak{a})$ are called Lagrangian extensions of $(\mathfrak{h}, \nabla)$.

Classification of Lagrangian extensions

Theorem

(i) Every Lagrangian extension $(\mathfrak{g}, \omega, \mathfrak{a})$ over a given flat LA $(\mathfrak{h}, \nabla)$ gives rise to an extension class

$$[\alpha] \in H^2_{L,\nabla}(\mathfrak{h}, \mathfrak{h}^*).$$

(ii) Two extensions over $(\mathfrak{h}, \nabla)$ are isomorphic iff they have the same extension class.

Remark

There is a natural map (neither injective nor surjective)

$$H^2_{L,\nabla}(\mathfrak{h}, \mathfrak{h}^*) \to H^2(\mathfrak{h}, \mathfrak{h}^*),$$

where $\mathfrak{h}^*$ is considered as an $\mathfrak{h}$-module with the representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}^*)$ dual to ∇.

Definition of the Lagrangian extension cohomology

The above cohomology is defined as follows:

- Put
  $C^1_L(\mathfrak{h},\mathfrak{h}^*) := \{\varphi \in C^1(\mathfrak{h},\mathfrak{h}^*) | \varphi(u)v - \varphi(v)u = 0 \; \forall u, v \in \mathfrak{h}\}$,
  $C^2_L(\mathfrak{h},\mathfrak{h}^*) := \{\alpha \in C^2(\mathfrak{h},\mathfrak{h}^*) | \alpha(u,v)w + \text{cycl.} = 0 \; \forall u, v, w \in \mathfrak{h}\}$.
- Then $\partial = \partial^{(1)} : C^1(\mathfrak{h},\mathfrak{h}^*) \to C^2(\mathfrak{h},\mathfrak{h}^*)$ maps $C^1_L(\mathfrak{h},\mathfrak{h}^*)$ into

$$Z^2_L(\mathfrak{h},\mathfrak{h}^*) := C^2_L(\mathfrak{h},\mathfrak{h}^*) \cap Z^2(\mathfrak{h},\mathfrak{h}^*).$$

- We define

$$B^2_L(\mathfrak{h},\mathfrak{h}^*) := \partial C^1_L(\mathfrak{h},\mathfrak{h}^*)$$

- and finally

$$H^2_{L,\nabla}(\mathfrak{h},\mathfrak{h}^*) := Z^2_L(\mathfrak{h},\mathfrak{h}^*)/B^2_L(\mathfrak{h},\mathfrak{h}^*).$$

Definition of the Lagrangian extension class

- Choose a Lagrangian complement N to $\mathfrak{a} \subset \mathfrak{g}$ and put

$$\alpha(u, v) := \pi_{\mathfrak{a}}([\tilde{u}, \tilde{v}]), \quad \forall u, v \in \mathfrak{h} = \mathfrak{g}/\mathfrak{a} \cong N,$$

  where $\tilde{u}, \tilde{v} \in N \subset \mathfrak{g}$ are the corresponding lifts and

$$\pi_{\mathfrak{a}} : \mathfrak{g} = N + \mathfrak{a} \to \mathfrak{a} \cong N^* \cong \mathfrak{h}^*.$$

- Then $\alpha \in Z^2_L(\mathfrak{h}, \mathfrak{h}^*)$ and represents a class $[\alpha] \in H^2_{L,\nabla}(\mathfrak{h}, \mathfrak{h}^*)$.

III Existence of Lagrangian ideals

Symplectic LAs with a Lagrangian ideal $\longleftrightarrow$ Flat LAs with a Lagrangian extension class

Problem

Which symplectic LAs admit a Lagrangian ideal?

Theorem

(i) $\exists$ 6-dimensional completely solvable symplectic LA without Lagrangian ideal.

(ii) $\exists$ 8-dim. 4-step n.p. symplectic LA without Lagrangian ideal.

(iii) These examples are of minimal dimension.

Definition

The symplectic rank of a symplectic LA is the maximal dimension of an isotropic ideal.

Existence of Lagrangian ideals

Theorem

(i) Every 2-step n.p. symplectic LA admits a Lagrangian ideal.

(ii) Every 3-step n.p. symplectic LA of dimension < 10 admits a Lagrangian ideal.

(iii) $\exists$ 10-dimensional 3-step nilpotent symplectic LA without Lagrangian ideal.

The 10-dimensional example $(\mathfrak{g}, \omega)$:

- Let $\mathfrak{g}$ be the 10-dimensional LA with the basis $(x, y, u_1, u_2, v_1, v_2, w_1, w_2, z_1, z_2)$ and non-zero brackets

$$[x, u_i] = v_i, \ [y, u_i] = w_i, \ [u_i, v_i] = -z_2, \ [u_i, w_i] = z_1, \ i = 1, 2.$$

- The symplectic form is

$$\omega = x^* \wedge z_1^* + y^* \wedge z_2^* + u_1^* \wedge u_2^* + v_1^* \wedge w_1^* + v_2^* \wedge w_2^*.$$

Existence of Lagrangian ideals

Theorem

Every n.p. symplectic LA $(\mathfrak{g}, \omega)$ which admits a central element H such that

$$\bar{\mathfrak{g}} = H^{\perp}/\langle H \rangle$$

is Abelian has a Lagrangian ideal.

Theorem

Every filiform n.p. symplectic LA of dimension $n = 2\ell$ has a unique Lagrangian ideal, namely $C^{\ell-1}\mathfrak{g}$.

Corollary

There is a bijection between isomorphism classes of filiform symplectic LAs $(\mathfrak{g}, \omega)$ and isomorphism classes of flat LAs $(\mathfrak{h}, \nabla, [\alpha])$ with filiform extension class $[\alpha]$.

Further results: Symplectic solvmanifolds

- It was asked by Guan (2010), whether given a compact symplectic solvmanifold $\Gamma\backslash G$, the solvability degree of G is always bounded by 3.
- This conjecture has been verified in dimensions ≤ 6 by Ovando (dim 4) and Macri (dim 6).
- We show that, contrary to the conjecture, the solvability degree of symplectic solvmanifolds is unbounded with increasing dimension.
- We prove this by giving a series of symplectic nilmanifolds, which has unbounded solvability degree.
- In particular, we have a 72-dimensional example of solvability degree $4 > 3$.

Further results: Obstruction to existence of symplectic structure

- Milenteva has introduced the invariants

$$\begin{aligned} \mu(\mathfrak{g}) &:= \max\{\dim \mathfrak{a} | \mathfrak{a} \subset \mathfrak{g} \text{ Abelian ideal}\} \leq \\ \nu(\mathfrak{g}) &:= \max\{\dim \mathfrak{a} | \mathfrak{a} \subset \mathfrak{g} \text{ Abelian subalgebra}\} \end{aligned}$$

  and shown (in 2008) that there are infinitely many 2-step nilpotent LAs with $\nu(\mathfrak{g}) < \frac{1}{2} \dim \mathfrak{g}$.
- These LAs do not admit any symplectic structure.
- In fact, since isotropic ideals are Abelian, every symplectic LA $(\mathfrak{g}, \omega)$ has

$$\sigma(\mathfrak{g}, \omega) \leq \mu(\mathfrak{g})$$

  and every 2-step n.p. symplectic LA has

$$\sigma(\mathfrak{g}, \omega) = \frac{1}{2} \dim \mathfrak{g}.$$