On the Dolbeault-Dirac Operator on Quantised Compact Hermitian Symmetric Spaces

Ulrich Krähmer

joint work with Matthew Tucker-Simmons (U Berkeley)

Seminar Sophus Lie 2014

Plan

There will be three self-contained blocks on fairly classical material:

- Compact Hermitian symmetric spaces $\rightsquigarrow$ quantum groups
- Dirac operators $\rightsquigarrow$ noncommutative differential geometry
- Koszul algebras $\rightsquigarrow$ noncommutative algebraic geometry

At the end I will then talk about a project with Matt Tucker-Simmons whose ultimate goal is a quantum group version of Parthasarathy's formula for the square of the Dirac operator on a symmetric space.

Quantised symmetric algebras

- Let $\mathfrak{k}$ be a complex simple Lie algebra and V be an irreducible d-dimensional representation of $\mathfrak{k}$.

Quantised symmetric algebras

- Let $\mathfrak{k}$ be a complex simple Lie algebra and V be an irreducible d-dimensional representation of $\mathfrak{k}$.
- The **Drinfel'd coboundary operator** $\sigma : V \otimes_{\mathbb{C}} V \to V \otimes_{\mathbb{C}} V$ of $U_q(\mathfrak{k})$ defines a **quantised symmetric algebra**

$$SV = TV / \langle \ker(\sigma + \mathrm{id}) \rangle.$$

σ has the same eigenvalues ± 1 as the tensor flip $v \otimes_{\mathbb{C}} w \mapsto w \otimes_{\mathbb{C}} v$, but the eigenvectors are deformed.

Quantised symmetric algebras

- Let $\mathfrak{k}$ be a complex simple Lie algebra and V be an irreducible d-dimensional representation of $\mathfrak{k}$.
- The **Drinfel'd coboundary operator** $\sigma : V \otimes_{\mathbb{C}} V \to V \otimes_{\mathbb{C}} V$ of $U_q(\mathfrak{k})$ defines a **quantised symmetric algebra**

$$SV = TV/\langle \ker(\sigma + \mathrm{id}) \rangle.$$

  σ has the same eigenvalues ± 1 as the tensor flip $v \otimes_{\mathbb{C}} w \mapsto w \otimes_{\mathbb{C}} v$, but the eigenvectors are deformed.
- Zwicknagl classified the V for which SV has the classical Hilbert-Poincaré series of a polynomial ring,

$$\sum_{i=0}^{\infty} \dim(SV_i) t^i = \sum_{i=0}^{\infty} \binom{d-1+i}{d-1} t^i = (1-t)^{-d}.$$

Quantised symmetric algebras

- Let $\mathfrak{k}$ be a complex simple Lie algebra and V be an irreducible d-dimensional representation of $\mathfrak{k}$.
- The **Drinfel'd coboundary operator** $\sigma : V \otimes_{\mathbb{C}} V \to V \otimes_{\mathbb{C}} V$ of $U_q(\mathfrak{k})$ defines a **quantised symmetric algebra**

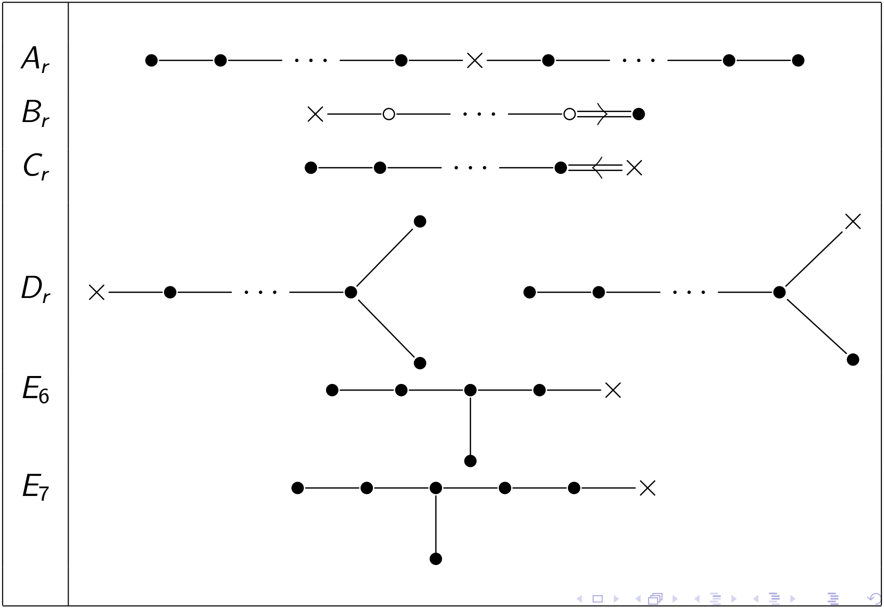
$$SV = TV / \langle \ker(\sigma + \mathrm{id}) \rangle.$$

  σ has the same eigenvalues ± 1 as the tensor flip $v \otimes_{\mathbb{C}} w \mapsto w \otimes_{\mathbb{C}} v$, but the eigenvectors are deformed.
- Zwicknagl classified the V for which SV has the classical Hilbert-Poincaré series of a polynomial ring,

$$\sum_{i=0}^{\infty} \dim(SV_i) t^i = \sum_{i=0}^{\infty} \binom{d-1+i}{d-1} t^i = (1-t)^{-d}.$$

- Example: for $\mathfrak{k} = \mathfrak{sl}(2, \mathbb{C})$ only the 2- and the 3-dimensional irreducible representations work!

The best manifolds ever



Simple Lie algebras and parabolic subalgebras

- $\mathfrak{g}$ - finite-dimensional complex simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$ and root system $\Delta \subseteq \mathfrak{h}^*$.
- Δ^+ - positive roots, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ - simple roots.

Simple Lie algebras and parabolic subalgebras

- $\mathfrak{g}$ - finite-dimensional complex simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$ and root system $\Delta \subseteq \mathfrak{h}^*$.
- Δ^+ - positive roots, $\Pi = \{\alpha_1, \dots, \alpha_r\}$ - simple roots.
- Given $\mathcal{S} \subseteq \Pi$, define

$$\Delta(\mathfrak{l}) = \mathrm{span}_{\mathbb{Z}}(\mathcal{S}) \cap \Delta, \quad \Delta(\mathfrak{u}_+) = \Delta^+ \setminus \Delta(\mathfrak{l}),$$

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{l})} \mathfrak{g}_\alpha, \quad \mathfrak{u}_\pm = \bigoplus_{\alpha\in\Delta(\mathfrak{u}_+)} \mathfrak{g}_{\pm\alpha}, \quad \text{and} \quad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+.$$

- Them one has

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, \quad [\mathfrak{u}_\pm, \mathfrak{u}_\pm] \subseteq \mathfrak{u}_\pm, \quad [\mathfrak{l}, \mathfrak{u}_\pm] \subseteq \mathfrak{u}_\pm.$$

Simple Lie algebras and parabolic subalgebras

- $\mathfrak{g}$ - finite-dimensional complex simple Lie algebra with fixed Cartan subalgebra $\mathfrak{h}$ and root system $\Delta \subseteq \mathfrak{h}^*$.
- Δ^+ - positive roots, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ - simple roots.
- Given $\mathcal{S} \subseteq \Pi$, define

$$\Delta(\mathfrak{l}) = \mathrm{span}_{\mathbb{Z}}(\mathcal{S}) \cap \Delta, \quad \Delta(\mathfrak{u}_+) = \Delta^+ \setminus \Delta(\mathfrak{l}),$$

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}_\alpha, \quad \mathfrak{u}_\pm = \bigoplus_{\alpha \in \Delta(\mathfrak{u}_+)} \mathfrak{g}_{\pm\alpha}, \quad \text{and} \quad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+.$$

- Them one has

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, \quad [\mathfrak{u}_\pm, \mathfrak{u}_\pm] \subseteq \mathfrak{u}_\pm, \quad [\mathfrak{l}, \mathfrak{u}_\pm] \subseteq \mathfrak{u}_\pm.$$

- $\mathfrak{p}$ is the **standard parabolic subalgebra** associated to $\mathcal{S}$, and $\mathfrak{l}$ is its Levi factor while $\mathfrak{u}_+$ is its nilradical. We put $\mathfrak{k} := [\mathfrak{l}, \mathfrak{l}]$.

An example

- For $\mathfrak{g} = \mathfrak{sl}(r+1, \mathbb{C}) = A_r$, $\mathfrak{p}$ can be any Lie subalgebra that contains all upper triangular matrices, for example for $r = 3$ the one containing all traceless matrices of the form

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

- Here $\mathcal{S} = \{\alpha_1, \alpha_3\}$ and we denote this choice by crossing out the missing α_2 in the Dynkin diagram of $\mathfrak{g}$,

The $\mathfrak{p}$ of cominuscule type

- The following are equivalent:
  (i) $\mathfrak{g}/\mathfrak{p}$ is a simple $\mathfrak{p}$-module (wrt the adjoint action);
  (ii) $\mathfrak{u}_+$ is a simple $\mathfrak{l}$-module;
  (iii) $\mathfrak{u}_+$ is an abelian Lie algebra;
  (iv) $\mathfrak{p}$ is maximal, i.e. $\mathcal{S} = \Pi \setminus \{\alpha_s\}$ for some $1 \leq s \leq r$, and moreover α_s has coefficient 1 in the highest root of $\mathfrak{g}$;
  (v) $(\mathfrak{g}, \mathfrak{l})$ is a symmetric pair, i.e. there is an involutive Lie algebra automorphism of $\mathfrak{g}$ whose invariants are $\mathfrak{l}$.
- Note: the highest root is just the highest weight of the adjoint representation (which is irreducible as $\mathfrak{g}$ is simple).

The $\mathfrak{p}$ of cominuscule type

- The following are equivalent:
  (i) $\mathfrak{g}/\mathfrak{p}$ is a simple $\mathfrak{p}$-module (wrt the adjoint action);
  (ii) $\mathfrak{u}_+$ is a simple $\mathfrak{l}$-module;
  (iii) $\mathfrak{u}_+$ is an abelian Lie algebra;
  (iv) $\mathfrak{p}$ is maximal, i.e. $\mathcal{S} = \Pi \setminus \{\alpha_s\}$ for some $1 \leq s \leq r$, and moreover α_s has coefficient 1 in the highest root of $\mathfrak{g}$;
  (v) $(\mathfrak{g}, \mathfrak{l})$ is a symmetric pair, i.e. there is an involutive Lie algebra automorphism of $\mathfrak{g}$ whose invariants are $\mathfrak{l}$.
- Note: the highest root is just the highest weight of the adjoint representation (which is irreducible as $\mathfrak{g}$ is simple).
- Zwicknagl's list contains exactly these $\mathfrak{u}_\pm$ as V plus somewhat mysteriously the first fundamental representation of C_r.

The compact Hermitian symmetric spaces

- If G and P are the Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{p}$, then G/P is called a **generalised flag manifold**.
- The subclass characterised on the previous slide are the irreducible **compact Hermitian symmetric spaces**.

The compact Hermitian symmetric spaces

- If G and P are the Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{p}$, then G/P is called a **generalised flag manifold**.
- The subclass characterised on the previous slide are the irreducible **compact Hermitian symmetric spaces**.
- These admit certain quantisations, e.g. for $G/P = \mathbb{C}P^1$ the **standard Podleś sphere.** What we are after is a quantisation not only of the space but of the Kähler metric using Alain Connes' framework of **spectral triples**.

Clifford algebras

- Let k be a commutative ring. To any k-module M equipped with a symmetric bilinear form

$$g : M \otimes_k M \to k$$

one associates its **Clifford algebra**

$$Cl(V, g) := TM / \langle m \otimes_k m - g(m, m) \mid m \in M \rangle.$$

Spin^c-manifolds

- Example: $k = C^\infty(X, \mathbb{C})$, X some $2d$-dimensional manifold, $M := \mathrm{Der}_{\mathbb{C}}(k)$, g a Riemannian metric.

Spin^c-manifolds

- Example: $k = C^\infty(X, \mathbb{C})$, X some $2d$-dimensional manifold, $M := \mathrm{Der}_{\mathbb{C}}(k)$, g a Riemannian metric.
- The Clifford algebra of X is an **Azumaya algebra** - its fibre in each pointi of X is a matrix algebra $M_{2^n}(\mathbb{C})$.

Spin^c-manifolds

- Example: $k = C^\infty(X, \mathbb{C})$, X some $2d$-dimensional manifold, $M := \mathrm{Der}_{\mathbb{C}}(k)$, g a Riemannian metric.
- The Clifford algebra of X is an **Azumaya algebra** - its fibre in each pointi of X is a matrix algebra $M_{2^n}(\mathbb{C})$.
- A **spinor module** is a k-module S such that

$$Cl(M, g) \simeq \mathrm{End}_k(S).$$

The elements of S are called **spinors** or **spinor fields**. A choice of a spinor module is called a **spin^c-structure** on X.

The Bass construction

- If V is any finitely generated projective module over a commutative ring k and $V^* := \mathrm{Hom}_k(V, k)$ is its dual, then

$$M := V \oplus V^*$$

carries a canonical nondegenerate symmetric bilinear form g:

$$g((x, \varphi), (y, \psi)) := \varphi(y) + \psi(x).$$

- By a result of Bass this always admits a spinor module, namely the exterior algebra of V,

$$Cl(V \oplus V^*, g) \simeq \mathrm{End}_k(\Lambda V).$$

Corollary: Hermitian manifolds are spin^c

- This applies to Hermitian manifolds X: local coordinates

$$z_1, \ldots, z_d : X \supseteq U \to \mathbb{C}$$

split $M = \mathrm{Der}_{\mathbb{C}}(k)$ into the $(1,0)$- and $(0,1)$-components

$$V = \Gamma(T^{1,0}), \quad V^* = \Gamma(T^{0,1})$$

which are locally spanned by

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d} \quad \text{respectively} \quad \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_d}.$$

The duality is given by the Hermitian metric.

- Hence a Hermitian manifold has a **canonical spin^c-structure** given by the exterior algebra of the holomorphic tangent bundle which again by the metric is identified with $\Omega^{0,\bullet}$.

The Dolbeault-Dirac operator

- The **Dolbeault operator** is the projection of Cartan's d to

$$\bar{\partial}\colon \Omega^{0,q} \to \Omega^{0,q+1}.$$

The Dolbeault-Dirac operator

- The **Dolbeault operator** is the projection of Cartan's d to

$$\bar{\partial}\colon \Omega^{0,q} \to \Omega^{0,q+1}.$$

- Using a Hermitian inner product one defines the Hilbert space H of square-integrable sections of $\Omega^{0,\bullet}$ and on H the **Dolbeault-Dirac operator**

$$D = \bar{\partial} + \bar{\partial}^*.$$

- Noncommutative differential geometry: one can reconstruct the Hermitian manifold X fully from the **spectral triple** (k, H, D).

Quadratic algebras

- A $\mathbb{C}$-algebra is **quadratic** if it is finitely generated with defining relations all homogeneous of degree 2,

$$A := A(V, R) := TV/\langle R \rangle, \quad R \subseteq V \otimes_{\mathbb{C}} V.$$

  Here V is the vector space spanned by the generators.
- Note: A is $\mathbb{N}$-graded and connected, $A_0 = \mathbb{C}$. We define

$$A_+ := \bigoplus_{i>0} A_i.$$

Quadratic algebras

- A $\mathbb{C}$-algebra is **quadratic** if it is finitely generated with defining relations all homogeneous of degree 2,

$$A := A(V,R) := TV/\langle R\rangle, \quad R \subseteq V \otimes_{\mathbb{C}} V.$$

  Here V is the vector space spanned by the generators.
- Note: A is $\mathbb{N}$-graded and connected, $A_0 = \mathbb{C}$. We define

$$A_+ := \bigoplus_{i>0} A_i.$$

- To any quadratic algebra one assigns its **quadratic dual** $A^!$ with the same number of generators but orthogonal relations,

$$A^! = A(V^*, R^\perp).$$

- Example: $SV^! = \Lambda V$.

Koszul algebras

- Given any quadratic algebra $A = A(V, R)$, one defines its **Koszul complex**. As graded vector space this is

$$K = A \otimes_{\mathbb{C}} (A^!)^*.$$

with grading induced by that of $A^!$.

- The differential is given by

$$\bar{\partial} := \sum_i X_i \otimes X^i : a \otimes f \mapsto \sum_i aX_i \otimes fX^i$$

where $\{X_i\}$ is a basis of V and $\{X^i\}$ is the dual basis of V^*.

- A quadratic algebra is a **Koszul algebra** if the Koszul complex is acyclic (thus providing a free resolution of the module A/A_+).

Spectral triples and differential calculi

- Heckenberger-Kolb: Studied covariant differential calculi over the quantised compact Hermitian symmetric spaces.

Spectral triples and differential calculi

- Heckenberger-Kolb: Studied covariant differential calculi over the quantised compact Hermitian symmetric spaces.
- Generalising results for $\mathbb{C}P^1$ due to Dąbrowski and Sitarz, I constructed a subspace of $U_q(\mathfrak{g})$ spanned by elements $X_1, \ldots, X_d$ that play the analogue of $\mathfrak{u}_+$, and argued that there is a quantum Clifford algebra Cl_q such that

$$D := \bar{\partial} + \bar{\partial}^*, \quad \bar{\partial} := \sum_i X_i \otimes X^i \in U_q(\mathfrak{g}) \otimes Cl_q$$

defines something.

Spectral triples and differential calculi

- Heckenberger-Kolb: Studied covariant differential calculi over the quantised compact Hermitian symmetric spaces.
- Generalising results for $\mathbb{C}P^1$ due to Dąbrowski and Sitarz, I constructed a subspace of $U_q(\mathfrak{g})$ spanned by elements $X_1, \ldots, X_d$ that play the analogue of $\mathfrak{u}_+$, and argued that there is a quantum Clifford algebra Cl_q such that

$$D := \bar{\partial} + \bar{\partial}^*, \quad \bar{\partial} := \sum_i X_i \otimes X^i \in U_q(\mathfrak{g}) \otimes Cl_q$$

defines something.
- Further work by Leipzig (Schmüdgen-Wagner), Oslo (Neshveyev-Tuset), Trieste (Dąbrowski-D'Andrea-Landi et al.) and others (e.g. O'Buachalla).

Applying Berenstein-Zwicknagl

- Fresh wind: They also embed $\mathfrak{u}_+$ into $U_q(\mathfrak{g})$ in order to generate $S\mathfrak{u}_+$ as a **twisted quantum Schubert cell**.
- Example: if $G/P = Gr(2,4)$, then $S\mathfrak{u}_+$ are the quantum 2×2-matrices that can indeed be embedded into $U_q(\mathfrak{sl}(4,\mathbb{C}))$.

Applying Berenstein-Zwicknagl

- Fresh wind: They also embed $\mathfrak{u}_+$ into $U_q(\mathfrak{g})$ in order to generate $S\mathfrak{u}_+$ as a **twisted quantum Schubert cell**.
- Example: if $G/P = Gr(2,4)$, then $S\mathfrak{u}_+$ are the quantum 2×2-matrices that can indeed be embedded into $U_q(\mathfrak{sl}(4,\mathbb{C}))$.
- If we define

$$Cl_q := \mathrm{End}_{\mathbb{C}}(\Lambda\mathfrak{u}_+),$$

then a quantum Bass construction gives an algebra factorisation

$$Cl_q \simeq \Lambda\mathfrak{u}_- \otimes \Lambda\mathfrak{u}_+$$

and viewing the Koszul boundary map as an element in $U_q(\mathfrak{g}) \otimes Cl_q$ produces my old Dolbeault operator.

The real task

- We now know
$$D^2 = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$
thanks to the identification of $\bar\partial$ with the Koszul boundary.

The real task

- We now know
$$D^2 = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$
thanks to the identification of $\bar\partial$ with the Koszul boundary.
- What we really want is a quantum **Parthasarathy formula** that expresses D^2 as a linear combination of Casimirs of $\mathfrak{g}$ and of $\mathfrak{l}$ plus a constant term. This would allow one to compute the spectrum of D and to prove it defines a class in the K-homology of the quantised function algebras of G/P.

The real task

- We now know
$$D^2 = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$
thanks to the identification of $\bar\partial$ with the Koszul boundary.
- What we really want is a quantum **Parthasarathy formula** that expresses D^2 as a linear combination of Casimirs of $\mathfrak{g}$ and of $\mathfrak{l}$ plus a constant term. This would allow one to compute the spectrum of D and to prove it defines a class in the K-homology of the quantised function algebras of G/P.
- Question: Given a noncommutative polynomial ring, is there a corresponding partner like $\mathfrak{u}_-$ for $\mathfrak{u}_+$, both sitting inside some quantum group, leading to a corresponding quantum Hermitian symmetric space?