Locally conformal symplectic Lie groups

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What is this about?

1. Locally conformal symplectic geometry
2. Structure results for lcs manifolds of the 1^{st} kind
3. Locally conformal symplectic Lie groups and Lie algebras
4. An example

Goal of the talk

Goal

The aim of the talk is the study of **locally conformal symplectic structures (lcs)**, with a particular emphasis on **Lie groups and Lie algebras**. We will show that there is an interplay between

symplectic, contact and lcs structures of the 1^{st} kind

Locally conformal symplectic structures

Definition

A **locally conformal symplectic structure (lcs)** on a manifold $M^{\geq 4}$ consists of a non-degenerate 2-form $\omega \in \Omega^2(M)$ and an open covering $\{U_\alpha\}$ of M such that there exist $f_\alpha \in C^\infty(U_\alpha)$ such that $d(e^{f_\alpha}\omega|_{U_\alpha}) = 0$. Equivalently, $d\omega = \theta \wedge \omega$ for some closed $\theta \in \Omega^1(M)$, the Lee form. We denote the lcs structure by (ω, θ).

- Since ω is non-degenerate, $\dim M = 2n$
- We always assume that θ is non-zero
- If (ω, θ) is a lcs structure with θ exact, then (ω, θ) is globally conformal to a symplectic structure
- Lcs manifolds can be seen as generalized phase spaces of Hamiltonian dynamical systems

Different kinds of lcs structures

Lichnerowicz cohomology

If M is a manifold and $\theta \in \Omega^1(M)$ is closed, define the Lichnerowicz differential $d_\theta \colon \Omega^*(M) \to \Omega^{*+1}(M)$ by

$$d_\theta \alpha = d\alpha - \theta \wedge \alpha$$

- $\omega \in \Omega^*(M^{2n})$ with $\omega^n \neq 0$ defines a lcs structure $\Leftrightarrow$ $\exists\ \theta \in \Omega^1(M)$, closed, with $d_\theta(\omega) = 0$
- (ω, θ) is **exact** if $\exists \eta \in \Omega^1(M)$ with $d_\theta(\eta) = \omega$; **non-exact** otherwise
- (ω, θ) is **of the first kind** if $\exists U \in \mathfrak{X}(M)$, the anti-Lee field, with $L_U \omega = 0$ and $\theta(U) = 1$; **of the second kind** otherwise
- Being exact is not an invariant of the conformal class of the lcs structure

Locally conformal symplectic structures of the first kind

Proposition

Let (ω, θ) be a lcs structure of the first kind on M^{2n} and let $U \in \mathfrak{X}(M)$ be the anti-Lee field. Set $\eta = -\imath_U \omega$. Then

- $d\eta$ has rank $2n - 2$
- $\theta \wedge \eta \wedge (d\eta)^{n-1} \neq 0$
- $\omega = d\eta - \theta \wedge \eta$

Conversely, let M^{2n} be a manifold endowed with two nowhere zero 1-forms θ and η with $d\theta = 0$, $\mathrm{rank}(d\eta) < 2n$ and such that $\theta \wedge \eta \wedge (d\eta)^{n-1} \neq 0$. If we set $\omega = d\eta - \theta \wedge \eta$, then (ω, θ) is a lcs structure of the first kind on M.

Examples of locally conformal symplectic structures

Examples

- Let Q be a manifold, let $\hat{\theta} \in \Omega^1(Q)$ be closed and let $\pi\colon T^*Q \to Q$ be the projection. If $\lambda_{can} \in \Omega^1(T^*Q)$ is the canonical 1-form, $\omega = -d\lambda_{can} + \theta \wedge \lambda_{can}$ defines a lcs structure on T^*Q; here $\theta = \pi^*\hat{\theta}$
- even-dimensional leaves of a transitive Jacobi structure have a lcs structure
- Let (g, J) be a Hermitian structure on M^{2n}, with Kähler form ω and Lee form $\theta = \frac{1}{n-1}(\delta\omega) \circ J$. (g, J) is locally conformal Kähler (lcK) if $\theta \neq 0$, $d\theta = 0$ and $d\omega = \theta \wedge \omega$. It is Vaisman if, in addition, $\nabla\theta = 0$. Every lcK structure has an underlying lcs structure.

Locally conformal Kähler structures

- LcK metrics on compact complex surfaces have been studied by Belgun
- LcK homogeneous structures are Vaisman (Alekseevski *et al.*)
- No known topological obstructions to the existence of lcK metrics on a compact manifold
- If M carries a Vaisman structure, then $b_1(M)$ is odd
- There exist lcK manifolds with b_1 even (Oeljeklaus - Toma)

Question (Ornea - Verbitski)

Are there compact manifolds with lcs structure but no lcK metrics?

Mapping tori and more examples

Let P be a smooth manifold and let $\varphi\colon P \to P$ be a diffeomorphism. The $\mathbb{Z}$-action $(p, t) \mapsto (\varphi^m(p), t + m)$ on $P \times \mathbb{R}$ is free and properly discontinuous, hence the quotient space P_φ is a smooth manifold.

Definition

P_φ is the **mapping torus** of P and φ. $P \to P_\varphi \to S^1$ is a fiber bundle.

Proposition

Let (P, η) be a contact manifold and let $\varphi\colon P \to P$ be a strict contactomorphism. Then the mapping torus P_φ has a lcs structure of the first kind.

More examples and a question

More examples

- The product of a compact contact manifold and a circle has a lcs structure of the first kind
- In particular, $S^3 \times S^1$ is a compact manifold with a lcs structure, but no symplectic structures
- [Bande-Kotschick] One can choose a 3-manifold P in such a way that $P \times S^1$ has no complex structure $\Rightarrow$ Positive answer to Ornea-Verbitski

Question

Are there compact **non-product** manifolds with a lcs structure but no lcK metrics?

Structure results for lcs manifolds of the 1st kind

Theorem (Banyaga)

Let (ω, θ) be a lcs structure of the first kind on M compact and let U be the anti-Lee field. Then M is fibered over S^1 and the restriction of $\eta = -\imath_U\omega$ to each fiber is a contact form.

- A compact manifold fibering over S^1 is diffeomorphic to a mapping torus
- Banyaga proves that M is diffeomorphic to the mapping torus of a strict contactomorphism of a fiber.

Structure results for lcs manifolds of the 1st kind

Definition

Let (ω, θ) be a lcs structure on M. The **Lee vector field** is $V \in \mathfrak{X}(M)$, defined by $\imath_V \omega = \theta$.

Theorem (Vaisman)

Let (ω, θ) be a lcs structure of the first kind on M compact; let U and V be the anti-Lee and Lee field. The distribution $\mathcal{D} = \langle U, V \rangle$ integrates to a foliation $\mathcal{G}$ on M. Under certain regularity assumptions, the space of leaves $N = M/\mathcal{G}$ has a symplectic structure and $p\colon M \to N$ is a principal T^2-bundle.

Structure results for lcs manifolds of the 1st kind

Theorem (−, Marrero)

Let M be a compact connected manifold endowed with a lcs structure of the first kind (ω, θ), let U be the anti-Lee vector field and write $\omega = d_\theta(\eta)$. If

- U is complete
- $\mathcal{F} = \{\theta = 0\}$ has a compact leaf L with inclusion $i\colon L \to M$,

then

- $i^*\eta$ is a contact form on L
- there exists a strict contactomorphism $\varphi\colon L \to L$
- the flow of U induces an isomorphism between L_φ and M.

Locally conformal symplectic Lie groups

Definition

A Lie group G of dimension $2n$ ($n \geq 2$) is a **locally conformal symplectic (lcs) Lie group** if there exist

- $\omega \in \Omega^2(G)^G$ with $\omega^n \neq 0$ and
- $\theta \in \Omega^1(G)^G$ closed

with $d\omega = \theta \wedge \omega$.
(ω, θ) is of the first kind if there exists $U \in \mathfrak{X}(G)^G$ with $L_U\omega = 0$ and $\theta(U) = 1$. Then $\eta = -\imath_U\omega \in \Omega^1(G)^G$ satisfies $\omega = d\eta - \theta \wedge \eta$.

Locally conformal symplectic Lie algebras

Definition

A Lie algebra $\mathfrak{g}$ of dimension $2n$ ($n \geq 2$) is a **locally conformal symplectic (lcs) Lie algebra** if there exist

- $\omega \in \Lambda^2 \mathfrak{g}^*$ with $\omega^n \neq 0$ and
- $\theta \in \mathfrak{g}^*$ with $d\theta = 0$

with $d\omega = \theta \wedge \omega$.

The lcs structure is of the first kind if there exists $U \in \mathfrak{g}$ with $L_U \omega = 0$ and $\theta(U) = 1$. Then $\eta = -\imath_U \omega \in \mathfrak{g}^*$ satisfies $\omega = d\eta - \theta \wedge \eta$.

Symplectic and contact structures on Lie algebras

Definition

Let $\mathfrak{s}$ be a Lie algebra, $\dim \mathfrak{s} = 2n$. A **symplectic structure** on $\mathfrak{s}$ is $\sigma \in \Lambda^2 \mathfrak{s}^*$, closed and non-degenerate, i. e. $\sigma^n \neq 0$

Definition

Let $\mathfrak{h}$ be a Lie algebra, $\dim \mathfrak{h} = 2n - 1$. A **contact structure** on $\mathfrak{h}$ is $\eta \in \mathfrak{h}^*$ such that $\eta \wedge (d\eta)^{n-1} \neq 0$. The Reeb vector is $R \in \mathfrak{h}$, defined by $\imath_R d\eta = 0$ and $\eta(R) = 1$.

A few facts

- A unimodular symplectic Lie algebra is solvable.
- The only semisimple Lie algebras admitting contact structures are $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$.

Reductive lcs Lie algebras

Proposition (Angella, –, Parton)

Let $(\mathfrak{g}, \omega, \theta)$ be a reductive lcs Lie algebra. Then $\mathfrak{g}$ is 4-dimensional. If $\mathfrak{g}$ is unimodular, then $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}$, otherwise $\mathfrak{g} = \mathfrak{sl}(2) \oplus \mathbb{R}$.

$\mathfrak{su}(2) \oplus \mathbb{R}$ is the Lie algebra of $S^3 \times S^1$.

Where to look for?

We concentrate therefore on **nilpotent** and, more generally, **solvable** lcs Lie algebras.

Lcs Lie algebras from contact Lie algebras

Proposition (–, Marrero)

Lcs Lie algebras of the first kind are in 1-1 correspondence with contact Lie algebras endowed with a contact derivation.

- If $(\mathfrak{h}, \eta)$ is a contact Lie algebra and $D\colon \mathfrak{h} \to \mathfrak{h}$ is a derivation with $D^*\eta = 0$, then $\mathfrak{g} = \mathfrak{h} \rtimes_D \mathbb{R}$ is endowed with a lcs structure of the first kind.
- Suppose $(\mathfrak{g}, \omega, \theta)$ is a lcs Lie algebra of the first kind with $\dim \mathfrak{g} = 2n$; let U be the anti-Lee field and $\eta = -\imath_U \omega$. Set $\mathfrak{h} = \ker(\theta)$ and let η be the restriction of η to $\mathfrak{h}$. Then $(\mathfrak{h}, \eta)$ is a contact Lie algebra, endowed with a derivation D with $D^*\eta = 0$. Moreover, $\mathfrak{g} \cong \mathfrak{h} \rtimes_D \mathbb{R}$.

Contact Lie algebras from symplectic Lie algebras

If $(\mathfrak{h}, \eta)$ is a contact Lie algebra, then:

- $\mathfrak{Z}(\mathfrak{h}) = 0$, or
- $\mathfrak{Z}(\mathfrak{h}) = \langle R \rangle$

We consider contact Lie algebras with non-trivial center.

Proposition

Contact Lie algebras with non-trivial center are in 1-1 correspondence with central extensions of symplectic Lie algebras.

- If $\sigma \in \Lambda^2 \mathfrak{s}^*$ is a symplectic structure on $\mathfrak{s}$, the central extension $\mathfrak{h} = \mathbb{R} \circledcirc_\sigma \mathfrak{s}$ of $\mathfrak{s}$ by $\sigma \in Z^2(\mathfrak{s}, \mathbb{R})$ has a contact structure
- if $(\mathfrak{h}, \eta)$ is a contact Lie algebra with Reeb vector R such that $\mathfrak{Z}(\mathfrak{h}) = \langle R \rangle$, then the Lie algebra $\mathfrak{s} = \mathfrak{h}/\langle R \rangle$ has a symplectic structure σ and $\mathfrak{h}$ is isomorphic to $\mathbb{R} \circledcirc_\sigma \mathfrak{s}$.

Lcs extensions

symplectic $\leftrightsquigarrow$ contact $\leftrightsquigarrow$ lcs of the 1st kind

Definition

A derivation D of $(\mathfrak{s},\sigma)$ is **symplectic** if

$$\sigma(DX,Y)+\sigma(X,DY)=0 \quad \forall\, X,Y\in\mathfrak{s}.$$

Theorem (–, Marrero)

There exists a 1-1 correspondence between lcs Lie algebras of the first kind $(\mathfrak{g},\omega,\theta)$ of dimension $2n+2$ with central Lee vector and symplectic Lie algebras $(\mathfrak{s},\sigma)$ of dimension $2n$ endowed with a symplectic derivation.

The correspondence

Take a symplectic Lie algebra $(\mathfrak{s}, \sigma)$ and a symplectic derivation $D\colon \mathfrak{s} \to \mathfrak{s}$. On $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ define the Lie bracket

$$[(a, X, \alpha), (b, Y, \beta)]_{\mathfrak{g}} = (\sigma(X, Y), \alpha D(Y) - \beta D(X) + [X, Y]_{\mathfrak{s}}, 0). \tag{1}$$

Then $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ is a Lie algebra. Define $\theta, \eta \in \mathfrak{g}^*$ by

$$\theta(a, X, \alpha) = \alpha \quad \text{and} \quad \eta(a, X, \alpha) = a. \tag{2}$$

Then (ω, θ), where $\omega = d_\theta(\eta)$, is a lcs structure of the first kind on $\mathfrak{g}$ with central Lee vector $V = (1, 0, 0) \in \mathfrak{g}$.

Definition

$\mathfrak{g} = \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ endowed with the Lie algebra structure (1) and the lcs structure of the first kind (2) is the lcs extension of $(\mathfrak{s}, \sigma)$ by the derivation D.

Symplectic double extensions (Dardié, Medina, Revoy)

You need:

- a symplectic Lie algebra $(\mathfrak{s}_0, \sigma_0)$
- a derivation $D_0 \colon \mathfrak{s}_0 \to \mathfrak{s}_0$
- a vector $Z_0 \in \mathfrak{s}_0$

Recipe:

- $D_0^*\sigma_0 \in Z^2(\mathfrak{s}_0, \mathbb{R})$; put $\mathfrak{h}_0 = \mathbb{R} \circledcirc_{D_0^*\sigma_0} \mathfrak{s}_0$
- the linear map $A \colon \mathfrak{h}_0 \to \mathfrak{h}_0$, $(a, X) \mapsto (-\sigma_0(Z_0, X), -D_0(X))$ is a derivation $\Leftrightarrow d(i_{Z_0}\sigma_0) = -(D_0^*)^2\sigma_0$
- assuming it is so, $\mathfrak{s} = \mathfrak{h}_0 \rtimes_A \mathbb{R}$ is a symplectic Lie algebra with symplectic form

$$\sigma((a, X, \alpha), (b, Y, \beta)) = a\beta - \alpha b + \sigma_0(X, Y)$$

Symplectic double extensions

Definition

$(\mathfrak{s}, \sigma)$ is the **symplectic double extension** of $(\mathfrak{s}_0, \sigma_0)$ by D_0 and Z_0.

Theorem (Dardié, Medina, Revoy)

Every symplectic Lie algebra can be obtained by a sequence of symplectic double extensions starting with the abelian Lie algebra $\mathbb{R}^2$.

Lcs nilpotent Lie algebras

Facts

- If $\mathfrak{g}$ is a nilpotent Lie algebra, $\mathcal{Z}(\mathfrak{g}) \neq 0$
- if (ω, θ) is a lcs structure on $\mathfrak{g}$ nilpotent, $V \in \mathcal{Z}(\mathfrak{g})$
- every lcs structure on a nilpotent Lie algebra is of the first kind

Theorem (—, Marrero)

1. Every lcs nilpotent Lie algebra of dimension $2n + 2$ may be obtained as the lcs extension of a $2n$-dimensional symplectic nilpotent Lie algebra $\mathfrak{s}$ by a symplectic nilpotent derivation.
2. In turn, the symplectic nilpotent Lie algebra $\mathfrak{s}$ may obtained by a sequence of $n - 1$ symplectic double extensions by nilpotent derivations from the abelian Lie algebra of dimension 2.

An example

- Start with the abelian Lie algebra $\mathbb{R}^2$ with symplectic form σ and symplectic derivation

$$D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- let $\mathfrak{g}$ be the lcs extension of $\mathbb{R}^2$ by D: $\mathfrak{g} = (\mathbb{R} \circledcirc_\sigma \mathbb{R}^2) \rtimes_D \mathbb{R}$
- G, the unique connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, has a lcs structure of the first kind
- the structure constants of $\mathfrak{g}$ are rational numbers. By a results of Malcev, there exists a lattice $\Gamma \subset G$. Then $N = \Gamma \backslash G$ is a nilmanifold with a lcs structure of the first kind.

Another answer to Ornea-Verbitski question

Theorem (−, Marrero)

$N = \Gamma\backslash G$ is a 4-dimensional nilmanifold endowed with a lcs structure of the first kind. It is not homeomorphic to a product $P \times S^1$. Moreover, it carries no locally conformally Kähler metric.

Final remarks

- The first example of a symplectic manifold with no Kähler structure (Thurston, 1976) is also a nilmanifold
- Conjecture: a nilmanifold N^{2n} with a lcK structure is a quotient of $H^{2n-1} \times \mathbb{R}$.

Thank you very much!

slides available at https://sites.google.com/site/gbazzoni/

- D. Angella, G. Bazzoni and M. Parton, *Four dimensional locally conformal symplectic Lie algebras, in preparation*
- G. Bazzoni and J. C. Marrero, *On locally conformal symplectic manifolds of the first kind, preprint*, `arXiv:1510.04947`