Castle Rauischholzhausen – March 2016

Lie Theory and Geometry

Spin and metaplectic structures on homogeneous spaces

(arXiv:1602.07968)

(joint work with D. V. Alekseevsky)

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- (M^n, g) connected oriented pseudo-Riemannian mnfd, signature (p, q)
- $\pi : P = \mathrm{SO}(M) \to M$ the $\mathrm{SO}_{p,q}$-principal bundle of positively oriented orthonormal frames.

$$TM = \mathrm{SO}(M) \times_{\mathrm{SO}_{p,q}} \mathbb{R}^n = \eta_- \oplus \eta_+,$$

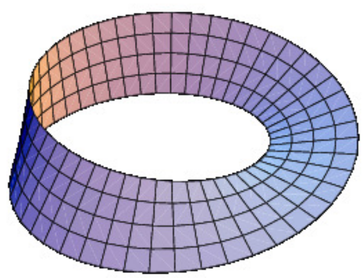
Def. (M^n, g) is called *time-oriented* (resp. *space-oriented*) if η_- (resp. η_+) is oriented.

- *time-oriented:* if and only if $H^1(M; \mathbb{Z}_2) \ni w_1(\eta_-) = 0$
- *space-oriented:* if and only if $H^1(M; \mathbb{Z}_2) \ni w_1(\eta_+) = 0$
- *oriented:* if and only if $H^1(M; \mathbb{Z}_2) \ni w_1(M) := w_1(TM) = w_1(\eta_-) + w_1(\eta_+) = 0$

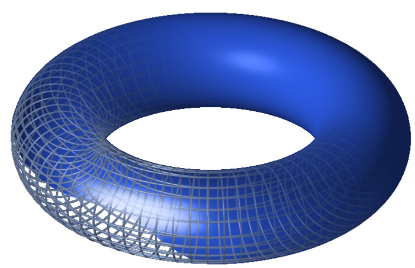
Examples: $\Rightarrow$ **Trivial line bundle** $M = \mathrm{S}^1 \times \mathbb{R}$, $w_1(M) = 0$,

$\Rightarrow$ **Torus** $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$, $w_1(\mathbb{T}^2) = 0$

$\not\Rightarrow$ **Möbius strip** $S = \mathrm{S}^1 \times_G \mathbb{R} \to \mathrm{S}^1$, $G = \mathbb{Z}_2$, $w_1(S) \neq 0$



$w_1 \neq 0$



$w_1 = 0$

Remark: Since $H^1(M; \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(M); \mathbb{Z}_2) \Rightarrow$ if M is simply-connected then it is oriented

- $\mathrm{C}\ell_{p,q} = \mathrm{C}\ell(\mathbb{R}^{p,q}) = \Sigma_{r=0}^{\infty} \otimes^r \mathbb{R}^{p,q} / < x \otimes x - \langle x, x \rangle \cdot 1 >$
- $\mathrm{C}\ell_{p,q} = \mathrm{C}\ell^0_{p,q} + \mathrm{C}\ell^1_{p,q}$
- $\mathrm{Spin}_{p,q} := \{x_1 \cdots x_{2k} \in \mathrm{C}\ell^0_{p,q} : x_j \in \mathbb{R}^{p,q}, \langle x_j, x_j \rangle = \pm 1\} \subset \mathrm{C}\ell^0_{p,q}$
- $\mathrm{Ad} : \mathrm{Spin}_{p,q} \to \mathrm{SO}_{p,q}$ double covering

Def. A $\mathrm{Spin}_{p,q}$-structure (shortly *spin structure*) on (M^n, g)

- a $\mathrm{Spin}_{p,q}$-principal bundle $\tilde{\pi} : Q = \mathrm{Spin}(M) \to M$ over M,
- a $\mathbb{Z}_2$-cover $\Lambda : \mathrm{Spin}(M) \to \mathrm{SO}(M)$ of $\pi : \mathrm{SO}(M) \to M$, such that:

$$\begin{array}{ccccc}
\mathrm{Spin}(M) \times \mathrm{Spin}_{p,q} & \longrightarrow & \mathrm{Spin}(M) & \overset{\tilde{\pi}}{\searrow} & \\
\Lambda \times \mathrm{Ad} \downarrow & & \downarrow \Lambda & & \\
\mathrm{SO}(M) \times \mathrm{SO}_{p,q} & \longrightarrow & \mathrm{SO}(M) & \xrightarrow{\pi} & M
\end{array}$$

- If such a pair (Q, Λ) exists, we shall call (M^n, g) a **pseudo-Riemannian spin manifold.**
- if the manifold is *time-oriented* and *space-oriented*, then a Spin^+-structure is a reduction Q^+ of the $\mathrm{SO}^+(n, k)$-principal bundle P^+ of positively time- and space-oriented orthonormal frames, onto the spin group $\mathrm{Spin}^+(n, k) = \mathrm{Ad}^{-1}(\mathrm{SO}^+(n, k))$, with $Q^+ \to P^+$ being the double covering.

Obstructions

- (M,g) **oriented** pseudo-Riemannian manifold [Karoubi'68, Baum'81]

$$\exists \text{ spin structure} \quad \Leftrightarrow \quad w_2(\eta_-) + w_2(\eta_+) = 0 \quad (*)$$

$$\Leftrightarrow w_2(M) = w_1(\eta_-) \smile w_1(\eta_+) \quad (**)$$

Remark: If $(*)$ or $(**)$ holds, $\Longrightarrow$ set of spin structures on (M,g) $\Longleftrightarrow$ elements in $H^1(M;\mathbb{Z}_2)$.

...a bit more special cases:

- (M,g) **time-oriented** + **space oriented** pseudo-Riemannian manifold

$$\exists \text{ spin structure} \quad \Leftrightarrow \quad w_2(M) = 0$$

- (M,g) **oriented** Riemannian manifold

$$\exists \text{ spin structure} \quad \Leftrightarrow \quad w_2(M) = 0$$

- (M,J) **compact (almost) complex** mnfd. Then, $c_1(M) := c_1(TM,J) = w_2(TM) \pmod 2$

$$\exists \text{ spin structure} \quad \Leftrightarrow \quad c_1(M) \text{ is } \textbf{even} \text{ in } H^2(M;\mathbb{Z}).$$

Examples

- $$\underbrace{\mathbb{C}P^1 = \mathrm{SU}_2\,/\,\mathrm{U}_1,\ \mathbb{C}P^3_{\mathrm{irr}} = \mathrm{SU}_4\,/\,\mathrm{U}_3,\ \mathbb{C}P^3 = \mathrm{SO}_5\,/\,\mathrm{U}_2,\ \mathbb{F}_{1,2} = \mathrm{SU}_3\,/\mathrm{T}_{\max},\ \mathrm{G}_2\,/\mathrm{T}_{\max}, \ldots}_{\Rightarrow\ w_1 = w_2 = 0}$$

- $\overset{\mathbb{C}P^2 = \mathrm{SU}_3\,/\,\mathrm{U}_2}{\text{Riemannian}}$, $\overset{\mathbb{C}P^2 - \text{point}}{\text{Lorentzian}}$, $\mathrm{G}_2\,/\,\mathrm{U}_2, \ldots \Rightarrow\ w_1 = 0$ but $w_2 \neq 0$

- parallelizable mnfds (e.g. Lie groups), etc

Classification results

- special structures often imply existence of a spin structure, e.g. G_2-mndfs, nearly-Kähler mnfds, Einstein-Sasaki mnfds, 3-Sasakian mnfds $\Rightarrow$ **spin** [Friedrich-Kath-Moroianu-Semmelmann '97, Boyer-Galicki '90]

  (but: also the spin structure defines the special structure, sometimes!)

- classification of spin symmetric spaces [Cahen-Gutt-Trautman '90]

- classification of spin pseudo-symmetric spaces & non-symmetric cyclic Riemannian mnfds $(G/L, g)$ [Gadea-González-Dávila-Oubiña '15]

  $$\mathfrak{S}_{X,Y,Z}\langle [X, Y]_{\mathfrak{m}}, Z\rangle = 0, \quad \forall\ X, Y, Z \in \mathfrak{m} \cong T_o G/K$$

  $\Leftrightarrow$ type $\mathcal{T}_1 \oplus \mathcal{T}_2$ (Vanhecke-Tricceri classification)

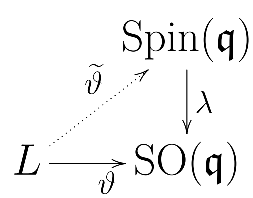
- What new we can say? $\Longrightarrow$ Classification of spin flag manifolds

Invariant spin structures

Def. A spin structure $\tilde{\pi} : Q \to M$ on a homogeneous pseudo-Riemannian manifold $(M = G/L, g)$ is called **G-invariant** if the natural action of G on the bundle $\pi : P \to M$ of positively oriented orthonormal frames, can be extended to an action on the $\mathrm{Spin}_{p,q} \equiv \mathrm{Spin}(\mathfrak{q})$-principal bundle $\tilde{\pi} : Q \to M$. Similarly for spin^{+} structures.

- Fix $(M = G/L, g)$ oriented homogeneous pseudo-Riemannian manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$.
- $\mathrm{Ad} : \mathrm{Spin}(\mathfrak{q}) \to \mathrm{SO}(\mathfrak{q})$.

Thm. [Cahen-Gutt '91] (a) Given a lift of the isotropy representation onto the spin group $\mathrm{Spin}(\mathfrak{q})$, i.e. a homomorphism $\widetilde{\vartheta} : L \to \mathrm{Spin}(\mathfrak{q})$ which makes the following diagram commutative, then M admits a G-invariant spin structure given by $Q = G \times_{\widetilde{\vartheta}} \mathrm{Spin}(\mathfrak{q})$.



(b) Conversely, if G is simply-connected and $(M = G/L, g)$ has a spin structure, then ϑ lifts to $\mathrm{Spin}(\mathfrak{q})$, i.e. the spin structure is G-invariant. Hence in this case there is a one-to-one correspondence between the set of spin structures on $(M = G/L, g)$ and the set of lifts of ϑ onto $\mathrm{Spin}(\mathfrak{q})$.

Metaplectic structures

- $(V = \mathbb{R}^{2n},\, \omega)$ symplectic vector space
- $\mathrm{Sp}(V) = \mathrm{Sp}_n(\mathbb{R}) := \mathrm{Aut}(V, \omega)$ the symplectic group.
- $\mathrm{Sp}_n(\mathbb{R})$ is a connected Lie group, with $\pi_1(\mathrm{Sp}_n(\mathbb{R})) = \mathbb{Z}$.
- Metaplectic group $\mathrm{Mp}_n(\mathbb{R})$ is the unique connected (double) covering of $\mathrm{Sp}_n(\mathbb{R})$
- (M^{2n}, ω) symplectic manifold, $\mathrm{Sp}(M) \to M$ is the $\mathrm{Sp}_n(\mathbb{R})$-principal bundle of symplectic frames

Def. A **metaplectic structure** on a symplectic manifold (M^{2n}, ω) is a $\mathrm{Mp}_n(\mathbb{R})$-equivariant lift of the symplectic frame bundle $\mathrm{Sp}(M) \to M$ with respect to the double covering $\rho : \mathrm{Mp}_n\mathbb{R} \to \mathrm{Sp}_n\mathbb{R}$.

- (M^{2n}, ω) **symplectic manifold**

$\exists$ metaplectic structure $\Leftrightarrow$ $w_2(M) = 0$

$\Leftrightarrow$ $c_1(M)$ is **even**

Remark: Then, the set of metaplectic structures on $(M^{2n}, \omega) \Longleftrightarrow$ elements in $H^1(M; \mathbb{Z}_2)$.

Compact homogeneous symplectic manifolds

• compact homogeneous symplectic manifold $(M^{2n} = G/H, \omega)$+alomost effective action of G connected. Then $\Rightarrow$

- $G = G' \times R$, $\quad G' =$ compact, semisimple, $\; R =$ solvable $\quad \Longrightarrow$

- $M = F \times N$, $\quad N =$ flag manifold, $\; N =$ solvmanifold with symplectic structure

$\longrightarrow$ In particular, any simply-connected compact homogeneous symplectic manifold $(M = G/H, \omega)$ is symplectomorphic to a flag manifold.

Prop. Simply-connected compact homogeneous symplectic manifolds admitting a metaplectic structure, are exhausted by **flag manifolds** $F = G/H$ of a compact simply-connected semisimple Lie group G such that $w_2(F) = 0$, or equivalently $c_1(F; J) =$ even, for some ω-compatible complex structure J.

• This is equivalent to say that the isotropy representation $\vartheta : H \to \mathrm{Sp}(\mathfrak{m})$ lifts to $\mathrm{Mt}(\mathfrak{m})$, i.e. there exists (*unique*) homomorphism $\tilde{\vartheta} : H \to \mathrm{Mt}(\mathfrak{m})$ such that

$$\begin{array}{ccc} & & \mathrm{Mt}(\mathfrak{m}) \\ & \overset{\tilde{\vartheta}}{\nearrow} & \downarrow \rho \\ H & \xrightarrow[\vartheta]{} & \mathrm{Sp}(\mathfrak{m}) \end{array}$$

Homogeneous fibrations and spin structures

... in the spirit of Borel-Hirzebruch

- Let $L \subset H \subset G$ be *compact & connected* subgroups of a compact connected Lie group G.
- $\pi : M = G/L \to F = G/H$ (homogeneous fibration), base space $F = G/H$, fibre H/K.
- Fix an Ad_L-invariant reductive decomposition for $M = G/L$,

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{q} = \mathfrak{l} + (\mathfrak{n} + \mathfrak{m}), \quad \mathfrak{q} := \mathfrak{n} + \mathfrak{m} = T_{eL}M.$$

such that:

- $\mathfrak{h} = \mathfrak{l} + \mathfrak{n}$ is a reductive decomposition of H/L,
- $\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{l} + \mathfrak{n}) + \mathfrak{m}$ is a reductive decomposition of $F = G/H$.
- An Ad_L-invariant (pseudo-Euclidean) metric $g_{\mathfrak{n}}$ in $\mathfrak{n}$ $\Rightarrow$ a (pseudo-Riemannian) invariant metric in H/L
- An Ad_H-invariant (pseudo-Euclidean) metric $g_{\mathfrak{m}}$ in $\mathfrak{m}$ $\Rightarrow$ a (pseudo-Riemannian) invariant metric in the base $F = G/H$.

$\Rightarrow$ The direct sum metric $g_{\mathfrak{q}} = g_{\mathfrak{n}} \oplus g_{\mathfrak{m}}$ in $\mathfrak{q}$ $\Rightarrow$ an invariant pseudo-Riemannian metric in $M = G/L$ such that $\pi : G/L \to G/H$ is a pseudo-Riemannian submersion with totally geodesic fibres.

- $N := H/L \overset{i}{\hookrightarrow} M := G/L \overset{\pi}{\to} F := G/H$

Prop. (i) The bundles $i^*(TM)$ and TN are stably equivalent.
(ii) The Stiefel-Whitney classes of the fiber $N = H/L$ are in the image of the homomorphism $i^* : H^*(M; \mathbb{Z}_2) \to H^*(N; \mathbb{Z}_2)$, induced by the inclusion map $i : N = H/L \hookrightarrow M = G/L$, and
(iii)

$$w_1(TM) = 0, \quad w_2(TM) = w_2(\tau_N) + \pi^*(w_2(TF)),$$

Hints:

- $\tau_N := G \times_L \mathfrak{n} \to G/L$ is the tangent bundle along the fibres (with fibres, the tangent spaces $\mathfrak{n} \cong T_{eL}N$ of the fibres $\pi^{-1}(x) \cong H/L := N$ $(x \in F)$).

$$TM = G \times_L \mathfrak{q} = G \times_L (\mathfrak{n} + \mathfrak{m}) = (G \times_L \mathfrak{n}) \oplus (G \times_L \mathfrak{m}) := \tau_N \oplus \pi^*(TF).$$

$\Rightarrow TN = H \times_L \mathfrak{n} \cong i^*(\tau_N)$, and

$$(i^* \circ \pi^*)(TF) = (\pi \circ i)^*(TF) = \epsilon^{\dim F},$$

$\epsilon^t :=$ trivial real vector bundle of rank t. Thus,

$$i^*(TM) = \epsilon^{\dim F} \oplus TN.$$

Due to naturality of Stiefel-Whitney classes we get that

$$w_j(i^*(TM)) = i^*(w_j(TM)) = w_j(TN),$$

or equivalently, $i^*(w_j(M)) = w_j(N)$.
Final step:

$$\begin{aligned} w_2(TM) &= w_2(\tau_N \oplus \pi^*(TF)) = w_2(\tau_N) + w_1(\tau_N) \smile w_1(\pi^*(TF)) + w_2(\pi^*(TF)) \\ &= w_2(\tau_N) + w_2(\pi^*(TF)) = w_2(\tau_N) + \pi^*(w_2(TF)). \end{aligned}$$

Corol. Let $N := H/L \overset{i}{\hookrightarrow} M := G/L \overset{\pi}{\to} F := G/H$, as above. Then:

$\alpha)$ If $F = G/H$ is spin, then $M = G/L$ is spin if and only if $N = H/L$ is spin.

$\beta)$ If $N = H/L$ is spin, then $M = G/L$ is spin if and only if

$$w_2(G/H) \equiv w_2(TF) \in \ker \pi^* \subset H^2(F; \mathbb{Z}_2),$$

where $\pi^* : H^2(F; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}_2)$ is the induced homomorphism by π.

- In particular, if N and F are spin, so is M with respect to any pseudo-Riemannian metric.

Hints: Consider the injection $i : N \hookrightarrow M$. Then $i^*(\tau_N) = TN$, $\tau_N = i_*(TN)$ and

$$TM = \tau_N \oplus \pi^* TF = i_*(TN) \oplus \pi^* TF.$$

Remark: If M is G-spin and N is H-spin, then $\pi^*(w_2(TF)) = w_2(\pi^*(TF)) = 0$, which in general does not imply the relation $w_2(TF) = 0$, i.e. F is not necessarily spin.

Example: Hopf fibration

$$\mathrm{S}^1 \to \mathrm{S}^{2n+1} = \mathrm{SU}_{n+1} / \mathrm{SU}_n \to \mathbb{C}P^n = \mathrm{SU}_{n+1} / \mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_n).$$

$\longrightarrow$Although the sphere S^{2n+1} is a spin manifold for any n (its tangent bundle is stably trivial), $\mathbb{C}P^n$ is spin only for $n =$ odd.

Generalized Flag manifolds: $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathrm{P}$

- G compact, connected, semisimple Lie group

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (Z(\mathfrak{h}) + \mathfrak{h}') + \mathfrak{m}$$

- $\mathrm{T}^{\ell} \subset \mathbf{H} \subset G$ maximal torus.
- H := centralizer of torus $S \subset G$
- $\mathfrak{a} = \mathrm{Lie}(\mathrm{T}^{\ell}) = T_e\mathrm{T}^{\ell}$ = max. abelian subalgebra $\Rightarrow$ $\mathfrak{a}^{\mathbb{C}}$ is a common CSA
- Set:

$$\mathfrak{a}_0 := i\mathfrak{a}, \quad \mathfrak{z} := Z(\mathfrak{h}), \quad \mathfrak{t} := i\mathfrak{z} \subset \mathfrak{a}_0$$

- Let $\mathbf{R}$, $\mathbf{R_H}$ be the root systems of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$, $(\mathfrak{h}^{\mathbb{C}}, \mathfrak{a}^{\mathbb{C}})$, respectively.
- $\mathbf{\Pi_W} = \{\alpha_1, \ldots, \alpha_{\mathbf{u}}\}$ fundamental system of $\mathbf{R_H}$.
- Extend to a fundamental system $\mathbf{\Pi}$ of $\mathbf{R}$,

$$\mathbf{\Pi} = \mathbf{\Pi_W} \sqcup \mathbf{\Pi_B} = \{\alpha_1, \ldots, \alpha_{\mathbf{u}}\} \sqcup \{\beta_1, \ldots, \beta_{\mathbf{v}}\}, \quad \ell = \mathbf{u} + \mathbf{v}.$$

- Consider the corresponding systems of positive roots $\mathbf{R}^+$ and $\mathbf{R_H}^+$.

Def.

- $\mathbf{\Pi_B} := \mathbf{\Pi} \backslash \mathbf{\Pi_W}$ = **black (simple) roots.**

- $\mathbf{R_F} := \mathbf{R} \backslash \mathbf{R_H}$ = **complementary roots.**

- $\mathfrak{h}^{\mathbb{C}} = Z(\mathfrak{h}^{\mathbb{C}}) \oplus \mathfrak{h}^{\mathbb{C}}_{ss} = \mathfrak{t}^{\mathbb{C}} \oplus (\mathfrak{h}')^{\mathbb{C}}$, where

$$\Longrightarrow \quad (\mathfrak{h}')^{\mathbb{C}} = \mathfrak{g}(\mathbf{R_H}) = \mathfrak{a}' + \sum_{\alpha \in \mathbf{R_H}} \mathfrak{g}_{\alpha}, \quad \mathfrak{a}' := \sum_{\alpha \in \mathbf{\Pi_W}} \mathbb{C}H_{\alpha} \subset \mathfrak{a}^{\mathbb{C}}.$$

$\to$ Then $\mathfrak{h} = i\mathfrak{t} + \mathfrak{h}'$ is the standard compact real form of the complex reductive Lie algebra $\mathfrak{h}^{\mathbb{C}}$.

• Let Λ_{β_i} (or simply by Λ_i) be the fundamental weights associated to the black simple roots $\beta_i \in \mathbf{\Pi_B}$.

• In terms of the splitting $\mathbf{\Pi} = \mathbf{\Pi_W} \sqcup \mathbf{\Pi_B}$

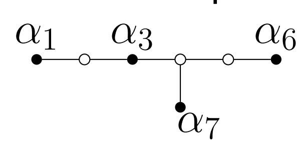
$$(\Lambda_i|\beta_j) := \frac{2(\Lambda_i, \beta_j)}{(\beta_j, \beta_j)} = \delta_{ij}, \quad (\Lambda_i|\alpha_k) = 0.$$

Lemma. The fundamental weights $(\Lambda_1, \cdots, \Lambda_v)$ associated with the black simple roots Π_B, form a basis of the space $\mathfrak{t}^* \cong \mathfrak{t}$

Def. By painting black in the Dynkin diagram of G the nodes corresponding to the black roots from $\mathbf{\Pi_B}$ we get the **painted Dynkin diagram** (PDD) of the flag manifold $F = G/\mathbf{H}$.

• The PDD graphically represents the splitting $\mathbf{\Pi} = \mathbf{\Pi_W} \sqcup \mathbf{\Pi_B}$. The subdiagram generated by the white nodes, i.e. the simple roots in $\mathbf{\Pi_W}$, defines the semisimple part $\mathbf{H}'$ of $\mathbf{H}$.

Example. Let $G = \mathrm{E}_7$ and consider the painted Dynkin diagram



$$\Bigg\} \quad = \mathrm{E}_7(2, 4, 5).$$

• It defines the flag manifold $F = \mathrm{E}_7 / \mathrm{SU}_3 \times \mathrm{SU}_2 \times \mathrm{U}_1^4$, with $\mathbf{\Pi_W} = \{\alpha_2, \alpha_4, \alpha_5\}$ and $\mathbf{\Pi_B} = \{\alpha_1, \alpha_3, \alpha_6, \alpha_7\}$, respectively. Hence $\dim \mathfrak{t} = 4 = \mathrm{rnk} R_T = b_2(F)$.

- Roots from $\mathbf{R_F} = \mathbf{R_F^+} \sqcup (-\mathbf{R_F^+})$ determine the complexified tangent space $(T_oF)^{\mathbb{C}} = \mathfrak{m}^{\mathbb{C}}$

$$\mathfrak{m}^{\mathbb{C}} := \mathfrak{m}^{10} + \mathfrak{m}^{01} = \sum_{\alpha\in\mathbf{R_F^+}} \mathbb{C}E_\alpha + \sum_{\alpha\in\mathbf{R_F^-}} \mathbb{C}E_\alpha, \quad \text{with } \overline{\mathfrak{m}^{10}} = \mathfrak{m}^{01},\ \overline{\mathfrak{m}^{01}} = \mathfrak{m}^{10}.$$

- This defines an (integrable) invariant complex structure J

$$J_oE_{\pm\alpha} = \pm iE_{\pm\alpha}, \quad \forall\alpha \in \mathbf{R_F^+},$$

$\Longrightarrow$ We identify $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathbf{P}$, where $\mathbf{H} = \mathbf{P} \cap G$, $\mathbf{P} \subset G^{\mathbb{C}}$ parabolic subgroup

$$\begin{aligned}\mathfrak{p}_{\Pi_\mathrm{W}} := \mathfrak{a}^{\mathbb{C}} + \sum_{\alpha\in\mathbf{R_H}\cup\mathbf{R_F^+}} \mathfrak{g}_\alpha &= \mathfrak{a}^{\mathbb{C}} + \sum_{\alpha\in\mathbf{R_H}} \mathfrak{g}_\alpha + \sum_{\alpha\in\mathbf{R_F^+}} \mathfrak{g}_\alpha \\ &= \mathfrak{h}^{\mathbb{C}} + \mathfrak{n}_+.\end{aligned}$$

$\Longrightarrow$ $B_+ \subset G^{\mathbb{C}}$ the Borel subgroup corresponding to the maximal solvable subalgebra

$$\mathfrak{b}^+ := \mathfrak{a}^{\mathbb{C}} + \sum_{\alpha\in\mathbf{R}^+} \mathfrak{g}_\alpha = \mathfrak{a}^{\mathbb{C}} + \mathfrak{g}(\mathbf{R}^+) \subset \mathfrak{g}^{\mathbb{C}}.$$

$\Longrightarrow$ $\Pi_\mathrm{W} = \emptyset$ and $\Pi_\mathrm{W} = \Pi$ define the spaces $\mathfrak{b}^+$ and $\mathfrak{g}^{\mathbb{C}}$, respectively.

Prop. [Borel-Hirzebruch '51, Alekseevsky '76] There is a 1-1 bijective correspondence between,

- Invariant complex structures on a flag manifold $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathbf{P}$
- extensions of a fixed fundamental system Π_W of the subalgebra $\mathfrak{h}^{\mathbb{C}}$ to a fundamental system Π of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.
- papabolic subalgebras $\mathfrak{p}_{\Pi_\mathrm{W}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{n}_+$ with reductive part $\mathfrak{h}^{\mathbb{C}}$

T-roots and applications

$$\begin{aligned} \mathfrak{t} &:= i\mathfrak{z} \subset \mathfrak{a}_0 \text{ where } \mathfrak{z} := Z(\mathfrak{h}) \\ &= \{X \in \mathfrak{a}_0 : \alpha_i(X) = 0, \text{ for all } \alpha_i \in \mathbf{\Pi_W}\}. \end{aligned}$$

$\Longrightarrow$ Consider the linear restriction map

$$\kappa : \mathfrak{a}^* \to \mathfrak{t}^*, \ \alpha \mapsto \alpha|_{\mathfrak{t}}$$

- Then: $\mathbf{R_H} = \{\alpha \in \mathbf{R}, \ \kappa(\alpha) = 0\}$.

Def.

$$R_T := \text{the restriction of } \mathbf{R_F} \text{ on } \mathfrak{t} = \kappa(\mathbf{R_F}) = \kappa(\mathbf{R}).$$

Elements in R_T are called **T-roots**. Notice that: $\mathbf{v} := \sharp(\mathbf{\Pi_B}) = \mathrm{rnk} R_T$.

Thm. [Siebenthal '64, Alekseevsky '76]
There exists an 1-1 correspondence between $\mathfrak{t}$-roots and complex irreducible $\mathbf{H}$-submodules $\mathfrak{f}_\xi$ of $\mathfrak{m}^{\mathbb{C}}$. This correspondence is given by

$$R_T \ni \xi \quad \leftrightarrow \quad \mathfrak{f}_\xi := \sum_{\alpha \in \mathbf{R_F} : \kappa(\alpha) = \xi} \mathbb{C} E_\alpha.$$

- Moreover, there is a natural 1-1 correspondence between positive T-roots $\xi \in R_T^+ = \kappa(\mathbf{R_F^+})$ and real pairwise inequivalent irreducible $\mathbf{H}$-submodules $\mathfrak{m}_\xi \subset \mathfrak{m}$, given by

$$R_T^+ \ni \xi \quad \longleftrightarrow \quad \mathfrak{m}_\xi := (\mathfrak{f}_\xi + \mathfrak{f}_{-\xi}) \cap \mathfrak{m} = (\mathfrak{f}_\xi + \mathfrak{f}_{-\xi})^\tau.$$

- Moreover, $\dim_{\mathbb{C}} \mathfrak{f}_\xi = \dim_{\mathbb{R}} \mathfrak{m}_\xi = d_\xi$ where $d_\xi := \sharp(\kappa^{-1}(\xi))$ is the cardinality of $\kappa^{-1}(\xi)$.

Invariant pseudo-Riemannian metrics

Corol. Any G-invariant pseudo-Riemannian metric g on a flag manifold $F = G/\mathbf{H}$ is defined by an $\mathrm{Ad}_{\mathbf{H}}$-invariant pseudo-Euclidean metric on $\mathfrak{m}$, given by

$$g_o := \sum_{i=1}^{d:=R_T^+} x_{\xi_i} B_{\xi_i}, \qquad (B_{\xi_i} := -B|_{\mathfrak{m}_i}),$$

where $x_{\xi_i} \neq 0$ are real numbers, for any $i = 1, \ldots, d := R_T^+$.
The signature of the metric g is $(2N_-, 2N_+)$, where

$$N_- := \sum_{\xi_i \in R_T^+ : x_{\xi_i} < 0} d_{\xi_i}, \ \ N_+ := \sum_{\xi_i \in R_T^+ : x_{\xi_i} > 0} d_{\xi_i}.$$

- In particular, the metric g is Riemannian if all $x_{\xi_i} > 0$, and no metric is *Lorentzian*.

How we deduce that a flag manifold has a spin structure or not?

... by computing the first Chern class for an invariant complex structure

- Consider the weight lattice associated to $\mathbf{R}$, that is

$$\mathcal{P} = \{\Lambda \in \mathfrak{a}_0^* \ : <\Lambda|\alpha> \in \mathbb{Z}, \ \forall \alpha \in \mathbf{R}\} = \mathrm{span}_{\mathbb{Z}}(\Lambda_1, \cdots, \Lambda_\ell) \subset \mathfrak{a}_0^*.$$

- Then set

$$\mathcal{P}_T := \{\lambda \in \mathcal{P}, \ (\lambda, \alpha) = 0, \ \forall \ \alpha \in \mathbf{R}_{\mathrm{H}}\}$$

Lemma. The **T-weight lattice** $\mathcal{P}_T$ is generated by the fundamental weights $\Lambda_1, \cdots, \Lambda_{\mathrm{v}}$ corresponding to the **black simple roots** $\mathbf{\Pi}_{\mathrm{B}} = \mathbf{\Pi} \backslash \mathbf{\Pi}_{\mathrm{W}}$.

Classical result: The group of characters $\mathcal{X}(\mathrm{T}^\ell) = \mathrm{Hom}(\mathrm{T}^\ell, \mathrm{T}^1) = \mathcal{X}(B_+)$ of the maximal torus $\mathrm{T}^\ell \subset \mathbf{H} \subset G$ is identified (when G is simply-connected) with the weight lattice $\mathcal{P} \subset \mathfrak{a}_0^*$, via the map

$$\mathcal{P} \ni \lambda \mapsto \chi_\lambda \in \mathcal{X}(\mathrm{T}^\ell) = \mathcal{X}(B_+), \quad \text{with} \quad \chi_\lambda(\exp X) = \exp(\frac{i\lambda(X)}{2\pi}), \quad \forall X \in \mathfrak{a}_0.$$

Extension: The following map is an isomorphism:

$$\mathcal{P}_T \ni \lambda \mapsto \chi_\lambda \in \mathcal{X}(\mathbf{H}) := \mathrm{Hom}(\mathbf{H}, \mathrm{T}^1).$$

- In particular, since $\mathbf{P} = \mathbf{H}^{\mathbb{C}} \cdot N_+$ any character $\chi = \chi_\lambda : \mathbf{H} \to \mathrm{T}^1$ has a natural extension to a character of the parabolic subgroup $\chi_\lambda^{\mathbb{C}} : \mathbf{P} \to \mathbb{C}^*$ and we get

$$\mathcal{P}_T \ni \lambda \mapsto \chi_\lambda^{\mathbb{C}} \in \mathcal{X}(\mathbf{P})$$

Line bundles and circle bundles

• For any T-weight $\lambda \in \mathcal{P}_T$ we assign a 1-dimensional $\mathbf{P}$-module $\mathbb{C}_\lambda$, where $\mathbf{P}$ acts on $\mathbb{C}_\lambda$ by the associated holomorphic character $\chi_\lambda^{\mathbb{C}} \in \mathcal{X}(\mathbf{P})$.
• We define the line bundle

$$\mathcal{L}_\lambda = G^{\mathbb{C}} \times_{\mathbf{P}} \mathbb{C}_\lambda = (G^{\mathbb{C}} \times \mathbb{C}_\lambda)/\sim$$

$$(g, z) \sim (gp, \chi_\lambda^{\mathbb{C}}(p^{-1})z), \quad (g, z) \in G^{\mathbb{C}} \times \mathbb{C}_\lambda,\ p \in \mathbf{P}.$$

• We also introduce the homogeneous circle bundle associated with the character $\chi : \mathbf{H} \to \mathrm{T}^1$,

$$F_\chi = G/H_\chi \to F = G/\mathbf{H}, \qquad H_\chi := \ker(\chi)$$

Prop. Let $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathbf{P}$ be a flag manifold endowed with a complex structure associated to a splitting $\mathbf{\Pi} = \mathbf{\Pi_W} \sqcup \mathbf{\Pi_B}$. Then, $\exists$ 1-1 correspondence between

- elements $\lambda \in \mathcal{P}_T = \mathrm{span}_{\mathbb{Z}}\{\Lambda_1, \ldots, \Lambda_{\mathbf{v}}\}$ of the T-weight lattice
- real characters $\chi = \chi_\lambda : \mathbf{H} \to \mathrm{T}^1$ (up to congugation),
- complex characters $\chi_\lambda^{\mathbb{C}} : \mathbf{P} \to \mathbb{C}^*$ (up to conjugation),
- holomorphic line bundles $\mathcal{L}_\lambda := G^{\mathbb{C}} \times_{\mathbf{P}} \mathbb{C}_\lambda \to F = G^{\mathbb{C}}/\mathbf{P}$ (up to conjugation)
- and homogeneous circle bundles $F_\chi := G/H_\chi \to F = G/\mathbf{H}$ (up to conjugation).

Prop. There is a natural isomorphism

$$\tau : \mathfrak{t}^* \to \Lambda^2_{cl}(\mathfrak{m}^*)^{\mathbf{H}} \cong H^2(\mathfrak{m}^*)^{\mathbf{H}} \simeq H^2(F, \mathbb{R})$$

between the space $\mathfrak{t}^*$ and the space $\Lambda^2_{cl}(\mathfrak{m}^*)^{\mathbf{H}}$ of $\mathrm{Ad}_{\mathbf{H}}$-invariant closed real 2-forms on $\mathfrak{m}$ (identified with the space of closed G-invariant real 2-forms on F), given by

$$\mathfrak{a}_0^* \supset \mathfrak{t}^* \ni \xi \mapsto \omega_\xi := \tfrac{i}{2\pi} d\xi = \tfrac{i}{2\pi} \textstyle\sum_{\alpha \in \mathbf{R_F}^+} (\xi|\alpha) \omega^\alpha \wedge \omega^{-\alpha} \in \Lambda^2_{cl}(\mathfrak{m}^*)^{\mathbf{H}}.$$

- $\tau(\mathcal{P}_T) \cong H^2(F, \mathbb{Z})$. Thus second Betti number of F equals to $b_2(F) = \dim \mathfrak{t} = \mathbf{v} = \mathrm{rnk} R_T$.

- In particular, the following maps are isomorphisms

$$\mathcal{P}_T \ni \lambda \mapsto \mathcal{L}_\lambda \in \mathcal{P}\mathrm{ic}(F) := H^1(G^{\mathbb{C}}/\mathbf{P}, \underline{\mathbb{C}}^*) \ni \mathcal{L}_\lambda \xrightarrow{c_1} c_1(\mathcal{L}_\lambda) \in H^2(F, \mathbb{Z}).$$

- The first Chern class $c_1(\mathcal{L}_{\xi_j})$ of the holomorphic line bundle $\mathcal{L}_{\xi_j}$ is the cohomology class of the associated curvature two-form

$$\omega_{\xi_j} = \frac{i}{2\pi} d\xi_j = \frac{i}{2\pi} \sum_{\alpha \in \mathbf{R}_\mathbf{F}^+} (\xi_j|\alpha) \omega^\alpha \wedge \omega^{-\alpha} \in \Lambda^2(\mathfrak{m}^*)^{\mathbf{H}} = \Omega^2_{cl}(F).$$

The first Chern class

- Let $\mathcal{P}^+ \subset \mathcal{P}$ be the subset of *strictly positive dominant weights*, and consider the 1-forms

$$\sigma_G = \frac{1}{2}\sum_{\alpha\in\mathbf{R}^+}\alpha, \quad \sigma_{\mathbf{H}} = \frac{1}{2}\sum_{\alpha\in\mathbf{R}^+_{\mathbf{H}}}\alpha.$$

Recall that $\sigma_G = \sum_{i=1}^{\ell}\Lambda_i \in \mathcal{P}^+$.

- We define the **Koszul form** associated to the flag manifold $(F = G^{\mathbb{C}}/\mathbf{P} = G/\mathbf{H}, J)$, by

$$\sigma^J := 2(\sigma_G - \sigma_H) = \sum_{\alpha\in\mathbf{R}^+_{\mathbf{F}}}\alpha.$$

$\Longrightarrow$ The **first Chern class** $c_1(J) \in H^2(F;\mathbb{Z})$ of the invariant complex structure J in F, associated with the decomposition $\mathbf{\Pi} = \mathbf{\Pi_W} \sqcup \mathbf{\Pi_B}$, is represented by the closed invariant 2-form $\gamma_J := \omega_{\sigma^J}$, i.e. the Chern form of the complex manifold (F, J).

Thm. [Alekseevsky '76, Alekseevsky-Perelomov '86] The Koszul form is a linear combination of the fundamental weights $\Lambda_1, \cdots, \Lambda_{\mathbf{v}}$ associated to the black roots, with positive integers coefficients, given as follows:

$$\sigma^J = \sum_{j=1}^{\mathbf{v}} k_j\Lambda_j = \sum_{j=1}^{\mathbf{v}}(2+b_j)\Lambda_j \in \mathcal{P}^+_T, \quad \text{where} \quad k_j = \frac{2(\sigma^J,\beta_j)}{(\beta_j,\beta_j)}, \ b_j = -\frac{2(2\sigma_{\mathbf{H}},\beta_j)}{(\beta_j,\beta_j)} \geqslant 0.$$

Def. The integers $k_j \in \mathbb{Z}_+$ are called **Koszul numbers** associated to the complex structure J on $F = G^{\mathbb{C}}/\mathbf{P} = G/\mathbf{H}$. They form the vector $\vec{k} := (k_1, \ldots, k_{\mathbf{v}}) \in \mathbb{Z}^{\mathbf{v}}_+$, which we shall call the **Koszul vector** associated to J.

Invariant spin structures

Thm. A flag manifold $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathbf{P}$ admits a G-invariant spin or metaplectic structure, if and only is the first Chern class $c_1(J)$ of an invariant complex structure J on F is **even**, that is all Koszul numbers are **even**. If this is the case, then such a structure will be unique.

Example Consider the manifold of full flags $F = G/\mathrm{T}^{\ell} = G^{\mathbb{C}}/B_+$.

- The Weyl group acts transitively on Weyl chambers $\implies$ $\exists$ unique (up to conjugation) invariant complex structure J.
- The canonical line bundle $\Lambda^n TF$ corresponds to the dominant weight $\sum_{\alpha \in R^+} \alpha = 2\sigma_G = 2(\Lambda_1 + \cdots + \Lambda_\ell)$.
- hence all the Koszul numbers equal to 2 and F admits a unique spin structure.

Corol. The divisibility by two of the Koszul numbers of an invariant complex structure J on a (pseudo-Riemannian) flag manifold $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathbf{P}$, does not depend on the complex structure.

Corol. On a spin or metaplectic flag manifold $F = G/\mathbf{H} = G^{\mathbb{C}}/\mathbf{P}$ with a fixed invariant complex structure J, there is a unique isomorphism class of holomorphic line bundles $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} = K_F$.

The computation of Koszul numbers-classical flag manifolds

Classical flag manifolds

- Flag manifolds of the groups $\mathrm{A}_n = \mathrm{SU}_{n+1}, \mathrm{B}_n = \mathrm{SO}_{2n+1}, \mathrm{C}_n = \mathrm{Sp}_n, \mathrm{D}_n = \mathrm{SO}_{2n}$ fall into four classes:

$$\begin{aligned} \mathrm{A}(\vec{n}) &= \mathrm{SU}_{n+1} \,/\, \mathrm{U}_1^{n_0} \times \mathrm{S}(\mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_s}), \\ \vec{n} &= (n_0, n_1, \cdots, n_s), \quad \sum n_j = n+1,\ n_0 \geqslant 0, n_j > 1; \end{aligned}$$

$$\begin{aligned} \mathrm{B}(\vec{n}) &= \mathrm{SO}_{2n+1} \,/\, \mathrm{U}_1^{n_0} \times \mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_s} \times \mathrm{SO}_{2r+1}, \\ \vec{n} &= \sum n_j + r,\ n_0 \geqslant 0, n_j > 1, r \geqslant 0; \end{aligned}$$

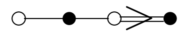
$$\begin{aligned} \mathrm{C}(\vec{n}) &= \mathrm{Sp}_n \,/\, \mathrm{U}_1^{n_0} \times \mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_s} \times \mathrm{Sp}_r, \\ \vec{n} &= \sum n_j + r,\ n_0 \geqslant 0, n_j > 1, r \geqslant 0; \end{aligned}$$

$$\begin{aligned} \mathrm{D}(\vec{n}) &= \mathrm{SO}_{2n} \,/\, \mathrm{U}_1^{n_0} \times \mathrm{U}_{n_1} \times \cdots \times \mathrm{U}_{n_s} \times \mathrm{SO}_{2r}, \\ \vec{n} &= \sum n_j + r,\ n_0 \geqslant 0, n_0 \geqslant 0, n_j > 1, r \neq 1, \end{aligned}$$

with $\vec{n} = (n_0, n_1, \cdots, n_s, r)$ for the groups $\mathrm{B}_n, \mathrm{C}_n$ and D_n.

The Koszul vector of classical flag manifolds

Example: Consider the flag manifold $F = \mathrm{SO}_9 \,/\, \mathrm{U}_1^2 \times \mathrm{SU}_2 \times \mathrm{SU}_2 = \mathrm{SO}_9 \,/\, \mathrm{U}_2 \times \mathrm{U}_2$



- It is $\mathbf{\Pi_B} = \{\alpha_2, \alpha_4\}$ and $\mathbf{\Pi_W} = \mathbf{R}^+_{\mathbf{H}} = \{\alpha_1, \alpha_3\}$.

$\Rightarrow$ $2\sigma_{\mathbf{H}} = \alpha_1 + \alpha_3$. Since $2\sigma_{\mathrm{SO}_9} = 7\alpha_1 + 12\alpha_2 + 15\alpha_3 + 16\alpha_4$, we conclude that

$$\sigma^{J_0} = 6\alpha_1 + 12\alpha_2 + 14\alpha_3 + 16\alpha_4.$$

- By the Cartan matrix of SO_9 we finally get $\sigma^{J_0} = 4\Lambda_2 + 4\Lambda_4$. Thus F admits a unique spin structure.

Thm. The Koszul vector $\vec{k} := (k_1, \cdots, k_v) \in \mathbb{Z}^v_+$ associated to the standard complex structure J_0 on a flag manifold $G(\vec{n})$ of classical type, is given by

$$\begin{array}{lll}
\mathrm{A}(\vec{n}): & \vec{k} = & (2, \cdots, 2, 1 + n_1, n_1 + n_2, \cdots, n_{s-1} + n_s), \\
\mathrm{B}(\vec{n}): & \vec{k} = & (2, \cdots, 2, 1 + n_1, n_1 + n_2, \cdots, n_{s-1} + n_s, n_s + 2r), \\
\mathrm{C}(\vec{n}): & \vec{k} = & (2, \cdots, 2, 1 + n_1, n_1 + n_2, \cdots, n_{s-1} + n_s, n_s + 2r + 1), \\
\mathrm{D}(\vec{n}): & \vec{k} = & (2, \cdots, 2, 1 + n_1, n_1 + n_2, \cdots, n_{s-1} + n_s, n_s + 2r - 1).
\end{array}$$

If $r = 0$, then the last Koszul number (over the end black root) is $2n_s$ for $\mathrm{B}(\vec{n})$, $n_s + 1$ for $\mathrm{C}(\vec{n})$ and $2(n_s - 1)$ for $\mathrm{D}(\vec{n})$.

Hence we conclude that (the same conclusions apply also for G-metaplectic structures):

Thm. (classification of spin or metaplectic structures)

α) The flag manifold $\mathrm{A}(\vec{n})$ with $n_0 > 0$ is G-spin if and only if all the numbers $n_1, \ldots, n_s$ are odd. If $n_0 = 0$, then $\mathrm{A}(\vec{n})$ is G-spin, if and only if the numbers $n_1, \ldots, n_s$ have the same parity, i.e. they are all odd or all even.

β) The flag manifold $\mathrm{B}(\vec{n})$ with $n_0 > 0$ and $r > 0$ does not admit a (G-invariant) spin structure. If $n_0 > 0$ and $r = 0$, then $\mathrm{B}(\vec{n})$ is G-spin, if and only if all the numbers $n_1, \ldots, n_s$ are odd. If $n_0 = 0$ and $r > 0$, then $\mathrm{B}(\vec{n})$ is G-spin if and only if all the numbers $n_1, \ldots, n_s$ are even. Finally, for $n_0 = 0 = r$, the flag manifold $\mathrm{B}(\vec{n})$ is G-spin if and only if all the numbers $n_1, \ldots, n_s$ have the same parity.

γ) The flag manifold $\mathrm{C}(\vec{n})$ with $n_0 > 0$ is G-spin, if and only if all the numbers $n_1, \ldots, n_s$ are odd, independently of r. The same holds if $n_0 = 0$.

δ) The flag manifold $\mathrm{D}(\vec{n})$ with $n_0 > 0$ is G-spin, if and only if all the numbers $n_1, \ldots, n_s$ are odd, independently of r. If $n_0 = 0$ and $r > 0$, then $\mathrm{D}(\vec{n})$ is G-spin, if and only if all the numbers $n_1, \ldots, n_s$ are odd. Finally, for $n_0 = 0 = r$, the flag manifold $\mathrm{D}(\vec{n})$ is G-spin, if and only if the numbers $n_1, \ldots, n_s$ have the same parity.

Table 1. Spin or metaplectic classical flag manifolds with $b_2 = 1, 2$.

$F = G/\mathrm{H}$ with $b_2(F) = 1$	conditions	d	$k_{\alpha_{i_o}} \in \mathbb{Z}_+$	G-spin ($\Leftrightarrow$)
$\mathrm{SU}_n \,/\, \mathrm{S}(\mathrm{U}_p \times \mathrm{U}_{n-p})$	$n \geqslant 2, 1 \leqslant p \leqslant n-1$	1	(n)	n even $\geqslant 2$
$\mathrm{Sp}_n \,/\, \mathrm{U}_n$	$n \geqslant 3$	1	$(n+1)$	n odd $\geqslant 3$
$\mathrm{SO}_{2n} \,/\, \mathrm{SO}_2 \times \mathrm{SO}_{2n-2}$	$n \geqslant 4$	1	$(2n-2)$	$\forall\ n \geqslant 4$
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_n$	$n \geqslant 3$	1	$(2n-4)$	$\forall\ n \geqslant 3$
$\mathrm{SO}_{2n+1} \,/\, \mathrm{U}_p \times \mathrm{SO}_{2(n-p)+1}$	$n \geqslant 2, 2 \leqslant p < n$	2	$(2n-p)$	p even $\geqslant 2$
$\mathrm{SO}_{2n+1} \,/\, \mathrm{U}_n$ (special case)	$n \geqslant 2$	2	$(2n)$	$\forall\ n \geqslant 2$
$\mathrm{Sp}_n \,/\, \mathrm{U}_p \times \mathrm{Sp}_{n-p}$	$n \geqslant 3, 1 \leqslant p \leqslant n-1$	2	$(2n-p+1)$	p odd $\geqslant 1$
$\mathrm{Sp}_n \,/\, \mathrm{U}_1 \times \mathrm{Sp}_{n-1} =: \mathbb{C}P^{2n-1}$	$n \geqslant 3$	2	$(2n)$	$\forall\ n \geqslant 3$
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_p \times \mathrm{SO}_{2(n-p)}$	$n \geqslant 4, 2 \leqslant p \leqslant n-2$	2	$(2n-p-1)$	p odd $\geqslant 2$
$F = G/\mathrm{H}$ with $b_2(F) = 2$	conditions	d	$\vec{k} \in \mathbb{Z}^2_+$	G-spin ($\Leftrightarrow$)
$\mathrm{SU}_n \,/\, \mathrm{U}_1 \times \mathrm{S}(\mathrm{U}_{p-1} \times \mathrm{U}_{n-p})$	$n \geqslant 3, 2 \leqslant p \leqslant n-2$	3	$(p, n-1)$	n odd & p even
$\mathrm{SU}_3 \,/ \mathrm{T}^2$ (special case)	-	3	$(2,2)$	yes
$\mathrm{SU}_n \,/\, \mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q \times \mathrm{U}_{n-p-q})$	$n \geqslant 5, 2 \leqslant p \leqslant n-2$ $4 \leqslant p+q \leqslant n$	3	$(p+q, n-p)$	p, q, n same parity
$\mathrm{SO}_5 \,/ \mathrm{T}^2$ (special case)	-	4	$(2,2)$	yes
$\mathrm{SO}_{2n+1} \,/\, \mathrm{U}_1 \times \mathrm{U}_{n-1}$	$n \geqslant 3$	5	$(n, 2(n-1))$	n even
$\mathrm{SO}_{2n+1} \,/\, \mathrm{U}_p \times \mathrm{U}_{n-p}$	$n \geqslant 4, 2 \leqslant p \leqslant n$	6	$(n, 2(n-p))$	n even
$\mathrm{SO}_{2n+1} \,/\, \mathrm{U}_p \times \mathrm{U}_q \times \mathrm{SO}_{2(n-p-q)+1}$	$n \geqslant 4, 2 \leqslant p \leqslant n-1$ $4 \leqslant p+q \leqslant n-1$	6	$(p+q, 2n-2p-q)$	p & q even
$\mathrm{Sp}_n \,/\, \mathrm{U}_p \times \mathrm{U}_{n-p}$	$n \geqslant 3, 1 \leqslant p \leqslant n-1$	4	$(n, n-p+1)$	n even & p odd
$\mathrm{Sp}_3 \,/ \mathrm{T}^2$ (special case)	-	4	$(2,2)$	yes
$\mathrm{Sp}_n \,/\, \mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{Sp}_{n-2}$	$n \geqslant 3$	6	$(2, 2(n-1))$	$\forall\ n \geqslant 3$
$\mathrm{Sp}_n \,/\, \mathrm{U}_p \times \mathrm{U}_q \times \mathrm{Sp}_{n-p-q}$	$n \geqslant 3, 1 \leqslant p \leqslant n-3$ $3 \leqslant p+q \leqslant n-1$	6	$(p+q, 2n-2p-q+1)$	p & q odd
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_1 \times \mathrm{U}_{n-1}$	$n \geqslant 4$	3	$(n, 2(n-2))$	n even
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{SO}_{2(n-2)}$	$n \geqslant 4$	4	$(2, 2(n-2))$	$\forall\ n \geqslant 4$
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_p \times \mathrm{U}_{n-p}$	$n \geqslant 4, 2 \leqslant p \leqslant n-2$	4	$(n, 2(n-p-1))$	n even
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_1 \times \mathrm{U}_p \times \mathrm{SO}_{2(n-p-1)}$	$n \geqslant 4, 2 \leqslant p \leqslant n-3$	5	$(1+p, 2n-p-3)$	p odd
$\mathrm{SO}_{2n} \,/\, \mathrm{U}_p \times \mathrm{U}_q \times \mathrm{SO}_{2(n-p-q)}$	$n \geqslant 5, 2 \leqslant p \leqslant n-4$ $4 \leqslant p+q \leqslant n-2$	6	$(p+q, 2n-2p-q-1)$	p & q odd

Spin structures on exceptional flag manifolds

- Given an exceptional Lie group $G \in \{\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, E_8\}$ with root system $\mathbf{R}$ and a basis of simple roots $\mathbf{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}$, we shall denote by

$$G(\alpha_1, \dots, \alpha_{\mathbf{u}}) \equiv G(1, \dots, \mathbf{u})$$

to denote the exceptional flag manifold $F = G/\mathbf{H}$ where the semisimple part $\mathfrak{h}'$ of the stability subalgebra $\mathfrak{h} = T_e\mathbf{H}$ corresponds to the simple roots $\mathbf{\Pi_W} := \{\alpha_1, \dots, \alpha_{\mathbf{u}}\}$.

- The remaining $\mathbf{v} := \ell - \mathbf{u}$ nodes in the Dynkin diagram $\Gamma(\Pi)$ of G have been painted black such that $\mathfrak{h} = \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1) \oplus \mathfrak{h}'$.

- There are 101 non-isomorphic flag manifolds $F = G/\mathbf{H}$ corresponding to a simple exceptional Lie group G.

G	$F = G/\mathbf{H}$	$b_2(F)$	$d = \sharp(R_T^+)$	σ^J
G_2	$\mathrm{G}_2(0) = \mathrm{G}_2 / \mathrm{T}^2$	2	6	$2(\Lambda_1 + \Lambda_2)$
	$\mathrm{G}_2(1) = \mathrm{G}_2 / \mathrm{U}_2^l$	1	3	$5\Lambda_2$
	$\mathrm{G}_2(2) = \mathrm{G}_2 / \mathrm{U}_2^s$	1	2	$3\Lambda_1$
F_4	$\mathrm{F}_4(0) = \mathrm{F}_4 / \mathrm{T}^4$	4	24	$2(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4)$
	$\mathrm{F}_4(1) = \mathrm{F}_4 / \mathrm{A}_1^l \times \mathrm{T}^3$	3	16	$3\Lambda_2 + 2(\Lambda_3 + \Lambda_4)$
	$\mathrm{F}_4(4) = \mathrm{F}_4 / \mathrm{A}_1^s \times \mathrm{T}^3$	3	13	$2(\Lambda_1 + \Lambda_2) + \Lambda_3$
	$\mathrm{F}_4(1,2) = \mathrm{F}_4 / \mathrm{A}_2^l \times \mathrm{T}^2$	2	9	$6\Lambda_3 + 2\Lambda_4$
	$\mathrm{F}_4(1,4) = \mathrm{F}_4 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	8	$3\Lambda_2 + 3\Lambda_3$
	$\mathrm{F}_4(2,3) = \mathrm{F}_4 / \mathrm{B}_2 \times \mathrm{T}^2$	2	6	$5\Lambda_1 + 6\Lambda_4$
	$\mathrm{F}_4(3,4) = \mathrm{F}_4 / \mathrm{A}_2^s \times \mathrm{T}^2$	2	6	$2\Lambda_1 + 4\Lambda_2$
	$\mathrm{F}_4(1,2,4) = \mathrm{F}_4 / \mathrm{A}_2^l \times \mathrm{A}_1^s \times \mathrm{T}$	1	4	$7\Lambda_3$
	$\mathrm{F}_4(1,3,4) = \mathrm{F}_4 / \mathrm{A}_2^s \times \mathrm{A}_1^l \times \mathrm{T}$	1	3	$5\Lambda_2$
	$\mathrm{F}_4(1,2,3) = \mathrm{F}_4 / \mathrm{B}_3 \times \mathrm{T}$	1	2	$11\Lambda_4$
	$\mathrm{F}_4(2,3,4) = \mathrm{F}_4 / \mathrm{C}_3 \times \mathrm{T}$	1	2	$8\Lambda_1$
E_6	$\mathrm{E}_6(0) = \mathrm{E}_6 / \mathrm{T}^6$	6	36	$2(\Lambda_1 + \cdots + \Lambda_6)$
	$\mathrm{E}_6(1) = \mathrm{E}_6 / \mathrm{A}_1 \times \mathrm{T}^5$	5	25	$3\Lambda_2 + 2(\Lambda_3 + \cdots + \Lambda_6)$
	$\mathrm{E}_6(3,5) = \mathrm{E}_6 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^4$	4	17	$2\Lambda_1 + 3\Lambda_2 + 4\Lambda_4 + 3\Lambda_6$
	$\mathrm{E}_6(4,5) = \mathrm{E}_6 / \mathrm{A}_2 \times \mathrm{T}^4$	4	15	$2(\Lambda_1 + \Lambda_2 + 2\Lambda_3 + \Lambda_6)$
	$\mathrm{E}_6(1,3,5) = \mathrm{E}_6 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^3$	3	11	$4(\Lambda_2 + \Lambda_4) + 3\Lambda_6$
	$\mathrm{E}_6(2,4,5) = \mathrm{E}_6 / \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{T}^3$	3	10	$3\Lambda_1 + 5\Lambda_3 + 2\Lambda_6$
	$\mathrm{E}_6(3,4,5) = \mathrm{E}_6 / \mathrm{A}_3 \times \mathrm{T}^3$	3	8	$2\Lambda_1 + 5(\Lambda_2 + \Lambda_6)$
	$\mathrm{E}_6(2,3,4,5) = \mathrm{E}_6 / \mathrm{A}_4 \times \mathrm{T}^2$	2	4	$6\Lambda_1 + 8\Lambda_6$
	$\mathrm{E}_6(1,3,4,5) = \mathrm{E}_6 / \mathrm{A}_3 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	5	$6\Lambda_2 + 5\Lambda_6$
	$\mathrm{E}_6(1,2,4,5) = \mathrm{E}_6 / \mathrm{A}_2 \times \mathrm{A}_2 \times \mathrm{T}^2$	2	6	$6\Lambda_3 + 2\Lambda_6$
	$\mathrm{E}_6(2,4,5,6) = \mathrm{E}_6 / \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	6	$3\Lambda_1 + 6\Lambda_3$
	$\mathrm{E}_6(2,3,4,6) = \mathrm{E}_6 / \mathrm{D}_4 \times \mathrm{T}^2$	2	3	$8(\Lambda_1 + \Lambda_5)$
	$\mathrm{E}_6(1,2,4,5,6) = \mathrm{E}_6 / \mathrm{A}_2 \times \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{T}$	1	3	$7\Lambda_3$
	$\mathrm{E}_6(1,2,3,4,5) = \mathrm{E}_6 / \mathrm{A}_5 \times \mathrm{T}$	1	2	$11\Lambda_6$
	$\mathrm{E}_6(1,3,4,5,6) = \mathrm{E}_6 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}$	1	2	$9\Lambda_2$
	$\mathrm{E}_6(2,3,4,5,6) = \mathrm{E}_6 / \mathrm{D}_5 \times \mathrm{T}$	1	1	$12\Lambda_1$

G	$F = G/\mathrm{H}$	$b_2(F)$	$d = \sharp(R_T^+)$	σ^J
E_7	$\mathrm{E}_7(0) = \mathrm{E}_7 / \mathrm{T}^7$	7	63	$2(\Lambda_1 + \cdots + \Lambda_7)$
	$\mathrm{E}_7(1) = \mathrm{E}_7 / \mathrm{A}_1 \times \mathrm{T}^6$	6	46	$3\Lambda_2 + 2(\Lambda_3 + \cdots + \Lambda_7)$
	$\mathrm{E}_7(4,6) = \mathrm{E}_7 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^5$	5	33	$2(\Lambda_1 + \Lambda_2) + 3(\Lambda_3 + \Lambda_7) + 4\Lambda_5$
	$\mathrm{E}_7(5,6) = \mathrm{E}_7 / \mathrm{A}_2 \times \mathrm{T}^5$	5	30	$2(\Lambda_1 + \cdots + \Lambda_4 + \Lambda_7)$
	$\mathrm{E}_7(1,3,5) = \mathrm{E}_7 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^4$ $[1,1]$	4	23	$4\Lambda_2 + 4\Lambda_4 + 3\Lambda_6 + 2\Lambda_7$
	$\mathrm{E}_7(1,3,7) = \mathrm{E}_7 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^4$ $[0,0]$	4	24	$4\Lambda_2 + 4\Lambda_4 + 2\Lambda_5 + 2\Lambda_6$
	$\mathrm{E}_7(3,5,6) = \mathrm{E}_7 / \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{T}^4$	4	21	$2\Lambda_1 + 3\Lambda_2 + 5\Lambda_4 + 2\Lambda_7$
	$\mathrm{E}_7(4,5,6) = \mathrm{E}_7 / \mathrm{A}_3 \times \mathrm{T}^4$	4	18	$2\Lambda_1 + 2\Lambda_2 + 5\Lambda_3 + 5\Lambda_7$
	$\mathrm{E}_7(1,2,3,4) = \mathrm{E}_7 / \mathrm{A}_4 \times \mathrm{T}^3$	3	10	$6\Lambda_5 + 2\Lambda_6 + 6\Lambda_7$
	$\mathrm{E}_7(1,2,3,5) = \mathrm{E}_7 / \mathrm{A}_3 \times \mathrm{A}_1 \times \mathrm{T}^3$ $[1,1]$	3	12	$6\Lambda_4 + 3\Lambda_6 + 2\Lambda_7$
	$\mathrm{E}_7(1,2,3,7) = \mathrm{E}_7 / \mathrm{A}_3 \times \mathrm{A}_1 \times \mathrm{T}^3$ $[0,0]$	3	13	$6\Lambda_4 + 2\Lambda_5 + 2\Lambda_6$
	$\mathrm{E}_7(1,2,4,5) = \mathrm{E}_7 / \mathrm{A}_2 \times \mathrm{A}_2 \times \mathrm{T}^3$	3	13	$6\Lambda_3 + 4\Lambda_6 + 4\Lambda_7$
	$\mathrm{E}_7(1,2,4,6) = \mathrm{E}_7 / \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^3$	3	14	$5\Lambda_3 + 4\Lambda_5 + 3\Lambda_7$
	$\mathrm{E}_7(1,3,5,7) = \mathrm{E}_7 / (\mathrm{A}_1)^4 \times \mathrm{T}^3$	3	16	$4\Lambda_2 + 5\Lambda_4 + 3\Lambda_6$
	$\mathrm{E}_7(3,4,5,7) = \mathrm{E}_7 / \mathrm{D}_4 \times \mathrm{T}^3$	3	9	$2\Lambda_1 + 8\Lambda_2 + 8\Lambda_6$
	$\mathrm{E}_7(1,2,3,4,5) = \mathrm{E}_7 / \mathrm{A}_5 \times \mathrm{T}^2$ $[1,1]$	2	5	$7\Lambda_6 + 10\Lambda_7$
	$\mathrm{E}_7(1,2,3,4,7) = \mathrm{E}_7 / \mathrm{A}_5 \times \mathrm{T}^2$ $[0,0]$	2	6	$10\Lambda_5 + 2\Lambda_6$
	$\mathrm{E}_7(1,2,3,4,6) = \mathrm{E}_7 / \mathrm{A}_4 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	6	$7\Lambda_5 + 6\Lambda_7$
	$\mathrm{E}_7(1,2,3,5,6) = \mathrm{E}_7 / \mathrm{A}_3 \times \mathrm{A}_2 \times \mathrm{T}^2$	2	7	$7\Lambda_4 + 2\Lambda_7$
	$\mathrm{E}_7(1,2,3,5,7) = \mathrm{E}_7 / \mathrm{A}_3 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	8	$7\Lambda_4 + 3\Lambda_6$
	$\mathrm{E}_7(1,3,4,5,7) = \mathrm{E}_7 / \mathrm{D}_4 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	6	$9\Lambda_2 + 4\Lambda_6$
	$\mathrm{E}_7(1,2,5,6,7) = \mathrm{E}_7 / \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^2$	2	8	$4\Lambda_3 + 5\Lambda_4$
	$\mathrm{E}_7(1,3,5,6,7) = \mathrm{E}_7 / \mathrm{A}_2 \times (\mathrm{A}_1)^3 \times \mathrm{T}^2$	2	9	$4\Lambda_2 + 6\Lambda_4$
	$\mathrm{E}_7(3,4,5,6,7) = \mathrm{E}_7 / \mathrm{D}_5 \times \mathrm{T}^2$	2	4	$2\Lambda_1 + 12\Lambda_2$
	$\mathrm{E}_7(1,2,3,4,5,6) = \mathrm{E}_7 / \mathrm{A}_6 \times \mathrm{T}$	1	2	$14\Lambda_7$
	$\mathrm{E}_7(2,3,4,5,6,7) = \mathrm{E}_7 / \mathrm{E}_6 \times \mathrm{T}$	1	1	$18\Lambda_1$
	$\mathrm{E}_7(1,3,4,5,6,7) = \mathrm{E}_7 / \mathrm{D}_5 \times \mathrm{A}_1 \times \mathrm{T}$	1	2	$13\Lambda_2$
	$\mathrm{E}_7(1,2,4,5,6,7) = \mathrm{E}_7 / \mathrm{A}_4 \times \mathrm{A}_2 \times \mathrm{T}$	1	3	$10\Lambda_3$
	$\mathrm{E}_7(1,2,3,5,6,7) = \mathrm{E}_7 / \mathrm{A}_3 \times \mathrm{A}_2 \times \mathrm{A}_1 \times \mathrm{T}$	1	4	$8\Lambda_4$
	$\mathrm{E}_7(1,2,3,4,6,7) = \mathrm{E}_7 / \mathrm{A}_5 \times \mathrm{A}_1 \times \mathrm{T}$	1	2	$12\Lambda_5$
	$\mathrm{E}_7(1,2,3,4,5,7) = \mathrm{E}_7 / \mathrm{D}_6 \times \mathrm{T}$	1	2	$17\Lambda_6$

Thm.
(1) For $G = \mathrm{G}_2$ there is a unique G-spin (or G-metaplectic) flag manifold, namely the full flag $\mathrm{G}_2(0) = \mathrm{G}_2 / \mathrm{T}^2$.
(2) For $G = \mathrm{F}_4$ the associated G-spin (of G-metaplectic) flag manifolds are the cosets defined by $\mathrm{F}_4(0)$, $\mathrm{F}_4(1,2)$, $\mathrm{F}_4(3,4)$, $\mathrm{F}_4(2,3,4)$, and the flag manifolds isomorphic to them. In particular:

- $\mathrm{F}_4(2,3,4) = \mathrm{F}_4 / \mathrm{C}_3 \times \mathrm{T}$ is the unique (up to equivalence) flag manifold of $G = \mathrm{F}_4$ with $b_2(F) = 1 = \mathrm{rnk}\, R_T$ which admits a G-invariant spin and metaplectic structure.

- There are not exist flag manifolds $F = G/H$ of $G = F_4$ with $b_2(F) = 3 = \mathrm{rnk}\, R_T$ carrying a (G-invariant) spin structure or a metaplectic structure.

(3) For $G = \mathrm{E}_6$ the associated G-spin (or G-metaplectic) flag manifolds are the cosets defined by $\mathrm{E}_6(0)$, $\mathrm{E}_6(4,5)$, $\mathrm{E}_6(2,3,4,5)$, $\mathrm{E}_6(1,2,4,5)$, $\mathrm{E}_6(2,3,4,6)$, $\mathrm{E}_6(2,3,4,5,6)$, and the flag manifolds isomorphic to them. In particular,

- $\mathrm{E}_6(4,5) = \mathrm{E}_6 / \mathrm{A}_2 \times \mathrm{T}^4$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_6$ with $b_2(F) = 4 = \mathrm{rnk}\, R_T$ which admits a G-invariant spin and metaplectic structure.

- $\mathrm{E}_6(2,3,4,5,6) = \mathrm{E}_6 / \mathrm{D}_5 \times \mathrm{T}$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_6$ with $b_2(F) = 1 = \mathrm{rnk}\, R_T$ which admits a G-invariant spin and metaplectic structure.

- There are not exist flag manifolds $F = G/H$ of $G = \mathrm{E}_6$ with $b_2(F) = 3 = \mathrm{rnk}\, R_T$ carrying a (G-invariant) spin or metaplectic structure.

Thm.
For $G = \mathrm{E}_7$ the associated G-spin (or G-metaplectic) flag manifolds are the cosets defined by $\mathrm{E}_7(0)$, $\mathrm{E}_7(5,6)$, $\mathrm{E}_7(1,3,7)$, $\mathrm{E}_7(1,2,3,4)$, $\mathrm{E}_7(1,2,3,7)$, $\mathrm{E}_7(1,2,4,5)$, $\mathrm{E}_7(3,4,5,7)$, $\mathrm{E}_7(1,2,3,4,7)$, $\mathrm{E}_7(1,3,5,6,7)$, $\mathrm{E}_7(3,4,5,6,7)$, $\mathrm{E}_7(1,2,3,4,5,6)$, $\mathrm{E}_7(2,3,4,5,6,7)$, $\mathrm{E}_7(1,2,4,5,6,7)$, $\mathrm{E}_7(1,2,3,5,6,7)$, $\mathrm{E}_7(1,2,3,4,6,7)$ and the flag manifolds isomorphic to them. In particular,

- $\mathrm{E}_7(5,6) = \mathrm{E}_7 / \mathrm{A}_2 \times \mathrm{T}^5$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_7$ with second Betti number $b_2(F) = 5 = \mathrm{rnk}\, R_T$, which admits a G-invariant spin and metaplectic structure.

- $\mathrm{E}_7(1,3,7) = \mathrm{E}_7 / \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{T}^4$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_7$ with second Betti number $b_2(F) = 4 = \mathrm{rnk}\, R_T$, which admits a G-invariant spin and metaplectic structure.

- There are not exist flag manifolds $F = G/H$ of $G = \mathrm{E}_7$ with $b_2(F) = \mathrm{rnk}\, R_T = 6$, carrying a ($G$-invariant) spin or metaplectic structure.

Thm.
For $G = \mathrm{E}_8$ the associated G-spin (or G-metaplectic) flag manifolds are the cosets defined by $\mathrm{E}_8(0)$, $\mathrm{E}_8(1,2)$, $\mathrm{E}_8(1,2,3,4)$, $\mathrm{E}_8(1,2,4,5)$, $\mathrm{E}_8(4,5,6,8)$, $\mathrm{E}_8(4,5,6,7,8)$, $\mathrm{E}_8(1,2,3,4,5,6)$, $\mathrm{E}_8(1,2,3,4,6,7)$, $\mathrm{E}_8(1,2,4,5,6,8)$, $\mathrm{E}_8(1,2,4,5,6,7,8)$ and the flag manifolds isomorphic to them. In particular,

- $\mathrm{E}_8(1,2) = \mathrm{E}_8 / \mathrm{A}_1 \times \mathrm{T}^6$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_8$ with second Betti number $b_2(F) = 6 = \mathrm{rnk}\, R_T$, which admits a G-invariant spin and metaplectic structure.

- $\mathrm{E}_8(4,5,6,7,8) = \mathrm{E}_8 / \mathrm{D}_5 \times \mathrm{T}^3$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_8$ with second Betti number $b_2(F) = 3 = \mathrm{rnk}\, R_T$, which admits a G-invariant spin and metaplectic structure.

- $\mathrm{E}_8(1,2,4,5,6,7,8) = \mathrm{E}_8 / \mathrm{D}_5 \times \mathrm{A}_2 \times \mathrm{T}$ is the unique (up to equivalence) flag manifold of $G = \mathrm{E}_8$ with second Betti number $b_2(F) = 1 = \mathrm{rnk}\, R_T$ which admits a G-invariant spin and metaplectic structure.

- There are not exist flag manifolds $F = G/H$ of $G = \mathrm{E}_8$ with $b_2(F) = \mathrm{rnk}\, R_T = 5$, or $b_2(F) = \mathrm{rnk}\, R_T = 7$, carrying a ($G$-invariant) spin or metaplectic structure.

... on the calculation of Koszul numbers

$\alpha)$ Consider the natural invariant ordering $\mathbf{R}_{\mathbf{F}}^{+} = R^{+} \backslash \mathbf{R_H}^{+}$ induced by the splitting $\mathbf{\Pi} = \mathbf{\Pi_W} \sqcup \mathbf{\Pi_B}$. Let us denote by J_0 the corresponding complex structure. Describe the root system $R_{\mathbf{H}}$ and compute

$$\sigma_{\mathbf{H}} := \frac{1}{2} \sum_{\beta \in R_H^+} \beta$$

$\beta)$ Apply the formula

$$2(\sigma_G - \sigma_{\mathbf{H}}) = \sum_{\gamma \in \mathbf{R}_{\mathbf{F}}^+} \gamma := \sigma^{J_0}.$$

In particular, for the exceptional simple Lie groups and with respect to the fixed bases of the associated roots systems, it is $2\sigma_{\mathrm{G}_2} = 6\alpha_1 + 10\alpha_2$,

$$\begin{aligned}
2\sigma_{\mathrm{F}_4} &= 16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4, \\
2\sigma_{\mathrm{E}_6} &= 16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 30\alpha_4 + 16\alpha_5 + 22\alpha_6, \\
2\sigma_{\mathrm{E}_7} &= 27\alpha_1 + 52\alpha_2 + 75\alpha_3 + 96\alpha_4 + 66\alpha_5 + 34\alpha_6 + 49\alpha_7, \\
2\sigma_{\mathrm{E}_8} &= 58\alpha_1 + 114\alpha_2 + 168\alpha_3 + 220\alpha_4 + 270\alpha_5 + 182\alpha_6 + 92\alpha_7 + 136\alpha_8.
\end{aligned}$$

$\gamma)$ Use the Cartan matrix $\mathcal{C} = (c_{i,j}) = \left(\frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}\right)$ associated to the basis Π (and its enumeration), to express the simple roots in terms of fundamental weights via the formula $\alpha_i = \sum_{j=1}^{\ell} c_{i,j} \Lambda_j$.

C-spaces and spin structures

• C-space is a compact, simply connected, homogeneous complex manifold $M = G/L$ of a compact semisimple Lie group G.
• stability group L is a closed connected subgroup of G whose semisimple part coincides with the semisimple part of the centralizer of a torus in G.
• Any C-space is the total space of a **homogeneous torus bundle** $M = G/L \to F = G/H$ over a flag manifold $F = G/H$.
• In particular, the fiber is a complex torus T^{2k} of real even dimension $2k$.

Well-know fact: Given a C-space $M = G/L$ the following are equivalent:

- $L = C(S)$, i.e. $M = G/L$ is a flag manifold,
- second Betti number of G/L is non-zero,
- the Euler characteristic of G/L is non-zero,

• Hence, non-Kählerian C-spaces may admit Lorentzian metric and complex structure with zero first Chern class $\Rightarrow$

such spaces may give examples of **homogeneous Calabi-Yau structures with torsion** [Fino-Grantcharov '04, Grantcharov '11]

• Consider a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (Z(\mathfrak{h}) + \mathfrak{h}') + \mathfrak{m},$$

associated with a flag manifold $F = G/H$ of G.

• We decompose

$$Z(\mathfrak{h}) = \mathfrak{t}_0 + \mathfrak{t}_1$$

into a direct sum of a (commutative) subalgebra $\mathfrak{t}_1$ of even dimension $2k$ and a complement $\mathfrak{t}_0$ which generates a closed toral subgroup T_0 of H, such that

$$\operatorname{rnk} G = \dim T_0 + \operatorname{rnk} H', \quad \text{and} \quad \operatorname{rnk} L = \dim T_1 + \operatorname{rnk} H'.$$

• Then, the homogeneous manifold $M = G/L = G/T_0 \cdot H'$ is a C-space and any C-space has such a form.

• Notice that $L \subset H$ is normal subgroup of H. In particular, H' (the semi-simple part of H) coincides with the simi-simple part of L.

Lemma. Any complex structure in $\mathfrak{t}_1$ together with an invariant complex structure J_F in $F = G/H = G/T_1 \cdot L$ defines an invariant complex structure J_M in $M = G/L = G/T_0 \cdot H'$ such that $\pi : M = G/L \to F = G/H$ is a holomorphic fibration with respect to the complex structures J_M and J_F. The fiber has the form $H/L = (T_1 \cdot L)/(T_0 \cdot H') \cong T_1$.

- Consider a homogeneous torus bundle $\pi : M = G/L \to F = G/H$

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{q} = (\mathfrak{h}' + \mathfrak{t}_0) + (\mathfrak{t}_1 + \mathfrak{m}), \quad \mathfrak{q} := (\mathfrak{t}_1 + \mathfrak{m}) \cong T_{eL}M.$$

- Let J_F be an invariant complex structure in F and J_M its extension to an invariant complex structure in M, defined by adding a complex structure $J_{\mathfrak{t}_1}$ in $\mathfrak{t}_1$. Then

Prop. The invariant Chern from $\gamma_{J_M} \in \Omega^2(M)$ of the complex structure J_M is the pull back of the invariant Chern form $\gamma_{J_F} \in \Omega^2(F)$ associated to the complex structure J_F on F, i.e. $\gamma_{J_M} = \pi^*\gamma_{J_F}$.

Corol. Given a C-space $M = G/L$ over flag manifold $F = G/H$, then

- $w_2(TM) = \pi^*(w_2(TF))$
- M is spin if and only if $w_2(TF)$ belongs to the kernel of $\pi^* : H^2(F; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}_2)$.
- If F is G-spin, then so is M.

Hints: Notice that

$$TM = G \times_L \mathfrak{q} = (G \times_L \mathfrak{t}_1) \oplus \pi^*(TF)$$

Thm. There are 45 non-biholomorphic C-spaces $M = G/L$ fibered over a spin flag manifold $F = G/H$ of an exceptional Lie group $G \in \{G_2, F_4, E_6, E_7, E_8\}$, and any such space carries a unique G-invariant spin structure. The associated fibrations are given as follows:

T^2	$\hookrightarrow$	G_2	$\longrightarrow$	G_2/T^2
T^4	$\hookrightarrow$	F_4	$\longrightarrow$	F_4/T^4
T^2	$\hookrightarrow$	F_4/T^2	$\longrightarrow$	F_4/T^4
T^2	$\hookrightarrow$	F_4/A_2^l	$\longrightarrow$	$F_4/A_2^l \times T^2$
T^2	$\hookrightarrow$	F_4/A_2^s	$\longrightarrow$	$F_4/A_2^s \times T^2$
T^6	$\hookrightarrow$	E_6	$\longrightarrow$	E_6/T^6
T^4	$\hookrightarrow$	E_6/T^2	$\longrightarrow$	E_6/T^6
T^2	$\hookrightarrow$	E_6/T^4	$\longrightarrow$	E_6/T^6
T^4	$\hookrightarrow$	E_6/A_2	$\longrightarrow$	$E_6/A_2 \times T^4$
T^2	$\hookrightarrow$	$E_6/A_2 \times T^2$	$\longrightarrow$	$E_6/A_2 \times T^4$
T^2	$\hookrightarrow$	E_6/A_4	$\longrightarrow$	$E_6/A_4 \times T^2$
T^2	$\hookrightarrow$	$E_6/A_2 \times A_2$	$\longrightarrow$	$E_6/A_2 \times A_2 \times T^2$
T^2	$\hookrightarrow$	E_6/D_4	$\longrightarrow$	$E_6/D_4 \times T^3$

T^6	$\hookrightarrow$	E_7/T	$\longrightarrow$	E_7/T^7
T^4	$\hookrightarrow$	E_7/T^3	$\longrightarrow$	E_7/T^7
T^2	$\hookrightarrow$	E_7/T^5	$\longrightarrow$	E_7/T^7
T^4	$\hookrightarrow$	$E_7/A_2 \times T$	$\longrightarrow$	$E_7/A_2 \times T^5$
T^2	$\hookrightarrow$	$E_7/A_2 \times T^3$	$\longrightarrow$	$E_7/A_2 \times T^5$
T^4	$\hookrightarrow$	$E_7/(A_1)^3$	$\xrightarrow{*}$	$E_7/(A_1)^3 \times T^4$
T^2	$\hookrightarrow$	$E_7/(A_1)^3 \times T^2$	$\xrightarrow{*}$	$E_7/(A_1)^3 \times T^4$
T^2	$\hookrightarrow$	$E_7/A_4 \times T$	$\longrightarrow$	$E_7/A_4 \times T^3$
T^2	$\hookrightarrow$	$E_7/A_3 \times A_1 \times T$	$\xrightarrow{*}$	$E_7/A_3 \times A_1 \times T^3$
T^2	$\hookrightarrow$	$E_7/A_2 \times A_2 \times T$	$\longrightarrow$	$E_7/A_2 \times A_2 \times T^3$
T^2	$\hookrightarrow$	$E_7/D_4 \times T$	$\longrightarrow$	$E_7/D_4 \times T^3$
T^2	$\hookrightarrow$	E_7/A_5	$\longrightarrow$	$E_7/A_5 \times T^2$
T^2	$\hookrightarrow$	$E_7/A_2 \times (A_1)^3$	$\xrightarrow{*}$	$E_7/A_2 \times (A_1)^3 \times$
T^2	$\hookrightarrow$	E_7/D_5	$\longrightarrow$	$E_7/D_5 \times T^2$

T^8	$\hookrightarrow$	E_8	$\longrightarrow$	E_8/T^8
T^6	$\hookrightarrow$	E_8/T^2	$\longrightarrow$	E_8/T^8
T^4	$\hookrightarrow$	E_8/T^4	$\longrightarrow$	E_8/T^8
T^2	$\hookrightarrow$	E_8/T^6	$\longrightarrow$	E_8/T^8
T^6	$\hookrightarrow$	E_8/A_2	$\longrightarrow$	$E_8/A_2 \times T^6$
T^4	$\hookrightarrow$	$E_8/A_2 \times T^2$	$\longrightarrow$	$E_8/A_2 \times T^6$
T^2	$\hookrightarrow$	$E_8/A_2 \times T^4$	$\longrightarrow$	$E_8/A_2 \times T^6$
T^4	$\hookrightarrow$	E_8/A_4	$\longrightarrow$	$E_8/A_4 \times T^4$
T^2	$\hookrightarrow$	$E_8/A_4 \times T^2$	$\longrightarrow$	$E_8/A_4 \times T^4$
T^4	$\hookrightarrow$	E_8/D_4	$\longrightarrow$	$E_8/D_4 \times T^4$
T^2	$\hookrightarrow$	$E_8/D_4 \times T^2$	$\longrightarrow$	$E_8/D_4 \times T^4$
T^4	$\hookrightarrow$	$E_8/A_2 \times A_2$	$\longrightarrow$	$E_8/A_2 \times A_2 \times T^4$
T^2	$\hookrightarrow$	$E_8/A_2 \times A_2 \times T^2$	$\longrightarrow$	$E_8/A_2 \times A_2 \times T^4$
T^2	$\hookrightarrow$	$E_8/D_5 \times T^1$	$\longrightarrow$	$E_8/D_5 \times T^3$
T^2	$\hookrightarrow$	E_8/A_6	$\longrightarrow$	$E_8/A_6 \times T^2$
T^2	$\hookrightarrow$	$E_8/A_4 \times A_2$	$\longrightarrow$	$E_8/A_4 \times A_2 \times T^2$
T^2	$\hookrightarrow$	$E_8/D_4 \times A_2$	$\longrightarrow$	$E_8/D_4 \times A_2 \times T^2$
T^2	$\hookrightarrow$	E_8/E_6	$\longrightarrow$	$E_8/E_6 \times T^2$