Holonomy in Supergeometry: Theory and Applications

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Marburg, March 11, 2016

Holonomy: All parallel transport operators around loops at a point.

- M a smooth manifold.
- E a vector bundle over M.
- ∇ a connection on E with curvature tensor R.
- $\gamma : [0,1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$; denoted $\gamma : x \to y$.
- $P_\gamma : E_x \to E_y \in \mathrm{End}(E_x, E_y)$ parallel transport wrt. ∇.

Definition and Theorem (Ambrose-Singer)

$\mathrm{Hol}_x := \{P_\gamma \mid \gamma : x \to x\}$ *is a Lie group with Lie algebra*

$$\mathrm{hol}_x = \left\langle P_\gamma^{-1} \circ R_y(u, v) \circ P_\gamma \,\middle|\, y \in M\,,\ \gamma : x \to y\ u, v \in T_y M \right\rangle$$

Theorem (Holonomy Principle)

A global section $X \in \Gamma(E)$ with $\nabla X = 0$ (parallel) is equivalent to a vector $X_x \in E_x$ with $\mathrm{Hol}_x \cdot X_x = X_x$.

Definition (Berezin-Leites)

A supermanifold is a super ringed space $M = (M_0, \mathcal{O}_M)$: M_0 a manifold, $\mathcal{O}_M$ a sheaf, locally: $\mathbb{R}^{n|m} := (\mathbb{R}^n, C^\infty(\mathbb{R}^n) \otimes \bigwedge \mathbb{R}^m)$.

Some concepts of differential geometry extend "straightforwardly": $\mathbb{Z}_2$-grading (even and odd) and sign rule $\xi^i \xi^j = (-1)^{|i||j|} \xi^j \xi^i$.

- A vector bundle $\mathcal{E}$ is a sheaf of locally free $\mathcal{O}_M$ supermodules, e.g. the tangent sheaf $\mathcal{T}M := \mathrm{Der}_{\mathcal{O}_M}$ of $\mathcal{O}_M$-superderivations.
- A connection on $\mathcal{E}$ is an $\mathbb{R}$-linear sheaf morphism $\nabla : \mathcal{E} \to \mathcal{T}M^* \otimes_{\mathcal{O}_M} \mathcal{E}$ with $\nabla(fe) = df \otimes_{\mathcal{O}_M} e + f \cdot \nabla e$
- ∇ induces curvature R and torsion T.

... others are non-trivial.

- Integration theory (differential $\neq$ integral forms; boundary...)
- Points do not completely characterise superfunctions etc.
- No obvious notion of parallel transport or holonomy!

1: "Superobject (S) = Classical (C) even + Infinitesimal (I) odd"

Example (Harish-Chandra Pair)

(S) a super Lie group: a supermanifold $\mathfrak{G}$ with $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} \ldots$

(C) a Lie group G.

(I) a super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

s.th. $G = \mathfrak{G}_0$, $\mathfrak{g}_0 = LA(G)$, with some compatibility conditions.

2: "Superobject (S) = Functor (F) to a Classical Category"

Example (Grothendieck-Yoneda)

(S) a super Lie group: a supermanifold $\mathfrak{G}$ with $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} \ldots$

(F) a representable group val. functor $G : (\mathrm{supermf}) \to (\mathrm{groups})$.

G the *functor of points* of $\mathfrak{g}$, i.e. on objects: $G(T) = \mathrm{Hom}(T, \mathfrak{G})$.
Conversely, m is determined by the $m_T : G(T) \times G(T) \to G(T)$.

A Construction using General Principle 1: (S)=(C)+(I).

- $x \in M_0$ a point. All vector spaces $x^*\mathcal{E}$ together form a bundle E over M_0, ∇ induces a classical connection ∇_0 on E.
- $P_{\gamma_0} : x^*\mathcal{E} \to y^*\mathcal{E}$ parallel transport along a classical path γ_0.
- Let $Y_1, \ldots, Y_r, Y, Z \in y^*\mathcal{T}M$, $\overline{\nabla}^r_{Y_r,\ldots,Y_1} R$ the r-fold derivative of R wrt. ∇ and an auxiliary connection $\overline{\nabla}$ on $\mathcal{T}M$ near y.

Definition (A. Galaev)

$\mathrm{Hol}^{\mathrm{Gal}}_x$ is the super Lie group defined by the Harish-Chandra pair

(C) $\mathrm{Hol}^{\nabla_0}_x$ the holonomy group wrt. ∇_0 on E.

(I) $\mathrm{hol}^{\mathrm{Gal}}_x := \left\langle P^{-1}_{\gamma_0} \circ \left(\overline{\nabla}^r_{Y_r,\ldots,Y_1} R\right)_y (Y, Z) \circ P_{\gamma_0} : x^*\mathcal{E} \to x^*\mathcal{E} \right\rangle$

- Main idea: Ambrose-Singer algebra with higher derivatives.
- Good properties: A version of the holonomy principle etc.

A Construction using General Principle 2: (S)$\stackrel{?}{=}$(F).

- For $S = \mathbb{R}^{0|L}$, consider $x : S \to M$ and $\gamma : S \times [0,1] \to M$.
- Parallel transport $P_\gamma : x^*\mathcal{E} \to y^*\mathcal{E}$ defined by $(\gamma^*\nabla)_{\partial_t} X = 0$.

Definition and Theorem (Ambrose-Singer)

$\mathrm{Hol}_x := \{P_\gamma \mid \gamma : x \to x\}$ *is a Lie group with LA* hol_x *generated by*

$$\{P_\gamma^{-1} \circ R_y(u, v) \circ P_\gamma \mid y : S \to M\,,\ \gamma : x \to y\ u, v \in (y^*\mathcal{T}M)_{\bar{0}}\}$$

- Consider other superpoints $T = \mathbb{R}^{0|L'}$ (equivalently $\bigwedge \mathbb{R}^{L'}$).
- $x_T : S \times T \to M$ defined by x and projection $S \times T \to S$.
- $T \mapsto \mathrm{Hol}_{x_T}$ defines a group-valued functor, denoted $\mathrm{Hol}_x(T)$.

$\mathrm{Hol}_x(T)$ *not* representable (examples), so $\mathrm{Hol}_x(T) \neq \mathrm{Hol}_x^{\mathrm{Gal}}(T)$.

Theorem (Holonomy Principle)

Let M be connected. Then the following are equivalent.

1. *A vector field $X \in \mathcal{E}$ s.th. $\nabla X = 0$.*
2. *A section $X_x \in x^*\mathcal{E}$ invariant under either holonomy.*

Both $\mathrm{Hol}_x(T)$ and $\mathrm{Hol}_x^{\mathrm{Gal}}$ contain the same amount of information.

Theorem (Comparison Theorem)

$\mathrm{hol}_x^{\mathrm{Gal}}$ *is the algebra of T-coefficient matrices of $\mathrm{hol}_x(T)$ for all T.*

- Technical main result, proved by an odd homotopy formula.

Theorem (Twofold Theorem)

Let $X_x \in x^*\mathcal{E}$. Then the following are equivalent.

1. $\mathrm{Hol}_x^{\nabla_0} \cdot X_x = X_x$ *and* $\mathrm{hol}_x^{\mathrm{Gal}} \cdot X_x = 0$.
2. $\mathrm{Hol}_x(T) \cdot X_x = X_x$ *for "sufficiently large" T.*

Definition

A semi-Riemannian supermanifold is a pair (M, g) with g an even, nondegenerate and supersymmetric bilinear form on $\mathcal{T}M$.

- A connection ∇ on $\mathcal{T}M$ is called metric if, for $X, Y, Z \in \mathcal{T}M$, $Xg(Y, Z) = g(\nabla_X Y, Z) + (-1)^{|X||Y|} g(Y, \nabla_X Z)$.
- ∇ is metric if and only if $\nabla g = 0$ (induced connection).
- Levi-Civita connection on $\mathcal{T}M$ (metric and torsion-free).

Proposition (Holonomy Principle)

The following are equivalent.

1. ∇ *is metric.*
2. $\mathrm{Hol}_x^{\nabla_0} \subseteq O(t, s) \times Sp(2m, \mathbb{R})$ *and* $\mathrm{hol}_x^{\mathrm{Gal}} \subseteq \mathfrak{osp}((t, s)|2m)$.
3. $\mathrm{Hol}_x(T) \in OSp_{(t,s)|2m}(\mathcal{O}_{S \times T})$ *for "sufficiently large"* T.

In particular, this is true for the Levi-Civita connection.

Definition

(M, g) is called Kähler if $\mathrm{Hol}_x(T) \subseteq U_{(t,s)|(k,l)}(\mathcal{O}_{S\times T})$.
(M, g) is called Calabi-Yau if $\mathrm{Hol}_x(T) \subseteq SU_{(t,s)|(k,l)}(\mathcal{O}_{S\times T})$.

- Calabi conjecture is wrong (conterexamples for $k + l = 2$)!

Proposition

A Calabi-Yau supermanifold has trivial holomorphic Berezinian ("canonical") bundle $\mathrm{Ber}(\mathcal{T}^{1,0}M)^*$.

Proof.

- ∇ canonically induces a connection $\tilde{\nabla}$ on $\mathrm{Ber}(\mathcal{T}^{1,0}M)^*$.
- Parallel transport wrt. $\tilde{\nabla}$ is $P_\gamma^{\tilde{\nabla}} = \mathrm{sdet}(P_\gamma^{\nabla})^{-1}$.
- $\mathrm{Hol}_x(T) \subseteq SL_{n|m}(\mathcal{O}_{S\times T})$ implies the holonomy of $\tilde{\nabla}$ is trivial.
- By the Holonomy Principle, there is a global parallel section.

□

An algebraic characterisation of triviality of the canonical bundle.

- A a supercommutative $\mathbb{C}$-algebra.
- $\Delta : A \to A$ an odd linear map, $\Delta^2 = 0$.
- $\beta \mapsto \pm\Delta(\alpha \cdot \beta) \mp \Delta(\alpha) \cdot \beta \mp \alpha \cdot \Delta(\beta)$ is a derivation $\forall \alpha \in A$.
- d an odd derivation, $d^2 = 0$, $[d, \Delta] = 0$.

Theorem (Barannikov-Kontsevich)

X complex supermanifold, simply connected. Then equivalent:

1. *The canonical bundle is trivial.*
2. *$\exists$ dGBV structure compatible with the Schouten bracket.*

1:1 correspondence between such dGBV and trivialising section ω.

- $A = \mathcal{A}^{0,*}(X, \bigwedge^* T^{1,0}X)$, $d = \overline{\partial}$.
- ∂ only defined on *integral forms*, corresponds to Δ via ω.

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Thank you very much for your attention!