Compact quotients of Cahen-Wallach spaces

Ines Kath, joint work with Martin Olbrich

Cahen-Wallach space = solvable Lorentzian symmetric space

quotient = quotient by a discrete subgroup of the isometry group

Compact quotients of manifolds

M manifold
G Lie group, acting on M

G acts *properly* $:\Leftrightarrow G \times M \to M \times M,\ (g, m) \mapsto (gm, m)$ proper

Fact

G-action is proper $\Rightarrow$ *(a) orbits are closed*
(b) $G \setminus M$ Hausdorff

Hence: G acts freely and properly on $M \Rightarrow$
$G \setminus M$ smooth manifold s. th. $M \to G \setminus M$ (C^∞)-submersion

Conversely: G acts freely, $G \setminus M$ smooth, $M \to G \setminus M$ submersion
$\Rightarrow$ G-action is proper

Compact quotients of homogeneous spaces

$X = G/H$ homogeneous space
$\Gamma \subset G$ discrete subgroup acting freely and properly on X
$\Gamma \setminus X$ quotient, also called *Clifford-Klein form*

Q: $X = G/H$, $\exists$ compact quotient?

- H **compact** ($\Rightarrow$ each discrete subgroup of G acts properly)
  yes, if there exists a cocompact lattice $\Gamma \subset G$,
  e.g., if X is a Riem. symmetric space (Borel 1963)
- H **non-compact**: difficult
  some results for **reductive** G (Benoist, Kobayashi)

Problem

Which Lorentzian symmetric spaces admit compact quotients?

- Minkowski space: yes
- de Sitter space $S^{1,n}$ (positive const. sectional curv.): no (Calabi, Markus)
- anti-de-Sitter $\tilde{H}^{1,n}$ (negative —): yes $\Leftrightarrow$ dimension is odd (Kulkarni)
- here: **Cahen-Wallach spaces**
  $X = G/G_+$, $G = Iso(X)$
  G not reductive, G_+ not compact !

Cahen-Wallach spaces

$$X_{p,q}(\lambda,\mu) := (\mathbb{R}^{p+q+2}, g_{\lambda,\mu})\,, \qquad \lambda \in (\mathbb{R}\setminus\{0\})^p,\ \ \mu \in (\mathbb{R}\setminus\{0\})^q$$

$z, x_1, \ldots, x_{p+q}, t$ coordinates

$$g_{\lambda,\mu} = 2dzdt + \left(\sum_{i=1}^{p} \lambda_i^2 x_i^2 - \sum_{j=1}^{q} \mu_j^2 x_{p+j}^2\right) dt^2 + \sum_{i=1}^{p+q} dx_i^2$$

$q = 0$: *real* type
$p = 0$: *imaginary* type
mixed type, otherwise

For which λ, μ do there exist compact quotients?

Idea

proven approach:

$X = G/H$ homogeneous space

- Find a (virtually) **connected** subgroup $U \subset G$ acting properly and cocompactly on X and a cocompact lattice $\Gamma \subset U$.
  Ex: $H^{1,2m} = \mathrm{SO}(2,2m)/\mathrm{SO}(1,2m) = \mathrm{U}(1,m)/\mathrm{U}(m)$
  $\Gamma \subset \mathrm{U}(1,m)$ torsion-free lattice
- Are all compact quotients of this form?
  Ex: no for $m = 1$ (Ghys, Goldman),
  conjecture: yes for $m \geq 2$ (Zeghib)

Idea

proven approach:

$X = G/H$ homogeneous space

- Find a (virtually) **connected** subgroup $U \subset G$ acting properly and cocompactly on X and a cocompact lattice $\Gamma \subset U$.
- Are all compact quotients of this form?

here:

X Cahen-Wallach space, $G = \mathrm{Iso}(X)$

proceed similarly,
but now $U \longrightarrow$ group with **infinite cyclic component group**

Discrete subgroups of $N \rtimes (\mathbb{R} \times K)$

$X = G/G_+$ Cahen-Wallach space, $G = \mathrm{Iso}(X)$

recall: $\hat{G} = H \rtimes_{\lambda,\mu} \mathbb{R}$ transvection group
$G = \hat{G} \rtimes (K \times \mathbb{Z}_2)$, K compact

more generally, we consider: $\boxed{G = N \rtimes (\mathbb{R} \times K)}$
N nilpotent
K compact
$\mathbb{R} \times K$ acts on N by semisimple automorphisms

$\Gamma \subset G$ discrete, $\Delta := \overline{\mathrm{pr}_{\mathbb{R}}(\Gamma)} \subset \mathbb{R}$

Proposition

Γ is a lattice in $(U \cdot \psi(\Delta)) \times C_K \subset G$, where $U \subset N$ connected, $\psi : \Delta \to G$ section, $C_K \subset K$ connected and abelian.

Compact quotients of Cahen-Wallach spaces

X Cahen-Wallach space
$N = H$ Heisenberg, $\quad G = \mathrm{Iso}(X) = H \rtimes (\mathbb{R} \times K)$

$\Gamma \subset G$ discrete, $\quad \Delta := \overline{\mathrm{pr}_{\mathbb{R}}(\Gamma)} \subset \mathbb{R}$

recall:

$\Gamma \subset (U \cdot \psi(\Delta)) \times C_K$ lattice, $U \subset N$ connected, $\psi : \Delta \to G$ section, $C_K \subset K$

Lemma: Γ acts properly and cocompactly iff $U \cdot \psi(\Delta)$ does so

$\Gamma \setminus X$ compact $\quad \Rightarrow \quad \Delta = \langle t_0 \rangle \quad$ or $\quad \Delta = \mathbb{R}$

$\Downarrow$

X group with biinvariant metric
$\exists$ cpt quotient with $\Delta = \langle t_0 \rangle$, too

assume $\Delta = \langle t_0 \rangle$ w.l.o.g.

Compact quotients of Cahen-Wallach spaces

Proposition

$X = X_{p,q}(\lambda, \mu)$ Cahen-Wallach space, $n := p + q$

X has a compact quotient iff there exist

(a) *an n-dimensional subspace $V \subset \mathfrak{a}$ such that $e^{tL} V \cap \mathfrak{a}_+ = \{0\}$ for all $t \in \mathbb{R}$;*

(b) *$t_0 \in \mathbb{R} \setminus \{0\}$, $\varphi_0 \in K$, $h_0 \in H^{t_0 \varphi_0}$ and a lattice Λ of $\mathfrak{z} \oplus V \subset H$ that is stable under conjugation by $h_0 t_0 \varphi_0$.*

real type	imaginary type
(a') $V \cap \mathfrak{a}_+ = 0$	(b') $t_0 \in \mathbb{R} \setminus \{0\}$, $\varphi_0 \in K$ s. t. $(e^{t_0 L} \varphi_0)\vert_V = \mathrm{id}_V$

Real type

Theorem

$X = X_{n,0}(\lambda)$ Cahen-Wallach space of **real type**

X admits a compact quotient iff

$\exists\ f \in \mathbb{Z}[x],\ f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x \pm 1$
with roots $\nu_1, \ldots, \nu_n \notin S^1$, such that

$$X \cong X_{n,0}(\ln|\nu_1|, \ldots, \ln|\nu_n|)$$

recipe that gives all possible spaces
however: for a given λ, hard to decide whether condition is satisfied

Imaginary type

$\underline{k} \in \mathbb{Z}^d$, $z^{\underline{k}} := \mathrm{diag}(z^{k_1}, \dots, z^{k_d})$

$\underline{k}$ admissible $:\Leftrightarrow$ $\exists$ d-dim real vector space $V \subset \mathbb{C}^d$ s.t.
$(z^{\underline{k}} \cdot V) \cap \mathbb{R}^d = 0$ for all $z \in S^1 \subset \mathbb{C}$.

Imaginary type

recall: $\underline{k} \in \mathbb{Z}^d$, $z^{\underline{k}} := \mathrm{diag}(z^{k_1}, \ldots, z^{k_d})$

$\underline{k}$ admissible $\Leftrightarrow$ $\exists\ C \in \mathrm{Mat}(d \times d, \mathbb{R})\ \forall z \in S^1 \subset \mathbb{C}:\ \det \Im(z^{\underline{k}} C) \neq 0$

Theorem

X Cahen-Wallach space of **imaginary type**

X admits a compact quotient iff

$\exists$ *admissible* $\underline{k} \in (\mathbb{Z}_{\neq 0})^d : X \cong X_{0,n}(k_1, \ldots, k_d, \underbrace{\mu_{d+1}, \ldots, \mu_n}_{\text{each } \mu_j \text{ with even multiplicity}})$

Rm. $\underline{k}$ admissible $\Rightarrow \sum(\pm k_i) = 0$

$$\begin{aligned} \det \Im(z^{\underline{k}} C) &= \det\left(\tfrac{1}{2i}(c_{lm} z^{k_l} - \overline{c_{lm}} z^{-k_l})_{l,m=1,\ldots,n}\right) \\ &= \textstyle\sum_\kappa d_\kappa z^{\kappa_1 k_1 + \cdots + \kappa_n k_n},\ \kappa = (\kappa_1, \ldots, \kappa_n) \in \{1, -1\}^n \\ &=: f_C(z) \qquad \overline{d_\kappa} = d_{-\kappa} \neq 0 \end{aligned}$$

$$\int_0^{2\pi} f_C(e^{it}) dt = 2\pi \sum_{\kappa_1 k_1 + \cdots + \kappa_n k_n = 0} d_\kappa$$

Imaginary type

recall: $\underline{k} \in \mathbb{Z}^d$, $z^{\underline{k}} := \mathrm{diag}(z^{k_1}, \dots, z^{k_d})$

$\underline{k}$ admissible $\Leftrightarrow$ $\exists\, C \in \mathrm{Mat}(d \times d, \mathbb{C})\ \forall z \in S^1 \subset \mathbb{C}:$ $\det \Im(z^{\underline{k}}\, C) \neq 0$

Theorem

X Cahen-Wallach space of **imaginary type**

X admits a compact quotient iff

$$\exists \text{ admissible } \underline{k} \in (\mathbb{Z}_{\neq 0})^d : X \cong X_{0,n}(k_1, \dots, k_d, \underbrace{\mu_{d+1}, \dots, \mu_n}_{\text{each } \mu_j \text{ with even multiplicity}})$$

Rm. $\underline{k}$ admissible $\Rightarrow \sum(\pm k_i) = 0$

$d \leq 4$: $\Leftarrow$

$d > 4$?? (only examples)