Hilbert Schemes, Symmetric Quotient Stacks, and Categorical Heisenberg Actions

Andreas Krug – University of Marburg

Lie Theory in Roischholzhausen 2016
Plan of the Talk

Theorem (Göttsche / Nakajima / Grojnowski)

The cohomology of the Hilbert schemes (Douady spaces) of points on a smooth quasi-projective surface carry the structure of the Fock space representation of a Heisenberg algebra.

In this Talk

Discuss three approaches to a categorification of this Heisenberg action:

1. Lift the Nakajima operators to the derived categories.
2. Lift other generators (half of the vertex operators).
3. Give the derived categories of the Hilbert schemes the structure of a Hopf category.
1 Preliminaries
- Symmetric Products and Hilbert Schemes of Points on Surfaces
- Cohomology of Hilbert Schemes and the Heisenberg Algebra
- Derived Categories and Grothendieck Groups
- McKay Correspondence

2 Three Constructions
- Nakajima $\mathbb{P}$-functors
- Lift of the Heisenberg Module Structure
- Categorical Hopf Algebras
Outline

1 Preliminaries
   • Symmetric Products and Hilbert Schemes of Points on Surfaces
   • Cohomology of Hilbert Schemes and the Heisenberg Algebra
   • Derived Categories and Grothendieck Groups
   • McKay Correspondence

2 Three Constructions
   • Nakajima $\mathbb{P}$-functors
   • Lift of the Heisenberg Module Structure
   • Categorical Hopf Algebras
Symmetric Quotient Varieties

Let $X$ be a smooth quasi-projective variety over $\mathbb{C}$. The symmetric group $\mathfrak{S}_n$ acts on $X^n$ by permutation of factors:

$$(x_1, \ldots, x_n) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

**Definition**

Quotient $X^{(n)} := X^n/\mathfrak{S}_n$ is called **symmetric quotient variety**.

**Examples (Curve Case)**

$\mathbb{C}^{(n)} \cong \mathbb{C}^n$ (Theorem on symmetric functions), $\mathbb{P}^{1(n)} \cong \mathbb{P}^n$.

**Non-Smoothness**

For $\dim(X) \geq 2$, the symmetric quotient variety is not smooth. The singular locus consists of the partial diagonals.
Preliminaries

Three Constructions

Symmetric Products and Hilbert Schemes of Points on Surfaces

Hilbert Schemes as Resolutions of Singularities

In the case that \( X \) is a surface, there is a resolution of singularities \( \mu : \ X^{[n]} \to X^{(n)} \) with very good properties: The Hilbert scheme of points on \( X \).

Example (n=2)

\[ \mu : \ X^{[2]} \to X^{(2)} \] is the blow-up along the diagonal.

Definition (General \( n \))

The Hilbert scheme (Douady space) \( X^{[n]} \) is the fine moduli space of \( n \)-Clusters on \( X \). The Hilbert–Chow morphism \( \mu : \ X^{[n]} \to X^{(n)} \) sends an \( n \)-Cluster to its weighted support.
Zero-Dimensional Subschemes (Clusters)

**Definition**
An $n$-Cluster on $X$ is a zero-dimensional closed subscheme $Z \subset X$ of length $\ell(Z) := \dim_{\mathbb{C}}(O(Z)) = n$.

**Examples of Clusters**
- **Collections of $n$ distinct points**: $Z = \{x_1, \ldots, x_n\} \subset X$, $\mu(Z) = x_1 + \cdots + x_n \in X^{(n)}$.
- **Fat points**: Non-reduced schemes concentrated in one point.
  - $n = 2$: Fat points are points with infinitesimal tangent direction. $Z \cong \text{Spec} (\mathbb{C}[\varepsilon]/\varepsilon^2)$, $\mu(Z) = 2x$.

Recall: $\mu : X^{[2]} \rightarrow X^{(2)}$ is blow-up along diagonal.
Outline

1 Preliminaries
   - Symmetric Products and Hilbert Schemes of Points on Surfaces
   - Cohomology of Hilbert Schemes and the Heisenberg Algebra
   - Derived Categories and Grothendieck Groups
   - McKay Correspondence

2 Three Constructions
   - Nakajima $\mathbb{P}$-functors
   - Lift of the Heisenberg Module Structure
   - Categorical Hopf Algebras
Fix smooth projective surface $X$

$\mathcal{H} := \bigoplus_{n \geq 0} H^*(X^{[n]}, \mathbb{C})$ is double graded vector space.

The Betti numbers are the dimensions of graded pieces $b_i(X^{[n]}) := \dim_{\mathbb{C}} H^i(X^{[n]}, \mathbb{C})$.

**Theorem (Göttsche, 1992)**

$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} b_i(X^{[n]}) s^{i-2n} t^n = \prod_{m=1}^{4} \prod_{j=0}^{(1 - (-1)^j s^{j-2} t^m)(-1)^{j+1} b_j(X)}$$

$\mathcal{F}$: Fock space representation of the Heisenberg Lie algebra $\mathfrak{h}_V$ associated to $V := H^*(X, \mathbb{C})$.

**Corollary**

$\mathcal{H} \simeq \mathcal{F}$ as double graded vector spaces.
Heisenberg Lie Algebra (Basic Case)

Definition of the (infinite dimensional) Heisenberg Lie algebra $\mathfrak{h}$:

- **As a vector space:**
  \[ \mathfrak{h} := \mathbb{C} \cdot c \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathbb{C} \cdot a(n). \]

- **Lie bracket** of $\mathfrak{h}$: $c$ is central, i.e. $[c, v] = 0 \forall v \in \mathfrak{h}$, and
  \[ [a(m), a(n)] = \delta_{m,-n} \cdot n \cdot c = \begin{cases} n \cdot c & \text{if } m = -n, \\ 0 & \text{else}. \end{cases} \]

**Alternative notation** $t^m := a(m)$: For $f, g \in \mathbb{C}[t, t^{-1}]$ have
\[ [f(t), g(t)] = \text{res}(f(t) \cdot \frac{\partial g}{\partial t}(t)) \cdot c. \]
Heisenberg Lie Algebra (General Case)

- **Given data:** Finite dimensional $\mathbb{C}$ vector space $V$ together with bilinear form $\langle _, _ \rangle$.
- **As a vector space:** $\mathfrak{h}_V := \mathbb{C} \cdot c \oplus V \oplus \mathbb{Z}\{0\}$.
- For $\beta \in V$ and $n \in \mathbb{Z}$, denote by $a_\beta(n) \in \mathfrak{h}_V$ the image of $\beta$ under the inclusion of the $n$-th summand.
- **Lie bracket:** $c$ is central and

\[
[a_\alpha(m), a_\beta(n)] = \delta_{m,-n} \cdot n \cdot \langle \alpha, \beta \rangle \cdot c .
\]

**Question (Geometric Interpretation of Göttsche’s Theorem)**

Let $V = H^*(X, \mathbb{C})$ together with **intersection form** $\langle \alpha, \beta \rangle = \int_X \alpha \cup \beta$. How to construct an action of $\mathfrak{h}_V$ on $H = \bigoplus_{n \geq 0} H^*(X^n, \mathbb{C})$ in a natural (geometric) way?

$\leadsto$ Constructions of Nakajima/Grojnowski (1996)
Nakajima Operators

- Nakajima correspondences: For \( \ell, n \in \mathbb{N} \) consider closed subscheme \( Z^{\ell, n} \subset X \times X^{[\ell]} \times X^{[\ell+n]} \)

\[
Z^{\ell, n} := \{(x, Z, Z') \mid Z \subset Z', Z \text{ and } Z' \text{ only differ in } x\}.
\]

\[\rightsquigarrow\text{ Induced operators } a(\ell, n): H^*(X \times X^{[\ell]}) \to H^*(X^{[\ell+n]}).\]

- Fix \( \beta \in H^*(X) \). Define Nakajima operator

\[
a_\beta(\ell, n) := a(\ell, n)(\beta \otimes _): \\
H^*(X) \otimes H^*(X^{[\ell]}) \xrightarrow{\text{K"unneth}} H^*(X \times X^{[\ell]}) \xrightarrow{a(\ell, n)} H^*(X^{[\ell+n]})
\]

\[
\beta \otimes _ \\
H^*(X^{[\ell]})
\]

\[
a_\beta(\ell, n)
\]
Set $a_\beta(n) := \bigoplus_{\ell \geq 0} a_\beta(\ell, n) : \mathbb{H} = \bigoplus_{\ell \geq 0} H^*(X[\ell]) \to \mathbb{H}[n]$.

For $n < 0$, define $a_\beta(n) : \mathbb{H}[n] \to \mathbb{H}$ as the **adjoint** of $a_\beta(-n)$ with respect to the intersection pairing on the cohomology of the Hilbert schemes.

**Theorem (Nakajima)**

The commutator relations between these operators satisfy

$$[a_\alpha(m), a_\beta(n)] = \delta_{m,-n} \cdot n \cdot \langle \alpha, \beta \rangle \cdot \text{id}_\mathbb{H}.$$

**Corollary**

The above operators equip $\mathbb{H}$ with the structure of a module over the Heisenberg Lie algebra $\mathfrak{h}_V$ associated to $V = H^*(X)$ where $c$ acts as the identity.
Outline

1 Preliminaries
   - Symmetric Products and Hilbert Schemes of Points on Surfaces
   - Cohomology of Hilbert Schemes and the Heisenberg Algebra
   - Derived Categories and Grothendieck Groups
   - McKay Correspondence

2 Three Constructions
   - Nakajima $\mathbb{P}$-functors
   - Lift of the Heisenberg Module Structure
   - Categorical Hopf Algebras
Lifting/Categorification Problem

Goal

Would like to lift the Heisenberg action from the cohomology of the Hilbert schemes to the level of Grothendieck groups or, even better, derived categories of coherent sheaves (‘categorification’).

\[
\mathbb{D} := \bigoplus_{\ell \geq 0} D(X[\ell]) \quad \xrightarrow{?} \quad \mathbb{D}
\]

\[
\mathbb{K} := \bigoplus_{\ell \geq 0} K(X[\ell]) \quad \xrightarrow{?} \quad \mathbb{K}
\]

\[
\mathbb{H} := \bigoplus_{\ell \geq 0} H^*(X[\ell])^a_{\beta}(n) \quad \xrightarrow{\text{ch}} \quad \mathbb{H}
\]
Coherent Sheaves and Complexes

- **Y**: smooth projective variety.
- **Coh(Y)**: abelian category of coherent sheaves.
- **Kom(Y)** := Kom(Coh(Y)) category of complexes

\[
\begin{align*}
\text{objects:} & \quad \cdots A_i^{-1} \rightarrow A_i \rightarrow A_i^{+1} \cdots \\
\text{morphisms:} & \quad \Downarrow \varphi_i^{-1} \quad \Downarrow \varphi_i \quad \Downarrow \varphi_i^{+1} \\
\cdots B_i^{-1} & \rightarrow B_i \rightarrow B_i^{+1} \cdots
\end{align*}
\]

- \(\varphi^\bullet: A^\bullet \rightarrow B^\bullet \leadsto \mathcal{H}^i(\varphi^\bullet): \mathcal{H}^i(A^\bullet) \rightarrow \mathcal{H}^i(B^\bullet)\).
- \(\varphi^\bullet\) is a quasi-isomorphism (qis)
  \(\iff\) \(\mathcal{H}^i(\varphi^\bullet)\) is an isomorphism \(\forall i \in \mathbb{Z}\).
Derived Category

**Definition (Derived Category)**

\[ D(Y) := D(\text{Coh}(Y)) := \text{Kom}(Y)[\text{qis}^{-1}] \]

- **Objects**: (Bounded) Complexes of coherent sheaves.
- **Morphisms**: Morphisms of complexes
  + Formal inverses of quasi-isomorphisms.

**Features of the Derived Category**

- **Shift** autoequivalence [1]: \( D(Y) \to D(Y) \).
- Fully faithful **embedding** \( \text{Coh}(Y) \hookrightarrow D(Y), E \mapsto E[0] \).
- **Graded Hom-spaces**
  \[ \text{Hom}^*(A^\bullet, B^\bullet) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D(Y)}(A^\bullet, B^\bullet[i])[−i]. \]
- For \( E, F \in \text{Coh}(Y) \):
  \[ \text{Hom}^*(E, F) \cong \text{Ext}^*(E, F). \]
Grothendieck Groups and Euler Characteristic

**Definition (Grothendieck Group and its Natural Bilinear Form)**

\[ K(Y) := K(\text{Coh}(Y)) := \mathbb{Z} \cdot \text{Coh}(Y)/\langle \text{short exact seq.} \rangle \]

is equipped with a bilinear form, the **Euler bicharacteristic**

\[ \langle [E], [F] \rangle := \chi(E, F) := \chi(\text{Ext}^*(E, F)) := \sum (-1)^i \text{Ext}^i(E, F). \]

For \( A^\bullet \in D(Y) \), set \( [A^\bullet] := \sum (-1)^i [\mathcal{H}^i(A^\bullet)] \in K(Y). \)

\[ \xymatrix{ D(Y) \times D(Y) \ar[r]^{\text{Hom}^*(\_, \_)} \ar[d]_{\square \times \square} & D(\text{VectorSpaces}) \ar[d]_{\square} \\
K(Y) \times K(Y) \ar[r]^{\langle \_, \_ \rangle} & \mathbb{Z} } \]

**Slogan:** \( \text{Hom}^*(\_, \_) \) categorifies \( \langle \_, \_ \rangle \).
Definition (Fourier–Mukai Transforms)

Given $P \in \mathcal{D}(Y \times Z)$, ‘Fourier–Mukai kernel’

$\rightsquigarrow$ \[ \text{FM}_P : D(Y) \to D(Z) \quad E \mapsto Rpr_*(P \otimes^L pr_Y^* E). \]

$\rightsquigarrow$ \text{induced correspondence operators:}

\begin{equation*}
\begin{array}{c}
\mathcal{D}(Y) \xrightarrow{\text{FM}_P} \mathcal{D}(Z) \\
\downarrow \Phi_{[P]} \downarrow \Phi_{ch(P)} \\
\mathcal{K}(Y) \xrightarrow{\text{ch}} \mathcal{K}(Z) \\
\downarrow \text{ch} \\
\mathcal{H}^*(Y) \xrightarrow{\Phi_{ch(P)}} \mathcal{H}^*(Z)
\end{array}
\end{equation*}
Approach to Categorification of Heisenberg Action

First Approach

Set $P = \mathcal{O}_{Z^{\ell,n}}$ (structure sheaf of Nakajima correspondence) and consider $A(\ell, n) := \text{FM}_P : \text{D}(X \times X^{[\ell]}) \to \text{D}(X^{[\ell+n]})$. Let $A(\ell, -n)$ be the (right) adjoint functor.

Problem

It is too hard to compute the compositions $A(\ell, -n) \circ A(\ell, n)$ since $Z^{n,\ell}$ is badly singular.

Solution: Derived McKay Correspondence

Translate the categorification question to an easier equivariant problem using the **McKay correspondence**

\[
\text{D}(X^{[n]}) \cong \text{D}_{\mathfrak{S}_n}(X^n).
\]
Outline

1 Preliminaries
- Symmetric Products and Hilbert Schemes of Points on Surfaces
- Cohomology of Hilbert Schemes and the Heisenberg Algebra
- Derived Categories and Grothendieck Groups
- McKay Correspondence

2 Three Constructions
- Nakajima \( P \)-functors
- Lift of the Heisenberg Module Structure
- Categorical Hopf Algebras
Set-up: Let $M$ smooth quasi-projective variety, $G \subset \text{Aut}(M)$ finite subgroup such that $\omega_M$ descends to a line bundle $\omega_{M/G}$.

Definition (Crepant Resolution)

A resolution of singularities $\mu : Y \rightarrow M/G$ is crepant if

\[ \mu^* \omega_{M/G} \cong \omega_Y \]

Crepant Resolution Principle (Conjecture)

The geometry of $Y$ should reflect the $G$-equivariant geometry of $M$.

More concretely: All invariants of $Y$ should agree with the corresponding invariants of the stack (orbifold) $[M/G]$. 
The **Hilbert–Chow morphism** \( \mu : X^{[n]} \to X^{(n)} = X^n / \mathfrak{S}_n \) is a crepant resolution.

**Theorem (Bridgeland–King–Reid + Haiman 2001)**

\( X \) smooth projective surface, \( n \in \mathbb{N} \). Then:

\[
D(X^{[n]}) \cong D_{\mathfrak{S}_n}(X^n) \cong D([X^n / \mathfrak{S}_n]).
\]

\( D_{\mathfrak{S}_n}(X^n) := D(\text{Coh}_{\mathfrak{S}_n}(X^n)) \) **Equivariant derived category**.

\( \text{Coh}_{\mathfrak{S}_n}(X^n) = \text{Coh}([X^n / \mathfrak{S}_n]) \) : Abelian category of \( \mathfrak{S}_n \)-equivariant sheaves.

**Objects**: Pairs \((E, \lambda)\) with \( E \in \text{Coh}(X^n) \) and

\[\lambda = (\lambda_{\sigma} : E \xrightarrow{=} \sigma^* E)_{\sigma \in \mathfrak{S}_n} \] a \( \mathfrak{S}_n \)-linearisation.

**Morphisms**: \( \text{Hom}_{\text{Coh}_{\mathfrak{S}_n}(X^n)}((E, \lambda), (F, \nu)) = \text{Hom}_{\text{Coh}(X^n)}(E, F) \).
Outline

1 Preliminaries
   - Symmetric Products and Hilbert Schemes of Points on Surfaces
   - Cohomology of Hilbert Schemes and the Heisenberg Algebra
   - Derived Categories and Grothendieck Groups
   - McKay Correspondence

2 Three Constructions
   - Nakajima $\mathbb{P}$-functors
   - Lift of the Heisenberg Module Structure
   - Categorical Hopf Algebras
Idea of Construction

Construct $A(\ell, n) : D_{\mathfrak{S}_n}(X \times X^{\ell}) \rightarrow D_{\mathfrak{S}_{\ell+n}}(X^{\ell+n})$ on equivariant side and use McKay correspondence.

Nakajima Correspondences in $X \times X^{[\ell]} \times X^{[\ell+n]}$

$$Z^{\ell, n} = \{(x, Z, Z') \mid Z \subset Z', Z \text{ and } Z' \text{ only differ in } x\}$$

Partial Diagonals in $X \times X^{\ell} \times X^{n+\ell}$

$$\Delta_0 = \{(x; x_1, \ldots, x_{\ell}; x_1, \ldots, x_{\ell}, x, \ldots, x)\} \cong X \times X^{\ell}$$

Example ($\ell = 0$)

$$A(0, n) = \delta_* : D(X) \rightarrow D_{\mathfrak{S}_n}(X^n) \cong D(X^{[n]}) \text{ is (equivariant) push-forward along embedding of small diagonal } \delta : X \hookrightarrow X^n.$$
Let $X$ be a smooth projective surface.

**Theorem** (\_

There exists a series $A(\ell, n) = \text{FM}_{P_{\ell,n}} : \text{D}(X \times X^{[\ell]}) \to \text{D}(X^{[\ell+n]})$ of Fourier–Mukai transforms with $\text{supp}(P_{\ell,n}) = Z_{\ell,n}$. For $n > \max\{\ell, 1\}$, the $A(\ell, n)$ are $\mathbb{P}$-functors. In particular, (for $\omega_X = \mathcal{O}_X$)

$$A(\ell, -n) \circ A(\ell, n) \cong \text{id} \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2(n - 1)]$$

where $A(\ell, -n): \text{D}(X^{[n+\ell]}) \to \text{D}(X \times X^{[\ell]})$ is the right-adjoint.

Addington (2011) defined $\mathbb{P}$-functors in order to construct non-standard autoequivalences of derived categories.
Comparison with Heisenberg Commutator Relations

- **Heisenberg relations:** \([a_\alpha(m), a_\beta(n)] = \delta_{m,-n} n\langle \alpha, \beta \rangle \cdot \text{id}_H.\)
- Set \(m = -n\) and consider degree \(\ell < n:\)
  \[a_\alpha(\ell, -n) a_\beta(\ell, n) = n\langle \alpha, \beta \rangle \cdot \text{id}_H^*(X[\ell]).\]

- **Fix** \(E \in D(X): A_E(\ell, n) := A(\ell, n) \circ i_E: D(X[\ell]) \to D(X^{n+\ell})\)
  with \(i_E: D(X[\ell]) \to D(X \times X[\ell]), B \mapsto E \boxtimes B.\)
  
  \[A_E(\ell, -n) A_F(\ell, n) = i_E^R A(\ell, -n) A(\ell, n) i_F \]
  \[= i_E^R i_F ([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)]) \]
  \[= \text{Hom}^*(E, F) ([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)]) \]

\[\downarrow \text{Descend to } K(X[\ell]), \text{ set } \alpha = [E], \beta = [F], a(\ell, n) = [A(\ell, n)] \]

\[a_\alpha(\ell, -n) a_\beta(\ell, n) = n\langle \alpha, \beta \rangle \cdot \text{id}_K(X[\ell]).\]
Induced Categorical Structures

No Categorification!

\( A(n, \ell) \) do not fulfil analogues of Heisenberg relations for \( n \neq m \).

Features of the Construction

- Get interesting autoequivalences of \( D_{\mathcal{G}_m}(X^m) \cong D(X^m) \) ‘characteristic functions of the stacky loci’.

- Construction makes sense for smooth \( X \) of arbitrary dimension (forget about \( X^m \) and only consider \( [X^m/\mathcal{G}_m] \)).

- Curve case: \( A(\ell, n) : D(C \times [C^\ell/\mathcal{G}_\ell]) \hookrightarrow D([C^{\ell+n}/\mathcal{G}_{\ell+n}]) \) fully faithful.

  \( \Rightarrow \) Characteristic autoequivalences of the stacky loci.

  \( \Rightarrow \) Semi-orthogonal decomposition which categorifies decomposition of orbifold cohomology.

- \( \exists \) analogous \( A(\ell, n) \) for generalised Kummer varieties.
Outline

1. Preliminaries
   - Symmetric Products and Hilbert Schemes of Points on Surfaces
   - Cohomology of Hilbert Schemes and the Heisenberg Algebra
   - Derived Categories and Grothendieck Groups
   - McKay Correspondence

2. Three Constructions
   - Nakajima $\mathbb{P}$-functors
   - Lift of the Heisenberg Module Structure
   - Categorical Hopf Algebras
Three Constructions

Theorem (\textit{\_\_\_})

For every smooth projective variety (stack) $X$ there exists a categorical Heisenberg action on $\mathbb{D} := \bigoplus_{\ell \geq 0} D([X^\ell / G^\ell])$.

Lifts other generators $p_{\beta}^{(n)}$, $q_{\beta}^{(n)}$ (and other relations) of Heisenberg algebra: \textit{halves of the vertex operators}

\[
\sum_{n \geq 0} p_{\beta}^{(n)} z^n = \exp \left( \sum_{\ell \geq 1} \frac{a_{\beta}(-\ell)}{\ell} z^\ell \right), \quad \sum_{n \geq 0} q_{\beta}^{(n)} z^n = \exp \left( \sum_{\ell \geq 1} \frac{a_{\beta}(\ell)}{\ell} z^\ell \right)
\]

Non-Integer Coefficients

Cannot reconstruct lifts of Nakajima operators from this.

Straight-forward generalisation of parts of constructions of Cautis–Licata ($X \to \mathbb{C}^2 / G$ resolution of Kleinian singularity) and Khovanov ($X$ a point).
Categorical Heisenberg Action

Definition/Lemma

For \( \chi \in \mathbb{C} \) and \( k \) a non-negative integer, we set
\[
s^k \chi := (\chi + k - 1)!(\chi + k - 2) \cdots (\chi + 1)\chi.
\]
For \( V^* \) a graded vector space:
\[
\chi(S^k V^*) = s^k(\chi(V^*)).
\]

\[
[q^{(m)}_\alpha, q^{(n)}_\beta] = 0 = [p^{(m)}_\alpha, p^{(n)}_\beta]
\]
\[
q^{(m)}_\alpha p^{(n)}_\beta = \sum_{k=0}^{\min\{m,n\}} s^k \langle \alpha, \beta \rangle \cdot p^{(n-k)}_\beta q^{(m-k)}_\alpha.
\]

Definition (Categorical Heisenberg Action)

A family of adjoint functors \( P^{(m)} : \mathbb{D} \leftrightarrows \mathbb{D} : Q^{(m)}_E \), for \( E \in \mathbb{D}(X) \)
and \( m \in \mathbb{Z} \), such that
\[
Q^{(m)}_E Q^{(n)}_F \cong Q^{(n)}_F Q^{(m)}_E, \quad P^{(m)}_E P^{(n)}_F \cong P^{(n)}_F P^{(m)}_E
\]
\[
Q^{(m)}_E P^{(n)}_F \cong \bigoplus_{k=0}^{\min\{m,n\}} S^k \text{Hom}^*(E, F) \otimes \mathbb{C} P^{(n-k)}_F Q^{(m-k)}_E
\]
\[\implies \text{Heisenberg module structure on } \mathbb{K}.\]
Outline

1. Preliminaries
   - Symmetric Products and Hilbert Schemes of Points on Surfaces
   - Cohomology of Hilbert Schemes and the Heisenberg Algebra
   - Derived Categories and Grothendieck Groups
   - McKay Correspondence

2. Three Constructions
   - Nakajima $\mathbb{P}$-functors
   - Lift of the Heisenberg Module Structure
   - Categorical Hopf Algebras
The Case that $X$ is a Point

**Note:** $\text{Coh}_{\mathbb{S}_n}(\text{pt}) \cong \text{Rep}(\mathbb{S}_n)$, $K([\text{pt}/\mathbb{S}_n]) \cong R(\mathbb{S}_n)$.

**Consider:** Graded vector space $R := \bigoplus_{n \geq 0} R(\mathbb{S}_n)$.

**Theorem (..., Zelevinsky)**

$R$ is a positive self-adjoint Hopf algebra (PSH). This means:

- **Bilinear form** $\langle V, W \rangle := \text{hom}(V, W)^G$.
- **Multiplication** $m = \text{Ind}: R \otimes R \rightarrow R$, on graded pieces: $R_a \otimes R_b \rightarrow R_{a+b}$, $V \otimes W \mapsto \bigoplus_{\mathbb{S}_{a+b}/(\mathbb{S}_a \times \mathbb{S}_b)} (V \boxtimes W)$.
- **Comultiplication** $\nabla = \text{Res}: R \rightarrow R \otimes R$ adjoint to $m$.
The General Case

- **Want:** \( \mathcal{D} := \bigoplus_{n \geq 0} \mathcal{D}([X^n/\mathcal{G}_n]) = \bigoplus_{n \geq 0} \mathcal{D}_{\mathcal{G}_n}(X^n) \) as a PSH category.

- **Idea:** For stacks \( \mathcal{Y}, \mathcal{Z} \) have \( \mathcal{D}(\mathcal{Y} \times \mathcal{Z}) \cong \mathcal{D}(\mathcal{Y}) \otimes \mathcal{D}(\mathcal{Z}) \) (can be made precise on level of dg-categories).

- \([X^a/\mathcal{G}_a] \times [X^b/\mathcal{G}_b] = [X^{a+b}/(\mathcal{G}_a \times \mathcal{G}_b)].\)

- \(E \in \mathcal{D}_{\mathcal{G}_{a+b}}(X^{a+b}) : \) \( \text{Ind}_{\mathcal{G}_a \times \mathcal{G}_b}^{\mathcal{G}_{a+b}}(E) = \bigoplus_{\mathcal{G}_{a+b}/(\mathcal{G}_a \times \mathcal{G}_b)} \sigma^* E. \)

- **Adjoint pair:**
  \[ m = \text{Ind} : \mathcal{D}_{\mathcal{G}_a \times \mathcal{G}_b}(X^a \times X^b) \leftrightarrow \mathcal{D}_{\mathcal{G}_{a+b}}(X^{a+b}) : \text{Res} = \nabla \]

\[
\begin{array}{ccc}
\mathcal{D} \otimes \mathcal{D} & \xrightarrow{m} & \mathcal{D} \\
\downarrow \nabla \otimes \nabla & & \downarrow \nabla \\
\mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} & \xrightarrow{\text{Res}} & \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \\
\end{array}
\]

**Slogan:** \( \mathcal{D} \) is a geometric PSH category.
A. Gal and E. Gal define **Heisenberg double** associated to every PSH category (in a stricter sense). Would like to do the same for geometric PSH category.

There exists the notion of **symmetric product of a (dg) category** by Ganter and Kapranov such that

\[ \text{Sym}^n(D(X)) \cong D([X^n/\mathcal{G}_n]). \]

**Question:** To which extend do the above constructions generalise to symmetric products of categories?