Positive energy representations of Hilbert loop algebras

Timothée Marquis
(joint with Karl-Hermann Neeb)

FAU Erlangen-Nuernberg

March 2016

Plan

Problematic and motivation

Lie algebra reformulation

Locally finite Lie algebras

Locally affine Lie algebras

Problematic and motivation

Problematic: Positive energy representations

- G Lie group with Lie algebra $\mathfrak{g} = \mathbb{L}(G)$.
  $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action on G.

Motivation

Problematic and motivation

Problematic: Positive energy representations

- G Lie group with Lie algebra $\mathfrak{g} = \mathbb{L}(G)$.
  $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action on G.
- $\pi\colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H})$ unitary representation on the Hilbert space $\mathcal{H}$.
  $\mathrm{d}\pi\colon \mathfrak{g} \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\infty)$ derived representation, $D := \frac{d}{dt}\big|_{t=0}\mathbb{L}(\alpha_t) \in \operatorname{der}\mathfrak{g}$.

Motivation

Problematic and motivation

Problematic: Positive energy representations

- G Lie group with Lie algebra $\mathfrak{g} = \mathbb{L}(G)$.
  $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action on G.
- $\pi\colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H})$ unitary representation on the Hilbert space $\mathcal{H}$.
  $\mathrm{d}\pi\colon \mathfrak{g} \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\infty)$ derived representation, $D := \frac{d}{dt}|_{t=0}\mathbb{L}(\alpha_t) \in \mathrm{der}\,\mathfrak{g}$.
- $(\pi, \mathcal{H})$ is a **positive energy representation (PER)** if the spectrum of the *Hamiltonian* $H := -i\mathrm{d}\pi(D)$ is bounded from below.

Motivation

Problematic and motivation

Problematic: Positive energy representations

- G Lie group with Lie algebra $\mathfrak{g} = \mathbb{L}(G)$.
  $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action on G.
- $\pi\colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H})$ unitary representation on the Hilbert space $\mathcal{H}$.
  $\mathrm{d}\pi\colon \mathfrak{g} \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\infty)$ derived representation, $D := \frac{d}{dt}|_{t=0}\mathbb{L}(\alpha_t) \in \operatorname{der} \mathfrak{g}$.
- $(\pi, \mathcal{H})$ is a **positive energy representation (PER)** if the spectrum of the *Hamiltonian* $H := -i\mathrm{d}\pi(D)$ is bounded from below.

Problem: Determine the irreducible PER $(\pi, \mathcal{H})$ of $G \rtimes_\alpha \mathbb{R}$.

Motivation

Problematic and motivation

Problematic: Positive energy representations

- G Lie group with Lie algebra $\mathfrak{g} = \mathbb{L}(G)$.
  $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action on G.
- $\pi\colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H})$ unitary representation on the Hilbert space $\mathcal{H}$.
  $\mathrm{d}\pi\colon \mathfrak{g} \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\infty)$ derived representation, $D := \frac{d}{dt}|_{t=0}\mathbb{L}(\alpha_t) \in \operatorname{der}\mathfrak{g}$.
- $(\pi, \mathcal{H})$ is a **positive energy representation (PER)** if the spectrum of the *Hamiltonian* $H := -i\mathrm{d}\pi(D)$ is bounded from below.

Problem: Determine the irreducible PER $(\pi, \mathcal{H})$ of $G \rtimes_\alpha \mathbb{R}$.

Motivation

- Problem $\approx$ "*Given a Lie group G and $d \in \mathfrak{g} = \mathbb{L}(G)$, determine all unitary representations $(\pi, \mathcal{H})$ of G for which* $\mathrm{Spec}(-i\mathrm{d}\pi(d))$ *is bounded from below.*"

Problematic and motivation

Problematic: Positive energy representations

- G Lie group with Lie algebra $\mathfrak{g} = \mathbb{L}(G)$.
  $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action on G.
- $\pi\colon G \rtimes_\alpha \mathbb{R} \to U(\mathcal{H})$ unitary representation on the Hilbert space $\mathcal{H}$.
  $\mathrm{d}\pi\colon \mathfrak{g} \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\infty)$ derived representation, $D := \frac{d}{dt}|_{t=0}\mathbb{L}(\alpha_t) \in \operatorname{der}\mathfrak{g}$.
- $(\pi, \mathcal{H})$ is a **positive energy representation (PER)** if the spectrum of the *Hamiltonian* $H := -i\mathrm{d}\pi(D)$ is bounded from below.

Problem: Determine the irreducible PER $(\pi, \mathcal{H})$ of $G \rtimes_\alpha \mathbb{R}$.

Motivation

- Problem ≈ "*Given a Lie group* G *and* $d \in \mathfrak{g} = \mathbb{L}(G)$, *determine all unitary representations* $(\pi, \mathcal{H})$ *of* G *for which* $\mathrm{Spec}(-i\mathrm{d}\pi(d))$ *is bounded from below.*"
- $\rightsquigarrow$ related to semibounded unitary representations (see [Neeb 2015, arXiv:1510.08695] for a recent survey).

Lie algebra reformulation (1/2)

Quadratic split Lie algebras

- A complex Lie algebra $\mathfrak{g}$ is **split** if there is a maximal abelian, ad-diagonalisable subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (a *Cartan subalgebra*).

Lie algebra reformulation (1/2)

Quadratic split Lie algebras

- A complex Lie algebra $\mathfrak{g}$ is **split** if there is a maximal abelian, ad-diagonalisable subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (a *Cartan subalgebra*).
- Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where

$$\forall \alpha \in \mathfrak{h}^*,\ \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\} \quad \textbf{root space},$$
$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\} \quad \textbf{root system}.$$

Lie algebra reformulation (1/2)

Quadratic split Lie algebras

- A complex Lie algebra $\mathfrak{g}$ is **split** if there is a maximal abelian, ad-diagonalisable subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (a *Cartan subalgebra*).
- Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where

$$\forall \alpha \in \mathfrak{h}^*,\ \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\} \quad \textbf{root space},$$
$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\} \quad \textbf{root system}.$$

- A root $\alpha \in \Delta$ is **integrable** if $\mathfrak{g}_{\pm\alpha} = \mathbb{C}x_{\pm\alpha}$, $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$, and $\operatorname{ad} x_{\pm\alpha}$ is locally nilpotent. Set $\Delta_i := \{\alpha \in \Delta \mid \alpha \text{ integrable}\}$.
  For $\alpha \in \Delta_i$, the unique $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(\alpha^\vee) = 2$ is the **coroot** of α.
  NB: $\mathfrak{g}_{-\alpha} + \mathbb{C}\alpha^\vee + \mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$ for all $\alpha \in \Delta_i$.

Lie algebra reformulation (1/2)

Quadratic split Lie algebras

- A complex Lie algebra $\mathfrak{g}$ is **split** if there is a maximal abelian, ad-diagonalisable subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (a *Cartan subalgebra*).
- Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where

$$\forall \alpha \in \mathfrak{h}^*,\ \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x\ \forall h \in \mathfrak{h}\} \quad \textbf{root space},$$
$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\} \quad \textbf{root system}.$$

- A root $\alpha \in \Delta$ is **integrable** if $\mathfrak{g}_{\pm\alpha} = \mathbb{C}x_{\pm\alpha}$, $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$, and $\operatorname{ad} x_{\pm\alpha}$ is locally nilpotent. Set $\Delta_i := \{\alpha \in \Delta \mid \alpha \text{ integrable}\}$.
  For $\alpha \in \Delta_i$, the unique $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(\alpha^\vee) = 2$ is the **coroot** of α.
  NB: $\mathfrak{g}_{-\alpha} + \mathbb{C}\alpha^\vee + \mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$ for all $\alpha \in \Delta_i$.
- $W := W(\mathfrak{g}, \mathfrak{h}) := \langle r_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^* : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha^\vee \mid \alpha \in \Delta_i \rangle \leq \mathrm{GL}(\mathfrak{h}^*)$ is the **Weyl group** of $\mathfrak{g}$.

Lie algebra reformulation (1/2)

Quadratic split Lie algebras

- A complex Lie algebra $\mathfrak{g}$ is **split** if there is a maximal abelian, ad-diagonalisable subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ (a *Cartan subalgebra*).
- Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where

$$\forall \alpha \in \mathfrak{h}^*,\ \mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x\ \forall h \in \mathfrak{h}\} \quad \textbf{root space},$$
$$\Delta := \Delta(\mathfrak{g},\mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\} \quad \textbf{root system}.$$

- A root $\alpha \in \Delta$ is **integrable** if $\mathfrak{g}_{\pm\alpha} = \mathbb{C}x_{\pm\alpha}$, $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$, and $\operatorname{ad} x_{\pm\alpha}$ is locally nilpotent. Set $\Delta_i := \{\alpha \in \Delta \mid \alpha \text{ integrable}\}$.
  For $\alpha \in \Delta_i$, the unique $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with $\alpha(\alpha^\vee) = 2$ is the **coroot** of α.
  NB: $\mathfrak{g}_{-\alpha} + \mathbb{C}\alpha^\vee + \mathfrak{g}_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$ for all $\alpha \in \Delta_i$.
- $W := W(\mathfrak{g},\mathfrak{h}) := \langle r_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^* : \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee\rangle \alpha^\vee \mid \alpha \in \Delta_i\rangle \leq \mathrm{GL}(\mathfrak{h}^*)$ is the **Weyl group** of $\mathfrak{g}$.
- $\mathfrak{g}$ is moreover **quadratic** if it possesses a non-degenerate symmetric bilinear form $\kappa \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which is *invariant*: $\kappa([x,y],z) = \kappa(x,[y,z])$.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a **positive system**: $\Delta = \Delta^+ \,\dot{\cup}\, -\Delta^+$ and the monoid $\mathbb{N}[\Delta^+] := \{\sum_{i=1}^k n_i \alpha_i \mid \alpha_i \in \Delta^+,\ n_i, k \in \mathbb{N}\}$ is free.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a **positive system**: $\Delta = \Delta^+ \dot\cup -\Delta^+$ and the monoid $\mathbb{N}[\Delta^+] := \{\sum_{i=1}^k n_i\alpha_i \mid \alpha_i \in \Delta^+,\ n_i, k \in \mathbb{N}\}$ is free.
- Let $\lambda \in \mathfrak{h}^*$. A $\mathfrak{g}$-module $V = V^\lambda$ is a **highest weight module** (HWM) with highest weight λ if there exists some nonzero $v_\lambda \in V$ such that
  - $h \cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$,
  - $\mathfrak{g}_\alpha \cdot v_\lambda = \{0\}$ for all $\alpha \in \Delta^+$,
  - $V = \mathcal{U}(\mathfrak{g}) \cdot v_\lambda$.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a **positive system**: $\Delta = \Delta^+ \dot{\cup} -\Delta^+$ and the monoid $\mathbb{N}[\Delta^+] := \{\sum_{i=1}^k n_i\alpha_i \mid \alpha_i \in \Delta^+,\ n_i, k \in \mathbb{N}\}$ is free.
- Let $\lambda \in \mathfrak{h}^*$. A $\mathfrak{g}$-module $V = V^\lambda$ is a **highest weight module** (HWM) with highest weight λ if there exists some nonzero $v_\lambda \in V$ such that
  - $h \cdot v_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$,
  - $\mathfrak{g}_\alpha \cdot v_\lambda = \{0\}$ for all $\alpha \in \Delta^+$,
  - $V = \mathcal{U}(\mathfrak{g}) \cdot v_\lambda$.
- For $\mu \in \mathfrak{h}^*$, $V_\mu := \{v \in V \mid h \cdot v = \mu(h)v\ \forall h \in \mathfrak{h}\}$ **weight space**. $\mathcal{P}_\lambda := \{\mu \in \mathfrak{h}^* \mid V_\mu \neq \{0\}\}$ set of **weights**.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a **positive system**: $\Delta = \Delta^+ \dot\cup -\Delta^+$ and the monoid $\mathbb{N}[\Delta^+] := \{\sum_{i=1}^k n_i\alpha_i \mid \alpha_i \in \Delta^+,\ n_i, k \in \mathbb{N}\}$ is free.
- Let $\lambda \in \mathfrak{h}^*$. A $\mathfrak{g}$-module $V = V^\lambda$ is a **highest weight module** (HWM) with highest weight λ if there exists some nonzero $v_\lambda \in V$ such that
  - $h \cdot v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$,
  - $\mathfrak{g}_\alpha \cdot v_\lambda = \{0\}$ for all $\alpha \in \Delta^+$,
  - $V = \mathcal{U}(\mathfrak{g}) \cdot v_\lambda$.
- For $\mu \in \mathfrak{h}^*$, $V_\mu := \{v \in V \mid h \cdot v = \mu(h)v\ \forall h \in \mathfrak{h}\}$ **weight space**. $\mathcal{P}_\lambda := \{\mu \in \mathfrak{h}^* \mid V_\mu \neq \{0\}\}$ set of **weights**.
- We assume the property *(Weight)*: $\mathcal{P}_\lambda = \mathrm{Conv}(W.\lambda) \cap (\lambda + \mathbb{Z}[\Delta])$.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a positive system.
- Let $\lambda \in \mathfrak{h}^*$ and $V = V^\lambda$ a HWM over $\mathfrak{g}$ with highest weight λ.
- We assume the property *(Weight)*: $\mathcal{P}_\lambda = \mathrm{Conv}(W.\lambda) \cap (\lambda + \mathbb{Z}[\Delta])$.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a positive system.
- Let $\lambda \in \mathfrak{h}^*$ and $V = V^\lambda$ a HWM over $\mathfrak{g}$ with highest weight λ.
- We assume the property *(Weight)*: $\mathcal{P}_\lambda = \mathrm{Conv}(W.\lambda) \cap (\lambda + \mathbb{Z}[\Delta])$.

Positive energy

- Consider the highest weight representation $\rho_\lambda \colon \mathfrak{g} \to \mathrm{End}(V^\lambda)$.
- Let $D \in \mathrm{der}(\mathfrak{g})$ be a *skew-symmetric* derivation: $\kappa(Dx, y) = -\kappa(x, Dy)$. Assume D is *diagonal*: $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a positive system.
- Let $\lambda \in \mathfrak{h}^*$ and $V = V^\lambda$ a HWM over $\mathfrak{g}$ with highest weight λ.
- We assume the property *(Weight)*: $\mathcal{P}_\lambda = \mathrm{Conv}(W.\lambda) \cap (\lambda + \mathbb{Z}[\Delta])$.

Positive energy

- Consider the highest weight representation $\rho_\lambda \colon \mathfrak{g} \to \mathrm{End}(V^\lambda)$.
- Let $D \in \mathrm{der}(\mathfrak{g})$ be a *skew-symmetric* derivation: $\kappa(Dx, y) = -\kappa(x, Dy)$. Assume D is *diagonal*: $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$.
- Then ρ_λ can be extended to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ by setting $\widetilde{\rho}_\lambda(D)v_\mu := i\chi(\mu - \lambda)v_\mu$ for all $\mu \in \mathcal{P}_\lambda$, $v_\mu \in V_\mu$.

Lie algebra reformulation (2/2)

Highest-weight representations

- $(\mathfrak{g}, \mathfrak{h}, \kappa)$ a quadratic split Lie algebra.
- $\Delta^+ \subseteq \Delta$ a positive system.
- Let $\lambda \in \mathfrak{h}^*$ and $V = V^\lambda$ a HWM over $\mathfrak{g}$ with highest weight λ.
- We assume the property *(Weight)*: $\mathcal{P}_\lambda = \mathrm{Conv}(W.\lambda) \cap (\lambda + \mathbb{Z}[\Delta])$.

Positive energy

- Consider the highest weight representation $\rho_\lambda \colon \mathfrak{g} \to \mathrm{End}(V^\lambda)$.
- Let $D \in \mathrm{der}(\mathfrak{g})$ be a *skew-symmetric* derivation: $\kappa(Dx, y) = -\kappa(x, Dy)$. Assume D is *diagonal*: $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$.
- Then ρ_λ can be extended to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ by setting $\widetilde{\rho}_\lambda(D)v_\mu := i\chi(\mu - \lambda)v_\mu$ for all $\mu \in \mathcal{P}_\lambda$, $v_\mu \in V_\mu$.
- Then

$$\begin{aligned}\widetilde{\rho}_\lambda \text{ is a PER} &\Leftrightarrow \text{Spectrum of } H := -i\widetilde{\rho}_\lambda(D) \text{ is bounded from below}\\ &\Leftrightarrow \inf \chi(\mathcal{P}_\lambda - \lambda) > -\infty\\ &\Leftrightarrow \inf \chi(W.\lambda - \lambda) > -\infty.\end{aligned}$$

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite**
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras

.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

- For a set J, consider the pre-Hilbert space $\mathbb{C}^{(J)} := \mathrm{vect}_{\mathbb{C}}\{e_j\}_{j \in J}$.

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

- For a set J, consider the pre-Hilbert space $\mathbb{C}^{(J)} := \mathrm{vect}_{\mathbb{C}}\{e_j\}_{j\in J}$.
- $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{C}) := \{A \in \mathrm{End}(\mathbb{C}^{(J)}) \mid A_{ij} := \langle Ae_j, e_i\rangle = 0 \ \forall'(i,j) \in J \times J\}$.
  Define $E_{jk} \in \mathfrak{g}$ by $E_{jk}(x) := \langle x, e_k\rangle e_j$ for all $x \in \mathbb{C}^{(J)}$.

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

- For a set J, consider the pre-Hilbert space $\mathbb{C}^{(J)} := \mathrm{vect}_{\mathbb{C}}\{e_j\}_{j \in J}$.
- $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{C}) := \{A \in \mathrm{End}(\mathbb{C}^{(J)}) \mid A_{ij} := \langle Ae_j, e_i \rangle = 0 \ \forall'(i,j) \in J \times J\}$.
  Define $E_{jk} \in \mathfrak{g}$ by $E_{jk}(x) := \langle x, e_k \rangle e_j$ for all $x \in \mathbb{C}^{(J)}$.
- $\mathfrak{h} := \{\text{diagonal matrices } \sum_j x_j E_{jj}\} \subseteq \mathfrak{g}$ is a Cartan subalgebra,
  $\mathfrak{h}^* = \{\sum_j x_j \varepsilon_j \mid x_j \in \mathbb{C}\}$ where $\varepsilon_k(E_{jj}) := \delta_{jk}$.

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

- For a set J, consider the pre-Hilbert space $\mathbb{C}^{(J)} := \mathrm{vect}_{\mathbb{C}}\{e_j\}_{j \in J}$.
- $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{C}) := \{A \in \mathrm{End}(\mathbb{C}^{(J)}) \mid A_{ij} := \langle Ae_j, e_i\rangle = 0 \ \forall'(i,j) \in J \times J\}$.
  Define $E_{jk} \in \mathfrak{g}$ by $E_{jk}(x) := \langle x, e_k\rangle e_j$ for all $x \in \mathbb{C}^{(J)}$.
- $\mathfrak{h} := \{\text{diagonal matrices } \sum_j x_j E_{jj}\} \subseteq \mathfrak{g}$ is a Cartan subalgebra,
  $\mathfrak{h}^* = \{\sum_j x_j \varepsilon_j \mid x_j \in \mathbb{C}\}$ where $\varepsilon_k(E_{jj}) := \delta_{jk}$.
- $[E_{\ell\ell}, E_{jk}] = (\delta_{\ell j} - \delta_{\ell k})E_{jk} = (\varepsilon_j - \varepsilon_k)(E_{\ell\ell})E_{jk} \Rightarrow \mathfrak{g}_{\varepsilon_j - \varepsilon_k} = \mathbb{C}E_{jk}$.
  $\rightsquigarrow \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $\Delta = \Delta(A_J) := \{\varepsilon_j - \varepsilon_k \mid j, k \in J,\ j \neq k\}$.

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

- For a set J, consider the pre-Hilbert space $\mathbb{C}^{(J)} := \mathrm{vect}_{\mathbb{C}}\{e_j\}_{j \in J}$.
- $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{C}) := \{A \in \mathrm{End}(\mathbb{C}^{(J)}) \mid A_{ij} := \langle Ae_j, e_i \rangle = 0 \ \forall'(i,j) \in J \times J\}$.
  Define $E_{jk} \in \mathfrak{g}$ by $E_{jk}(x) := \langle x, e_k \rangle e_j$ for all $x \in \mathbb{C}^{(J)}$.
- $\mathfrak{h} := \{\text{diagonal matrices } \sum_j x_j E_{jj}\} \subseteq \mathfrak{g}$ is a Cartan subalgebra,
  $\mathfrak{h}^* = \{\sum_j x_j \varepsilon_j \mid x_j \in \mathbb{C}\}$ where $\varepsilon_k(E_{jj}) := \delta_{jk}$.
- $[E_{\ell\ell}, E_{jk}] = (\delta_{\ell j} - \delta_{\ell k}) E_{jk} = (\varepsilon_j - \varepsilon_k)(E_{\ell\ell}) E_{jk} \Rightarrow \mathfrak{g}_{\varepsilon_j - \varepsilon_k} = \mathbb{C} E_{jk}$.
  $\rightsquigarrow \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $\Delta = \Delta(A_J) := \{\varepsilon_j - \varepsilon_k \mid j, k \in J,\ j \neq k\}$.
- $r_{\varepsilon_j - \varepsilon_k} = (j, k) \in S_J \Rightarrow W = W(\mathfrak{g}, \mathfrak{h}) = S_{(J)} \leq S_J$ finite permutations of J.

Locally finite Lie algebras (1/3)

Locally finite Lie algebras

- $\mathfrak{g}$ is **locally finite** simple
  $\Leftrightarrow$ Every finite subset of $\mathfrak{g}$ generates a finite dimensional Lie subalgebra.
  $\Leftrightarrow$ $\mathfrak{g}$ is the directed union of its finite dimensional subalgebras that are simple $\rightsquigarrow$ of type A_n, B_n, C_n, D_n.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally finite root system* Δ of type A_J, B_J, C_J or D_J for some infinite set J.

Example: $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$

- For a set J, consider the pre-Hilbert space $\mathbb{C}^{(J)} := \mathrm{vect}_{\mathbb{C}}\{e_j\}_{j \in J}$.
- $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{C}) := \{A \in \mathrm{End}(\mathbb{C}^{(J)}) \mid A_{ij} := \langle Ae_j, e_i \rangle = 0 \ \forall'(i,j) \in J \times J\}$.
  Define $E_{jk} \in \mathfrak{g}$ by $E_{jk}(x) := \langle x, e_k \rangle e_j$ for all $x \in \mathbb{C}^{(J)}$.
- $\mathfrak{h} := \{\text{diagonal matrices } \sum_j x_j E_{jj}\} \subseteq \mathfrak{g}$ is a Cartan subalgebra,
  $\mathfrak{h}^* = \{\sum_j x_j \varepsilon_j \mid x_j \in \mathbb{C}\}$ where $\varepsilon_k(E_{jj}) := \delta_{jk}$.
- $[E_{\ell\ell}, E_{jk}] = (\delta_{\ell j} - \delta_{\ell k}) E_{jk} = (\varepsilon_j - \varepsilon_k)(E_{\ell\ell}) E_{jk} \Rightarrow \mathfrak{g}_{\varepsilon_j - \varepsilon_k} = \mathbb{C} E_{jk}$.
  $\rightsquigarrow \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, $\Delta = \Delta(A_J) := \{\varepsilon_j - \varepsilon_k \mid j, k \in J, \ j \neq k\}$.
- $r_{\varepsilon_j - \varepsilon_k} = (j, k) \in S_J \Rightarrow W = W(\mathfrak{g}, \mathfrak{h}) = S_{(J)} \leq S_J$ finite permutations of J.
- $\kappa(x, y) := \mathrm{tr}(xy)$ non-degenerate invariant symmetric bilinear form.
  NB: $\mathfrak{g}$ has an antilinear involution $*\colon \ \mathfrak{g} \to \mathfrak{g} : E_{ij} \mapsto E_{ij}^* := E_{ji}$.

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.

Example: infinite wedge representations

Locally finite Lie algebras (2/3)

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- Facts (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.

Example: infinite wedge representations

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w\rangle = \langle v, X^* \cdot w\rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- Facts (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.
  - $L(\lambda, \Delta_+)$ is unitary $\Leftrightarrow$ λ *dominant integral*, i.e., $\lambda(\alpha^\vee) \in \mathbb{N}\ \forall \alpha \in \Delta_i$.

Example: infinite wedge representations

Locally finite Lie algebras (2/3)

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- Facts (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.
  - $L(\lambda, \Delta_+)$ is unitary $\Leftrightarrow$ λ *dominant integral*, i.e., $\lambda(\alpha^\vee) \in \mathbb{N}\ \forall \alpha \in \Delta_i$.
  - If $L(\lambda, \Delta_+)$ is unitary, then Property (Weight) is satisfied.

Example: infinite wedge representations

Locally finite Lie algebras (2/3)

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- Facts (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.
  - $L(\lambda, \Delta_+)$ is unitary $\Leftrightarrow$ λ *dominant integral*, i.e., $\lambda(\alpha^\vee) \in \mathbb{N}\ \forall \alpha \in \Delta_i$.
  - If $L(\lambda, \Delta_+)$ is unitary, then Property (Weight) is satisfied.

Example: infinite wedge representations

- Let $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$ with $J = \mathbb{Z}$, and choose the positive system $\Delta_+ = \{\varepsilon_j - \varepsilon_k \mid j < k\} \subseteq \Delta$.

Locally finite Lie algebras (2/3)

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- Facts (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.
  - $L(\lambda, \Delta_+)$ is unitary $\Leftrightarrow$ λ *dominant integral*, i.e., $\lambda(\alpha^\vee) \in \mathbb{N}\ \forall \alpha \in \Delta_i$.
  - If $L(\lambda, \Delta_+)$ is unitary, then Property (Weight) is satisfied.

Example: infinite wedge representations

- Let $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$ with $J = \mathbb{Z}$, and choose the positive system $\Delta_+ = \{\varepsilon_j - \varepsilon_k \mid j < k\} \subseteq \Delta$.
- $V := \mathrm{vect}_{\mathbb{C}} \langle e_{j_0} \wedge e_{j_{-1}} \wedge \ldots \mid j_0 > j_{-1} > \ldots \text{ and } j_s = s\ \forall s << 0 \rangle \subseteq \Lambda^\infty(\mathbb{C}^{(J)})$.
  $\psi := e_0 \wedge e_{-1} \wedge e_{-2} \wedge \cdots \in V$ *vacuum*.

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- <u>Facts</u> (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.
  - $L(\lambda, \Delta_+)$ is unitary $\Leftrightarrow$ λ *dominant integral*, i.e., $\lambda(\alpha^\vee) \in \mathbb{N}\ \forall \alpha \in \Delta_i$.
  - If $L(\lambda, \Delta_+)$ is unitary, then Property (Weight) is satisfied.

Example: infinite wedge representations

- Let $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$ with $J = \mathbb{Z}$, and choose the positive system $\Delta_+ = \{\varepsilon_j - \varepsilon_k \mid j < k\} \subseteq \Delta$.
- $V := \mathrm{vect}_{\mathbb{C}}\langle e_{j_0} \wedge e_{j_{-1}} \wedge \dots \mid j_0 > j_{-1} > \dots \text{ and } j_s = s\ \forall s << 0\rangle \subseteq \Lambda^\infty(\mathbb{C}^{(J)})$.
  $\psi := e_0 \wedge e_{-1} \wedge e_{-2} \wedge \dots \in V$ *vacuum*.
- $\mathfrak{g} \curvearrowright V$: $A(e_{j_0} \wedge e_{j_{-1}} \wedge \dots) := (Ae_{j_0}) \wedge e_{j_{-1}} \wedge \dots + e_{j_0} \wedge (Ae_{j_{-1}}) \wedge \dots + \dots$

$$\Rightarrow E_{jk}(\psi) = \begin{cases} e_0 \wedge \dots \wedge e_{k+1} \wedge e_j \wedge e_{k-1} \wedge \dots & \text{if } k \leq 0 < j, \\ 0 & \text{otherwise.} \end{cases}$$

Locally finite Lie algebras (2/3)

Unitary highest weight representations

- A $\mathfrak{g}$-module V is **unitary** if it has a contravariant positive definite hermitian form: $\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$ for all $X \in \mathfrak{g}$, $v, w \in V$.
- Facts (Neeb '98): Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra.
  - $\forall \lambda \in \mathfrak{h}^*$, $\exists!$ irreducible HWM $L(\lambda, \Delta_+)$ over $\mathfrak{g}$.
  - $L(\lambda, \Delta_+)$ is unitary $\Leftrightarrow$ λ *dominant integral*, i.e., $\lambda(\alpha^\vee) \in \mathbb{N}\ \forall \alpha \in \Delta_i$.
  - If $L(\lambda, \Delta_+)$ is unitary, then Property (Weight) is satisfied.

Example: infinite wedge representations

- Let $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$ with $J = \mathbb{Z}$, and choose the positive system $\Delta_+ = \{\varepsilon_j - \varepsilon_k \mid j < k\} \subseteq \Delta$.
- $V := \mathrm{vect}_{\mathbb{C}}\langle e_{j_0} \wedge e_{j_{-1}} \wedge \dots \mid j_0 > j_{-1} > \dots \text{ and } j_s = s\ \forall s << 0\rangle \subseteq \Lambda^\infty(\mathbb{C}^{(J)})$.
  $\psi := e_0 \wedge e_{-1} \wedge e_{-2} \wedge \dots \in V$ *vacuum*.
- $\mathfrak{g} \curvearrowright V$: $A(e_{j_0} \wedge e_{j_{-1}} \wedge \dots) := (Ae_{j_0}) \wedge e_{j_{-1}} \wedge \dots + e_{j_0} \wedge (Ae_{j_{-1}}) \wedge \dots + \dots$
  $$\Rightarrow E_{jk}(\psi) = \begin{cases} e_0 \wedge \dots \wedge e_{k+1} \wedge e_j \wedge e_{k-1} \wedge \dots & \text{if } k \leq 0 < j, \\ 0 & \text{otherwise.} \end{cases}$$
  $\Rightarrow E_{jj}(\psi) = \delta_{j\leq 0}\psi = \lambda(E_{jj})\psi$ for $\lambda\colon \mathfrak{h} \to \mathbb{R} : E_{jj} \mapsto \lambda(E_{jj}) := \delta_{j\leq 0}$.
  $\rightsquigarrow V \cong L(\lambda, \Delta_+)$ with $V_\lambda = \mathbb{C}\psi$.

Locally finite Lie algebras (3/3)

Setting

- Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra, and let $\rho_\lambda \colon \mathfrak{g} \to \mathfrak{u}(V^\lambda)$ be a unitary HWR. Extend ρ_λ to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ for some $D \in \mathrm{der}(\mathfrak{g})$ given by $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$. Thus $\widetilde{\rho}_\lambda$ is a PER $\Leftrightarrow \inf \chi(W.\lambda - \lambda) > -\infty$.
- Δ of type A_J, B_J, C_J or D_J, can be realised inside $\mathrm{span}_{\mathbb{Z}}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.
- We assume that $\lambda = \sum_{j \in J} \lambda_j \varepsilon_j \colon \mathfrak{h} \to \mathbb{Z}$ is *bounded*: $\sup_{j \in J} |\lambda_j| < \infty$.

Locally finite Lie algebras (3/3)

Setting

- Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra, and let $\rho_\lambda \colon \mathfrak{g} \to \mathfrak{u}(V^\lambda)$ be a unitary HWR. Extend ρ_λ to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ for some $D \in \mathrm{der}(\mathfrak{g})$ given by $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$. Thus $\widetilde{\rho}_\lambda$ is a PER $\Leftrightarrow \inf \chi(W.\lambda - \lambda) > -\infty$.
- Δ of type A_J, B_J, C_J or D_J, can be realised inside $\mathrm{span}_{\mathbb{Z}}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.
- We assume that $\lambda = \sum_{j \in J} \lambda_j \varepsilon_j \colon \mathfrak{h} \to \mathbb{Z}$ is *bounded*: $\sup_{j \in J} |\lambda_j| < \infty$.

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\min} + \chi_{\mathrm{sum}}$ with $\inf \chi_{\min}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathrm{sum}}(\varepsilon_j)| < \infty$.

Locally finite Lie algebras (3/3)

Setting

- Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra, and let $\rho_\lambda \colon \mathfrak{g} \to \mathfrak{u}(V^\lambda)$ be a unitary HWR. Extend ρ_λ to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ for some $D \in \mathrm{der}(\mathfrak{g})$ given by $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$. Thus $\widetilde{\rho}_\lambda$ is a PER $\Leftrightarrow$ $\inf \chi(W.\lambda - \lambda) > -\infty$.
- Δ of type A_J, B_J, C_J or D_J, can be realised inside $\mathrm{span}_\mathbb{Z}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.
- We assume that $\lambda = \sum_{j\in J} \lambda_j \varepsilon_j \colon \mathfrak{h} \to \mathbb{Z}$ is *bounded*: $\sup_{j \in J} |\lambda_j| < \infty$.

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\min} + \chi_{\mathrm{sum}}$ with $\inf \chi_{\min}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathrm{sum}}(\varepsilon_j)| < \infty$.

Example: Unitary group $U_1(\mathcal{H})$ of Schatten class 1

- $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$, $\mathcal{H}$ Hilbert-space completion of $\mathbb{C}^{(J)}$ with onb $\{e_j\}_{j \in J}$. $B_1(\mathcal{H})$ completion of $\mathfrak{g}$ wrt the norm $\|A\|_1 := \mathrm{Tr}|A|$ (*trace-class operators*). Set $\mathfrak{u}_1(\mathcal{H}) = \{X \in B_1(\mathcal{H}) \mid X = -X^*\}$ and $U_1(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathfrak{u}_1(\mathcal{H}))$.

Locally finite Lie algebras (3/3)

Setting

- Let $(\mathfrak{g}, \mathfrak{h})$ be a locally finite simple Lie algebra, and let $\rho_\lambda \colon \mathfrak{g} \to \mathfrak{u}(V^\lambda)$ be a unitary HWR. Extend ρ_λ to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ for some $D \in \mathrm{der}(\mathfrak{g})$ given by $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$. Thus $\widetilde{\rho}_\lambda$ is a PER $\Leftrightarrow \inf \chi(W.\lambda - \lambda) > -\infty$.
- Δ of type A_J, B_J, C_J or D_J, can be realised inside $\mathrm{span}_{\mathbb{Z}}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.
- We assume that $\lambda = \sum_{j \in J} \lambda_j \varepsilon_j \colon \mathfrak{h} \to \mathbb{Z}$ is *bounded*: $\sup_{j \in J} |\lambda_j| < \infty$.

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\mathrm{min}} + \chi_{\mathrm{sum}}$ with $\inf \chi_{\mathrm{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathrm{sum}}(\varepsilon_j)| < \infty$.

Example: Unitary group $U_1(\mathcal{H})$ of Schatten class 1

- $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$, $\mathcal{H}$ Hilbert-space completion of $\mathbb{C}^{(J)}$ with onb $\{e_j\}_{j \in J}$. $B_1(\mathcal{H})$ completion of $\mathfrak{g}$ wrt the norm $\|A\|_1 := \mathrm{Tr}|A|$ (*trace-class operators*). Set $\mathfrak{u}_1(\mathcal{H}) = \{X \in B_1(\mathcal{H}) \mid X = -X^*\}$ and $U_1(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathfrak{u}_1(\mathcal{H}))$.
- Fact (Neeb '98): If λ is bounded, then ρ_λ lifts to a unitary representation $\widehat{\rho}_\lambda \colon U_1(\mathcal{H}) \to U(\mathcal{H}^\lambda)$, where $\mathcal{H}^\lambda$ is the Hilbert-space completion of V^λ.

Locally finite Lie algebras (3/3)

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\mathsf{min}} + \chi_{\mathsf{sum}}$ with $\inf \chi_{\mathsf{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathsf{sum}}(\varepsilon_j)| < \infty$.

Example: Unitary group $U_1(\mathcal{H})$ of Schatten class 1

- $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$, $\mathcal{H}$ Hilbert-space completion of $\mathbb{C}^{(J)}$ with onb $\{e_j\}_{j \in J}$. $B_1(\mathcal{H})$ completion of $\mathfrak{g}$ wrt the norm $\|A\|_1 := \mathrm{Tr}|A|$ (*trace-class operators*). Set $\mathfrak{u}_1(\mathcal{H}) = \{X \in B_1(\mathcal{H}) \mid X = -X^*\}$ and $U_1(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathfrak{u}_1(\mathcal{H}))$.
- Fact (Neeb '98): If λ is bounded, then ρ_λ lifts to a unitary representation $\widehat{\rho}_\lambda\colon\ U_1(\mathcal{H}) \to U(\mathcal{H}^\lambda)$, where $\mathcal{H}^\lambda$ is the Hilbert-space completion of V^λ.

Locally finite Lie algebras (3/3)

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\text{min}} + \chi_{\text{sum}}$ with $\inf \chi_{\text{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\text{sum}}(\varepsilon_j)| < \infty$.

Example: Unitary group $U_1(\mathcal{H})$ of Schatten class 1

- $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$, $\mathcal{H}$ Hilbert-space completion of $\mathbb{C}^{(J)}$ with onb $\{e_j\}_{j \in J}$. $B_1(\mathcal{H})$ completion of $\mathfrak{g}$ wrt the norm $\|A\|_1 := \mathrm{Tr}|A|$ (*trace-class operators*). Set $\mathfrak{u}_1(\mathcal{H}) = \{X \in B_1(\mathcal{H}) \mid X = -X^*\}$ and $U_1(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathfrak{u}_1(\mathcal{H}))$.
- Fact (Neeb '98): If λ is bounded, then ρ_λ lifts to a unitary representation $\widehat{\rho}_\lambda \colon U_1(\mathcal{H}) \to U(\mathcal{H}^\lambda)$, where $\mathcal{H}^\lambda$ is the Hilbert-space completion of V^λ.
- $\alpha \colon \mathbb{R} \to U_1(\mathcal{H}) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action $\rightsquigarrow \alpha_t(g) = e^{itA} g e^{-itA}$ for some self-adjoint operator $A \in B(\mathcal{H})$. We assume A is diagonalisable: $Ae_j = d_j e_j \ \forall j \in J$. Then $\widehat{\rho}_\lambda$ extends to $U_1(\mathcal{H}) \rtimes_\alpha \mathbb{R} \to U(\mathcal{H}^\lambda)$.

Locally finite Lie algebras (3/3)

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\mathsf{min}} + \chi_{\mathsf{sum}}$ with $\inf \chi_{\mathsf{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathsf{sum}}(\varepsilon_j)| < \infty$.

Example: Unitary group $U_1(\mathcal{H})$ of Schatten class 1

- $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$, $\mathcal{H}$ Hilbert-space completion of $\mathbb{C}^{(J)}$ with onb $\{e_j\}_{j \in J}$. $B_1(\mathcal{H})$ completion of $\mathfrak{g}$ wrt the norm $\|A\|_1 := \mathrm{Tr}|A|$ (*trace-class operators*). Set $\mathfrak{u}_1(\mathcal{H}) = \{X \in B_1(\mathcal{H}) \mid X = -X^*\}$ and $U_1(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathfrak{u}_1(\mathcal{H}))$.
- <u>Fact</u> (Neeb '98): If λ is bounded, then ρ_λ lifts to a unitary representation $\widehat{\rho}_\lambda \colon U_1(\mathcal{H}) \to U(\mathcal{H}^\lambda)$, where $\mathcal{H}^\lambda$ is the Hilbert-space completion of V^λ.
- $\alpha \colon \mathbb{R} \to U_1(\mathcal{H}) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action $\rightsquigarrow \alpha_t(g) = e^{itA} g e^{-itA}$ for some self-adjoint operator $A \in B(\mathcal{H})$. We assume A is diagonalisable: $Ae_j = d_j e_j \; \forall j \in J$. Then $\widehat{\rho}_\lambda$ extends to $U_1(\mathcal{H}) \rtimes_\alpha \mathbb{R} \to U(\mathcal{H}^\lambda)$.
- Lie algebra level: $\widetilde{\rho}_\lambda \colon \mathfrak{u}_1(\mathcal{H}) \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\lambda)$ with "$D = \mathrm{ad}(iA)$", that is, $D(E_{jk}) = i(d_j - d_k)E_{jk} = i\chi(\varepsilon_j - \varepsilon_k)E_{jk} \rightsquigarrow \chi \colon \mathbb{Z}[\Delta] \to \mathbb{R} : \varepsilon_j \mapsto d_j$.

Locally finite Lie algebras (3/3)

Theorem 1 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\mathrm{min}} + \chi_{\mathrm{sum}}$ with $\inf \chi_{\mathrm{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathrm{sum}}(\varepsilon_j)| < \infty$.

Example: Unitary group $U_1(\mathcal{H})$ of Schatten class 1

- $\mathfrak{g} = \mathfrak{gl}(J, \mathbb{C})$, $\mathcal{H}$ Hilbert-space completion of $\mathbb{C}^{(J)}$ with onb $\{e_j\}_{j \in J}$. $B_1(\mathcal{H})$ completion of $\mathfrak{g}$ wrt the norm $\|A\|_1 := \mathrm{Tr}|A|$ (*trace-class operators*). Set $\mathfrak{u}_1(\mathcal{H}) = \{X \in B_1(\mathcal{H}) \mid X = -X^*\}$ and $U_1(\mathcal{H}) = U(\mathcal{H}) \cap (\mathbb{1} + \mathfrak{u}_1(\mathcal{H}))$.
- <u>Fact</u> (Neeb '98): If λ is bounded, then ρ_λ lifts to a unitary representation $\widehat{\rho}_\lambda \colon U_1(\mathcal{H}) \to U(\mathcal{H}^\lambda)$, where $\mathcal{H}^\lambda$ is the Hilbert-space completion of V^λ.
- $\alpha \colon \mathbb{R} \to U_1(\mathcal{H}) : t \mapsto \alpha_t$ continuous $\mathbb{R}$-action $\rightsquigarrow \alpha_t(g) = e^{itA} g e^{-itA}$ for some self-adjoint operator $A \in B(\mathcal{H})$. We assume A is diagonalisable: $Ae_j = d_j e_j \ \forall j \in J$. Then $\widehat{\rho}_\lambda$ extends to $U_1(\mathcal{H}) \rtimes_\alpha \mathbb{R} \to U(\mathcal{H}^\lambda)$.
- Lie algebra level: $\widetilde{\rho}_\lambda \colon \mathfrak{u}_1(\mathcal{H}) \rtimes \mathbb{R}D \to \mathfrak{u}(\mathcal{H}^\lambda)$ with "$D = \mathrm{ad}(iA)$", that is, $D(E_{jk}) = i(d_j - d_k)E_{jk} = i\chi(\varepsilon_j - \varepsilon_k)E_{jk} \rightsquigarrow \chi \colon \mathbb{Z}[\Delta] \to \mathbb{R} : \varepsilon_j \mapsto d_j$.
- Hence $\chi = \chi_{\mathrm{min}} + \chi_{\mathrm{sum}} \Leftrightarrow A = A_{\mathrm{min}} + A_{\mathrm{sum}}$ with $A_{\mathrm{min}}, A_{\mathrm{sum}} \in B(\mathcal{H})$ such that $iA_{\mathrm{sum}} \in \mathfrak{u}_1(\mathcal{H})$ and A_{min} yields a *minimal energy representation* $\Leftrightarrow \alpha_t = \alpha_t^{\mathrm{min}} \alpha_t^{\mathrm{sum}} = \alpha_t^{\mathrm{sum}} \alpha_t^{\mathrm{min}}$ with α_t^{sum} inner automorphism of $U_1(\mathcal{H})$.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$.
    $\Rightarrow \mathcal{L}_{\varphi}(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$. $\Rightarrow \mathcal{L}_\varphi(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.
  - There is a non-degenerate invariant bilinear form on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$, given by $\kappa(t^r \otimes x, t^s \otimes y) := \delta_{r,-s}\mathring{\kappa}(x, y)$.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$. $\Rightarrow \mathcal{L}_\varphi(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.
  - There is a non-degenerate invariant bilinear form on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$, given by $\kappa(t^r \otimes x, t^s \otimes y) := \delta_{r,-s}\mathring{\kappa}(x, y)$.
  - Let $D \in \mathrm{der}(\mathcal{L}_\varphi(\mathring{\mathfrak{g}}))$ be a skew-symmetric derivation. Then $\omega_D(x, y) := \kappa(Dx, y)$ is a 2-cocycle on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$. Extend D to the derivation $\widetilde{D}(z, x) = (0, Dx)$ of the corresponding central extension $\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$.
    $\Rightarrow \mathcal{L}_\varphi(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.
  - There is a non-degenerate invariant bilinear form on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$, given by $\kappa(t^r \otimes x, t^s \otimes y) := \delta_{r,-s}\mathring{\kappa}(x, y)$.
  - Let $D \in \mathrm{der}(\mathcal{L}_\varphi(\mathring{\mathfrak{g}}))$ be a skew-symmetric derivation. Then $\omega_D(x, y) := \kappa(Dx, y)$ is a 2-cocycle on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$. Extend D to the derivation $\widetilde{D}(z, x) = (0, Dx)$ of the corresponding central extension $\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.
  - $\mathfrak{g} = (\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})) \rtimes_{\widetilde{D}}$ **affinisation of** $\mathring{\mathfrak{g}}$, with Lie bracket $[(z, x, t), (z', x, t')] = (\omega_D(x, x'), [x, x'] + tDx' - t'Dx, 0)$.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$. $\Rightarrow \mathcal{L}_\varphi(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.
  - There is a non-degenerate invariant bilinear form on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$, given by $\kappa(t^r \otimes x, t^s \otimes y) := \delta_{r,-s}\mathring{\kappa}(x, y)$.
  - Let $D \in \mathrm{der}(\mathcal{L}_\varphi(\mathring{\mathfrak{g}}))$ be a skew-symmetric derivation. Then $\omega_D(x, y) := \kappa(Dx, y)$ is a 2-cocycle on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$. Extend D to the derivation $\widetilde{D}(z, x) = (0, Dx)$ of the corresponding central extension $\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.
  - $\mathfrak{g} = (\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})) \rtimes_{\widetilde{D}}$ **affinisation of** $\mathring{\mathfrak{g}}$, with Lie bracket $[(z, x, t), (z', x, t')] = (\omega_D(x, x'), [x, x'] + tDx' - t'Dx, 0)$.
  - $\mathfrak{h} := \mathbb{C} \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$ for some Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}^\varphi \subseteq \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$. $\Rightarrow \mathcal{L}_\varphi(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.
  - There is a non-degenerate invariant bilinear form on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$, given by $\kappa(t^r \otimes x, t^s \otimes y) := \delta_{r,-s}\mathring{\kappa}(x, y)$.
  - Let $D \in \mathrm{der}(\mathcal{L}_\varphi(\mathring{\mathfrak{g}}))$ be a skew-symmetric derivation. Then $\omega_D(x, y) := \kappa(Dx, y)$ is a 2-cocycle on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$. Extend D to the derivation $\widetilde{D}(z, x) = (0, Dx)$ of the corresponding central extension $\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.
  - $\mathfrak{g} = (\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})) \rtimes_{\widetilde{D}}$ **affinisation of** $\mathring{\mathfrak{g}}$, with Lie bracket $[(z, x, t), (z', x, t')] = (\omega_D(x, x'), [x, x'] + tDx' - t'Dx, 0)$.
  - $\mathfrak{h} := \mathbb{C} \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$ for some Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}^\varphi \subseteq \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.
  - $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \{0\} \times (\Delta(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}}) \cup \{0\}) \times \mathbb{C} \subseteq \mathfrak{h}^* \approx \mathbb{C} \times \mathring{\mathfrak{h}}^* \times \mathbb{C}$.

Locally affine Lie algebras (1/2)

Locally affine Lie algebras

- $\mathfrak{g}$ is **locally affine** $\Leftrightarrow$ $\mathfrak{g}$ direct limit of affine Kac–Moody algebras.
- $\rightsquigarrow$ $\mathfrak{g}$ has a *locally affine root system* Δ of type $A_J^{(1)}$, $B_J^{(1)}$, $C_J^{(1)}$, $D_J^{(1)}$, $B_J^{(2)}$, $C_J^{(2)}$ or $BC_J^{(2)}$ for some infinite set J.
- $\mathfrak{g}$ can be constructed as double extension of a (twisted) loop algebra over a locally finite simple Lie algebra $\mathring{\mathfrak{g}}$:
  - Set $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. Let $\varphi \in \mathrm{Aut}(\mathring{\mathfrak{g}})$ of finite order $N \in \mathbb{N}$ and let $\zeta = e^{2i\pi/N} \in \mathbb{C}$. Extend φ to $\widehat{\varphi} \in \mathrm{Aut}(\mathcal{L}(\mathring{\mathfrak{g}}))$ by $\widehat{\varphi}(t \otimes x) := \zeta t \otimes \varphi(x)$. $\Rightarrow \mathcal{L}_\varphi(\mathring{\mathfrak{g}}) := \mathcal{L}(\mathring{\mathfrak{g}})^{\widehat{\varphi}}$ **twisted loop algebra**.
  - There is a non-degenerate invariant bilinear form on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$, given by $\kappa(t^r \otimes x, t^s \otimes y) := \delta_{r,-s}\mathring{\kappa}(x, y)$.
  - Let $D \in \mathrm{der}(\mathcal{L}_\varphi(\mathring{\mathfrak{g}}))$ be a skew-symmetric derivation. Then $\omega_D(x, y) := \kappa(Dx, y)$ is a 2-cocycle on $\mathcal{L}_\varphi(\mathring{\mathfrak{g}})$. Extend D to the derivation $\widetilde{D}(z, x) = (0, Dx)$ of the corresponding central extension $\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.
  - $\mathfrak{g} = (\mathbb{C} \oplus_{\omega_D} \mathcal{L}_\varphi(\mathring{\mathfrak{g}})) \rtimes_{\widetilde{D}}$ **affinisation of** $\mathring{\mathfrak{g}}$, with Lie bracket $[(z, x, t), (z', x, t')] = (\omega_D(x, x'), [x, x'] + tDx' - t'Dx, 0)$.
  - $\mathfrak{h} := \mathbb{C} \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$ for some Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}^\varphi \subseteq \mathcal{L}_\varphi(\mathring{\mathfrak{g}})$.
  - $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \{0\} \times (\Delta(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}}) \cup \{0\}) \times \mathbb{C} \subseteq \mathfrak{h}^* \approx \mathbb{C} \times \mathring{\mathfrak{h}}^* \times \mathbb{C}$.
  - $\Delta(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$ is of type A_J, B_J, C_J, D_J or BC_J, and can be realised inside $\mathrm{span}_{\mathbb{Z}}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.

Locally affine Lie algebras (2/2)

Setting

- Let $(\mathfrak{g}, \mathfrak{h})$ be a locally affine Lie algebra, and let $\rho_\lambda \colon \mathfrak{g} \to \mathfrak{u}(V^\lambda)$ be a unitary HWR (these exist for λ integral, non-vanishing on the center of $\mathfrak{g}$, cf. [Neeb '10 and '14]).
- Extend ρ_λ to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ for some $D \in \mathrm{der}(\mathfrak{g})$ given by $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$. Then $\widetilde{\rho}_\lambda$ is a PER $\Leftrightarrow \inf \chi(W.\lambda - \lambda) > -\infty$.
- $\Delta \subseteq \{0\} \times \Delta(X_J) \times \mathbb{C}$ for some $X \in \{A, B, C, D, BC\}$, where $\Delta(X_J)$ can be realised inside $\mathrm{span}_{\mathbb{Z}}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.

Locally affine Lie algebras (2/2)

Setting

- Let $(\mathfrak{g}, \mathfrak{h})$ be a locally affine Lie algebra, and let $\rho_\lambda \colon \mathfrak{g} \to \mathfrak{u}(V^\lambda)$ be a unitary HWR (these exist for λ integral, non-vanishing on the center of $\mathfrak{g}$, cf. [Neeb '10 and '14]).
- Extend ρ_λ to a representation $\widetilde{\rho}_\lambda \colon \mathfrak{g} \rtimes \mathbb{C}D \to \mathrm{End}(V^\lambda)$ for some $D \in \mathrm{der}(\mathfrak{g})$ given by $D(x_\alpha) = i\chi(\alpha)x_\alpha$ for all $\alpha \in \Delta$, $x_\alpha \in \mathfrak{g}_\alpha$, for some character $\chi \colon \mathbb{Z}[\Delta] \to \mathbb{R}$. Then $\widetilde{\rho}_\lambda$ is a PER $\Leftrightarrow \inf \chi(W.\lambda - \lambda) > -\infty$.
- $\Delta \subseteq \{0\} \times \Delta(X_J) \times \mathbb{C}$ for some $X \in \{A, B, C, D, BC\}$, where $\Delta(X_J)$ can be realised inside $\mathrm{span}_{\mathbb{Z}}\{\varepsilon_j\}_{j \in J} \subseteq \mathfrak{h}^*$.

Theorem 2 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\mathrm{min}} + \chi_{\mathrm{sum}}$ with $\inf \chi_{\mathrm{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\mathrm{sum}}(\varepsilon_j)| < \infty$.

Locally affine Lie algebras (2/2)

Theorem 2 (M., Neeb '15):

The representation $\widetilde{\rho}_\lambda$ is a PER if and only if $\chi = \chi_{\text{min}} + \chi_{\text{sum}}$ with $\inf \chi_{\text{min}}(W.\lambda - \lambda) = 0$ and $\sum_{j \in J} |\chi_{\text{sum}}(\varepsilon_j)| < \infty$.

Methods

- Use explicit descriptions of the Weyl group and root system for the 7 standard affinisations, corresponding to "minimal" realisations of the root systems $X_J^{(1)}$, $Y_J^{(2)}$ for $X \in \{A, B, C, D\}$ and $Y \in \{B, C, BC\}$.
- Describe an explicit isomorphism from an arbitrary affinisation to a standard affinisation, as a deformation between two twists compatible with the root space decompositions.

Thank you for your attention!