Arakalov Inequalities
or
A Course on Higgs Bundles

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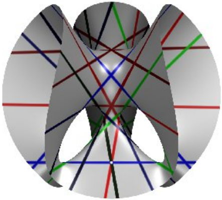

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Goal of this lecture

Some applications of Higgs bundle in complex geometry, in order to study e.g.

- Arakelov Inequalities
- Totally Geodesic (Special) Subvarieties in Period Domains

Special Subvarieties are usually arithmetic and characterized by (Periods and) extra Hodge Classes.

Literature

History: Nigel Hitchin, Carlos Simpson (+ analysis of Hermitian Yang-Mills equation)

Book: Jim Carlson/SMS/Chris Peters (Cambridge, 2003/2017)

Articles: in particular by Eckart Viehweg and Kang Zuo since $\sim$2000, partially together with Jürgen Jost, Martin Möller and (to a lesser extent) myself.

Set-Up

$f : A \to X$ smooth, projective holomorphic map between complex manifolds A and X, extending to compactifications:

$$\begin{array}{ccc} A & \hookrightarrow & \bar{A} \\ f \downarrow & & \downarrow \bar{f} \\ X & \hookrightarrow & \bar{X} \end{array}$$

$D = \bar{X} \setminus X$: Set of singular fibers of $\bar{f}$.

We will construct Higgs bundles on $\bar{X}$ arising from f:

$$(E, \vartheta) : E \xrightarrow{\vartheta} E \otimes \Omega^1_{\bar{X}}(\log D), \quad \vartheta \wedge \vartheta = 0$$

$\vartheta \in \mathrm{End}(E) \otimes \Omega^1_{\bar{X}}(\log D)$ Higgs field.

Monodromy

Fix $w \in \mathbb{N}$ (weight). The fibers A_t are all diffeomorphic (Ehresmann). Therefore:

The cohomology groups $H^w(A_t, \mathbb{C})$ form a local system $\mathbb{V} = R^w f_* \mathbb{C}$ of complex vector spaces.

$\mathbb{V}$ corresponds to a monodromy representation $\rho : \pi_1(X, *) \to GL_n(\mathbb{C})$, where $n = \dim_{\mathbb{C}} H^w(A_t, \mathbb{C})$.

The local monodromies around the divisor D at infinity are denoted by T.

Unipotency

Theorem (Borel, Landman)

T is always quasi-unipotent:

$$(T^\nu - 1)^{w+1} = 0.$$

We will often assume that $\nu = 1$, hence the local monodromy T is unipotent:

$$T = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Semistable implies Unipotent

Theorem

A semistable family has unipotent monodromies.

$\bar{f} : \bar{A} \to \bar{X}$ is called semistable, if $\bar{f}$ is a flat morphism, $D \subset \bar{X}$ is a normal crossing divisor and the inverse image $\bar{f}^{-1}(D)$ is also a normal crossing divisor in $\bar{A}$.

Semistable Reduction Theorem

If X is a curve, then – after passing to a finite cover of X – we may assume that f is semistable.

Gauß–Manin connection

We have a vector bundle $V = \mathbb{V} \otimes \mathcal{O}_X$ on X.

Gauß–Manin connection:

$$\nabla : V \to V \otimes \Omega^1_X, \ \nabla^2 = 0$$

is $\mathbb{C}$–linear. There is a Hodge filtration

$$V = F^0 \supset F^1 \supset \cdots$$

By Griffiths transversality we have $\mathcal{O}_X$–linear maps

$$\vartheta^p = \mathrm{Gr}^p \nabla : F^p / F^{p+1} \to F^{p-1} / F^p \otimes \Omega^1_X.$$

Example: Families of Abelian Varieties (Riemann)

An Abelian Variety of dimension g: compact torus $A_\tau = \mathbb{C}^g/\Lambda$.
$\Lambda =$ columns of $g \times 2g$-matrix $(\mathbb{I}\ \tau)$, with $\tau \in \mathbb{H}_g$ (Siegel space).

Cohomology: $H^1(A_\tau, \mathbb{Z}) = \mathbb{Z}^{2g}$ and $H^m(A_\tau, \mathbb{Z}) = \Lambda^m H^1(A_\tau, \mathbb{Z})$.

Hodge bundles for smooth family $f : A \to X$ of abelian varieties:

$$\mathbb{V} = F^0 = R^1 f_* \mathbb{C},\ F^1 = f_* \Omega^1_{A/X}.$$

Specialty: $\Omega^1_{\mathbb{H}_g,\tau} = S^2 H^0(A_\tau, \Omega^1_{A_\tau})$.

Higgs field: $H^0(A_\tau, \Omega^1_{A_\tau}) \longrightarrow \underbrace{H^1(A_\tau, \mathcal{O}_{A_\tau})}_{\text{dual of } H^0(A_\tau, \Omega^1_{A_\tau})} \otimes S^2 H^0(A_\tau, \Omega^1_{A_\tau})$.

Deligne extension, Higgs bundles

Theorem (Deligne 70)

V and the Hodge bundles F^p have extensions as vector bundles to $\bar{X}$ such that

$$\mathrm{Gr}^p\nabla : F^p/F^{p+1} \to F^{p-1}/F^p \otimes \Omega^1_{\bar{X}}(\log D).$$

are maps of vector bundles, i.e., $\mathcal{O}_{\bar{X}}$-linear.

The bundles $E^{p,w-p} = F^p/F^{p+1}$ form the Higgs bundle

$$E = \bigoplus_{p+q=w} E^{p,q}$$

with Higgs field $\vartheta = \bar{\nabla} : E \to E \otimes \Omega^1_{\bar{X}}(\log D)$. One has $\vartheta \wedge \vartheta = 0$.

The case $w = 1$ in general

Assume we have a semistable family $f : A \to X$.

Then $\mathbb{V} = R^1 f_* \mathbb{C}$ has the extended Hodge bundles $F^1 = \bar{f}_* \Omega^1_{\bar{A}/\bar{X}}(\log \bar{f}^{-1} D)$ and one has $F^0/F^1 = R^1 \bar{f}_* \mathcal{O}_{\bar{X}}$.

The associated Higgs bundle is

$$E = E^{1,0} \oplus E^{0,1} = \bar{f}_* \Omega^1_{\bar{A}/\bar{X}}(\log \bar{f}^{-1} D) \oplus R^1 \bar{f}_* \mathcal{O}_{\bar{X}}.$$

The Higgs field $\vartheta : E^{1,0} \to E^{0,1} \otimes \Omega^1_{\bar{X}}(\log D)$

comes pointwise from (adjoint of) Kodaira-Spencer map:

$$H^0(A_t, \Omega^1_{A_t}) \longrightarrow H^1(A_t, \mathcal{O}_{A_t}) \otimes H^1(A_t, T_{A_t})^\vee$$

First application: Arakelov inequalities

Theorem (Arakelov 71, Faltings 83, Deligne 87, Peters 00, Viehweg-Zuo 01/04, Jost-Zuo 02)

$\bar{f} : \bar{A} \to \bar{X}$ semistable family of abelian varieties of dimension g over a curve $\bar{X}$, $E = E^{1,0} \oplus E^{0,1}$ associated Higgs bundle, then

$$\deg(E^{1,0}) \leq \frac{g}{2} \deg \Omega^1_{\bar{X}}(\log D) = \frac{g}{2}(2g(\bar{X}) - 2 + \sharp D).$$

Corollary

$\bar{X} = \mathbb{P}^1$, $g = 1$, f not isotrivial, then $\sharp D \geq 4$.

Proof

By Simpson correspondence: After splitting off the maximal unitary local subsystem in $\mathbb{V}$, may assume $\theta : E^{1,0} \stackrel{\cong}{\to} B \otimes \Omega^1_{\bar{X}}(\log D)$ with $B \subset E^{0,1}$.

$E^{1,0} \oplus B \subseteq E$ sub Higgs bundle $\stackrel{\text{stability}}{\Rightarrow} \deg(E^{1,0} \oplus B) \leq 0$.

$$\begin{aligned} \text{Hence, } \deg(E^{1,0}) &= \deg(B) + \mathrm{rk}(B) \cdot \deg \Omega^1_{\bar{X}}(\log D) \\ &\leq -\deg(E^{1,0}) + g \cdot \deg \Omega^1_{\bar{X}}(\log D). \end{aligned}$$

Equality implies $E^{0,1} = B$, i.e., in the non-flat part θ is an isomorphism (maximal Higgs field). $\square$

Equality

Theorem (Viehweg-Zuo 2004)

Equality in the theorem holds iff in the non-flat part θ is an isomorphism. This implies (if $D \neq \emptyset$) up to an étale cover that $f : A \to X$ is a product $Z \times E \times_X E \times_X \cdots \times_X E$, where $E \to X$ is a modular family of elliptic curves and Z is a constant abelian variety.

Sketch of proof: Equality $\Rightarrow$ local system is $\mathbb{L} \otimes \mathbb{U}_1 \oplus \mathbb{U}_2$ with $\mathbb{U}_i$ unitary. $\mathbb{L}$ Higgs bundle of rank two, uniformizing: $\tilde{\varphi} : X \to \mathbb{H}$ period map. θ maximal $\Rightarrow$ $\tilde{\varphi}$ locally biholomorphic, hence isomorphism and $X = \Gamma \backslash \mathbb{H}$. □

Upshot: Extremal cases in Arakelov inequalities lead to special arithmetically defined families (also if $D = \emptyset$).

A Generalization: Hyperbolicity

Theorem (Viehweg-Zuo 01)

$\bar{f} : \bar{A} \to \bar{X}$ semistable family of m-folds over a curve $\bar{X}$, then for all $\nu \geq 1$ with $f_* \omega^{\nu}_{\bar{A}/\bar{X}} \neq 0$ one has

$$\frac{\deg(f_* \omega^{\nu}_{\bar{A}/\bar{X}})}{\mathrm{rank}(f_* \omega^{\nu}_{\bar{A}/\bar{X}})} \leq \frac{m \cdot \nu}{2} \deg \Omega^1_{\bar{X}}(\log D).$$

Corollary

$\bar{X} = \mathbb{P}^1$, f not isotrivial, then $\sharp D \geq 3$ (since left side is > 0).

Example

Non-isotrivial families of Calabi-Yau manifolds ($\nu = 1$, called minimal in case of equality), here local Torelli holds.

Generalization to base X a surface

Theorem (Viehweg-Zuo 2005)

$f : \bar{A} \to \bar{X}$ semistable family of abelian varieties of dimension g over a surface $\bar{X}$, smooth over $X = \bar{X} \setminus D$, and with period map $\varphi : X \to A_g$ generically finite. Then:

$$c_1(f_* \omega_{\bar{A}/\bar{X}}) \cdot c_1(\omega_{\bar{X}}(D)) \leq \frac{g}{4} c_1^2(\omega_{\bar{X}}(D)).$$

Here $A_g := \Gamma \backslash \mathbb{H}_g$, $\Gamma \subset Sp_g(\mathbb{Z})$, where $\mathbb{H}_g = Sp_g(\mathbb{R})/U(g)$ is Siegel upper half space. A_g parametrizes all abelian varieties.

Equality

If one has equality, and the Griffiths–Yukawa Coupling

$$\tau^g : \Lambda^g E^{1,0} \to \Lambda^{g-1} E^{1,0} \otimes E^{0,1} \otimes \Omega^1_{\bar{X}}(\log S) \to \cdots \to \Lambda^g E^{0,1} \otimes S^g \Omega^1_{\bar{X}}(\log S)$$

does not vanish, then X is a generalized Hilbert modular surface.

If, in the above, $g = 3$ and $\tau^3 = 0$, then

$$c_1(f_*\omega_{\bar{A}/\bar{X}}) \cdot c_1(\omega_{\bar{X}}(D)) \leq \frac{2}{3} c_1^2(\omega_{\bar{X}}(D)),$$

and X is a generalized Picard modular surface (ball quotient).

The proof uses again stability arguments.

Shimura varieties and special subvarieties

Shimura variety: $X = \Gamma\backslash G(\mathbb{R})/K$, Γ arithmetic subgroup.

G semisimple, adjoint algebraic group/$\mathbb{Q}$ of Hermitian type, i.e., $G(\mathbb{R})/K$ Hermitian symmetric domain.

Examples: $G = SL_2$ and $G = SO(2,2)$: modular curves and Hilbert modular surfaces

$SO(2,19)$: Moduli space of polarized K3 surfaces

Sp_{2g}: A_g

$SU(1,n)$: Ball quotients

Special subvariety: Image of $\Gamma'\backslash H(\mathbb{R})/K' \xrightarrow{H\subset G} \Gamma\backslash G(\mathbb{R})/K$.
Totally geodesic + CM-point !

Problems for special subvarieties

André-Oort Conjecture: Let $Y^0 \subset A_g$ be a smooth algebraic subvariety. If there are sufficiently many special subvarieties $C^0 \subset Y^0$, then Y^0 itself is special.

Solved by Jacob Tsimerman in 2015 for a dense infinite set of CM-points in A_g.

We want to use finitely many special curves.

Results for A_g

Theorem (SMS/Viehweg/Zuo 2009, 2011, 2015)

Let Y^0 be a smooth, algebraic subvariety of A_g such that Y^0 has unipotent monodromies at infinity. Assume there is a finite set I of compactified special curves C_i with:

(BIG) The $\mathbb{Q}$-Zariski closure H of the monodromy representation of $\pi_1(\bigcup_{i\in I} C_i^0, y)$ in $G = \mathrm{Sp}_{2g}$ equals the Zariski closure of the representation of $\pi_1(Y^0, y)$.

(LIE) H is of Hermitian type, and its Lie algebra $\mathfrak{h} = \mathrm{Lie}\, H(\mathbb{R})$ has Hodge decomposition $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}^{-1,1} \oplus \mathfrak{h}^{0,0} \oplus \mathfrak{h}^{1,-1}$ such that $\mathfrak{h}^{-1,1} = T_{Y^0,y}$ for the holomorphic tangent space of Y^0 at y.

(RPC) All special curves C_i^0 satisfy relative proportionality.

Then, Y^0 is a special subvariety of A_g.

Remarks: (LIE) and (RPC) are necessary. Condition (BIG) ?

Relative Proportionality for A_g

Definition (Relative Proportionality Condition)

Let $C \subset Y \subset \bar{A}_g$ be an irreducible special curve with logarithmic normal bundle $N_{C/Y}$ and Harder-Narasimhan filtration $N^{\bullet}_{C/Y}$. Then one has the relative proportionality inequality

$$\deg N_{C/Y} \leq \frac{\mathrm{rank}(N^1_{C/Y}) + \mathrm{rank}(N^0_{C/Y})}{2} \cdot \deg T_C(-\log S_C).$$

If C^0 and Y^0 are special subvarieties of A_g, then equality holds (RPC).

Example: C^0 special curve on a Hilbert modular surfaces, then

$$(K_X + D).C + 2C^2 = 4\delta + 2\varepsilon.$$

Sketch of Proof of the Theorem

Using H, we define a special subvariety

$$Z^0 = \Gamma\backslash H(\mathbb{R})/K \subset A_g,$$

where Γ is the image of $\pi_1(\bigcup_{i\in I} C_i^0, y)$ in $G = \mathrm{Sp}_{2g}$.

(RPC) $\Rightarrow$ $T_Y(-\log S_Y)|_C \cong T_C(-\log S_C) \oplus N_{C|Y}$ (thickening).

(RPC)+(BIG) $\Rightarrow$ we know all (p, p)-classes surviving over all points $y \in Y^0$ (coming from the C_i).

(RPC)+(BIG)+(LIE) $\Rightarrow$ $Y^0 = Z^0$ for dimension reasons. $\square$

Results for Mumford-Tate domains

Theorem (Abolfazl/SMS/Zuo 2015)

Let $X = \Gamma \backslash D$ be a Mumford-Tate variety, i.e., a locally symmetric quotient of a Mumford-Tate domain D associated to the Mumford-Tate group G. Let Y^0 be a smooth, horizontal algebraic subvariety of X such that Y^0 has unipotent monodromies at infinity. Assume (BIG), (LIE) and (RPC) as before.

Then, Y^0 is a special subvariety of X of Shimura type.

Relative Proportionality for Mumford-Tate domains

Definition (Relative Proportionality Condition (RPC))

The curve $\varphi : C \to Y$ satisfies (RPC), if the slope inequalities

$$\mu(N^i_{C/Y}/N^{i-1}_{C/Y}) \leq \mu(N^i_{C/X}/N^{i-1}_{C/X}),\ \ i = 0, ..., s$$

are equalities. The sheaves $N^i_{C/X}$ come from the HN-filtration.

The length s depends on the Lie group G.

Modular Curves

A Modular Curve is a (non-compact) quotient $X = \Gamma \backslash \mathbb{H}$, where

$$\Gamma \subset SL_2(\mathbb{Z})$$

is an discrete, torsion–free, "arithmetic" subgroup.

Γ can be a Congruence Subgroup: The curves

$$X(N) = \Gamma(N) \backslash \mathbb{H}, \quad X_1(N) = \Gamma_1(N) \backslash \mathbb{H}, \quad X_0(N) = \Gamma_0(N) \backslash \mathbb{H},$$

parametrize elliptic curves with additional structures:

$$X(N) = \{(E, \varphi) \mid \varphi : E_{\mathrm{N-tor}} \cong (\mathbb{Z}/N\mathbb{Z})^2\},$$

$$X_1(N) = \{(E, P) \mid N \cdot P = 0\}, \; X_0(N) = \{(E, C) \mid C \cong \mathbb{Z}/N\mathbb{Z}\}.$$

For $N \geq 3$ there is a universal family $f : A(N) \to X(N)$ of elliptic curves over $X(N)$, everything defined over $\mathbb{Q}$.

Hilbert modular surfaces

F totally real number field of degree d.

Lie group is $G = \mathrm{Res}_{F/\mathbb{Q}} SL_2 \subset SL_2 \times SL_2 \times \cdots \times SL_2$ (d–fold product)

$\Gamma \subset SL_2(\mathcal{O}_F)$ (torsion-free) arithmetic subgroup.

Hilbert modular variety: $X = \Gamma \backslash G(\mathbb{R})/K$ carries a family of d–dim. abelian varieties with extra endomorphisms.

Hilbert modular surface: $X = \Gamma \backslash \mathbb{H} \times \mathbb{H}$.

Ball quotients/Picard modular surfaces

$\mathcal{O}$ ring of integers for imaginary quadratic number field, e.g. $\mathcal{O} = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}] \subset \mathbb{Q}(\sqrt{-3})$.

Picard modular surfaces: $X = \Gamma \backslash \mathbb{B}_2$, where $\Gamma \subset U(2,1;\mathcal{O})$ arithmetic subgroup, i.e., X is a special surface in A_3.