The evolution of positively curved invariant Riemannian metrics on the Wallach spaces under the Ricci flow

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This talk is bases on our recent paper

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The study of Riemannian manifolds with positive sectional curvature has a long history. There are very few known examples, many of them are homogeneous: compact rank one symmetric spaces and certain homogeneous spaces in dimensions 6, 7, 12, 13 and 24 due to Berger [6], Wallach [33], and Aloff – Wallach [4]. The homogeneous spaces that admit invariant metrics with positive sectional curvature have been classified by Bérard Bergery, Berger, and Wallach [5, 6, 33]. As was recently observed by J. A. Wolf and M. Xu [35], there is a gap in Bérard Bergery's classification of odd dimensional positively curved homogeneous spaces in the case of the Stiefel manifold $Sp(2)/U(1) = SO(5)/SO(2)$. A refined proof of the suitable result was obtained by B. Wilking, see Theorem 5.1 in [35]. The recent paper [34] by B. Wilking and W. Ziller gives a new and short proof of the classification of homogeneous manifolds of positive curvature. A detailed exposition of various results on the set of invariant metrics with positive sectional curvature, the best pinching constant, and full connected isometry groups could be found in the papers by Püttmann, Shankar, Valiev, Verdiani, Ziller, and Vol'per [25, 27, 28, 29, 30, 31, 32].

It is a natural type of problems to investigate whether or not the positiveness of the sectional curvature or positiveness of the Ricci curvature is preserved under the Ricci flow. This idea is based on original results of R. Hamilton [15].
A recent survey on the evolution of positively curved Riemannian metrics under the Ricci flow could be found in the survey by Lei Ni [20]. Interesting results on the evolution of invariant Riemannian metrics could also be found in the papers by Böhm, Wilking, Buzano, Jablonski, Lafuente, Lauret, Payne, Cheung, and Wallach [8, 10, 11, 14, 17, 18, 24, 13] and the references therein. Sometimes it is helpful to use the (volume) normalized Ricci flow, see details e. g. on pp. 259–260 of [15]. The main object of our study in this paper are the Wallach spaces

$$\begin{aligned} W_6 &:= SU(3)/T_{\max}, \\ W_{12} &:= Sp(3)/Sp(1)\times Sp(1)\times Sp(1), \\ W_{24} &:= F_4/Spin(8) \end{aligned} \tag{1}$$

that admit invariant Riemannian metrics of positive sectional curvature [33].
Note that the Wallach spaces are the total spaces of the following submersions:

$$S^2 \to W_6 \to \mathbb{CP}^2, \quad S^4 \to W_{12} \to \mathbb{HP}^2, \quad S^8 \to W_{24} \to \mathrm{Ca}\mathbb{P}^2.$$

In the papers [2] and [3] by N. A. Abiev, A. Arvanitoyeorgos, Yu. G. Nikonorov, and P. Siasos, the authors studied the normalized Ricci flow equation

$$\frac{\partial}{\partial t}\mathbf{g}(t) = -2\mathrm{Ric}_{\mathbf{g}} + 2\mathbf{g}(t)\frac{S_{\mathbf{g}}}{n} \qquad (2)$$

on one special class of Riemannian manifolds M^n called generalized Wallach spaces (or three-locally-symmetric spaces in other terms) according to the definitions of [19] and [23], where $\mathbf{g}(t)$ means a 1-parameter family of Riemannian metrics, $\mathrm{Ric}_{\mathbf{g}}$ is the Ricci tensor and $S_{\mathbf{g}}$ is the scalar curvature of the Riemannian metric $\mathbf{g}$.

Generalized Wallach spaces are characterized as compact homogeneous spaces G/H whose isotropy representation decomposes into a direct sum

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \quad (\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p})$$

of three $\mathrm{Ad}(H)$-invariant irreducible modules satisfying

$$[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}, \quad i \in \{1, 2, 3\},$$

see e. g. [19, 21].

The complete classification of generalized Wallach spaces is obtained recently (independently) in the papers by Z. Chen, Y. Kang, K. Liang [12] and by Yu. G. Nikonorov [22].

For a fixed bi-invariant inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ of the Lie group G, any G-invariant Riemannian metric g on G/H is determined by an $\operatorname{Ad}(H)$-invariant inner product

$$(\cdot,\cdot) = x_1\langle\cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2\langle\cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3\langle\cdot,\cdot\rangle|_{\mathfrak{p}_3}\,, \qquad (3)$$

where x_1, x_2, x_3 are positive real numbers.
Therefore, the space of such metrics is 2-dimensional up to a scale factor. Any metric with $x_1 = x_2 = x_3$ is called normal, whereas the metric with $x_1 = x_2 = x_3 = 1$ is called standard or Killing.

For the Wallach spaces W_6, W_{12}, W_{24}, we have

$$\dim(\mathfrak{p}_1) = \dim(\mathfrak{p}_2) = \dim(\mathfrak{p}_3) =: \mathbf{d}\,,$$

where $\mathbf{d}$ is equal to 2, 4, 8 respectively.

On the given Wallach space G/H, the subspace of invariant metrics satisfying $x_i = x_j$ for some $i \neq j$, is invariant under the normalized Ricci flow, because these special metrics have a larger connected isometry group. Indeed, such a metric (x_1, x_2, x_3) admits additional isometries generated by the right action of the group $K \subset G$ with the Lie algebra

$$\mathfrak{k} := \mathfrak{h} \oplus \mathfrak{p}_k, \quad \{i, j, k\} = \{1, 2, 3\},$$

see details in [22].

All such metrics are related to the above mentioned submersions of the form $K/H \to G/H \to G/K$, coming from inclusions $H \subset K \subset G$, see e. g. [7, Chapter 9]. In what follows we call these metrics *exceptional* or submersion metrics. These metrics constitute three one-parameter families up to a homothety. All other metrics we call generic or non-exceptional.

We briefly describe the evolution of submersion metrics under the normalized Ricci flow. Without loss of generality we may consider the family

$$(x_1, x_2, x_3) = (x^{-1/3}, x^{-1/3}, x^{2/3}), \quad x \in \mathbb{R}_+.$$

It comes from changing the scaling of the fibre and the base with keeping of the volume. Here x is the ratio of the multiples of the normal metric on the fibre and on the base. Since this family is invariant under the Ricci flow, the behavior of the Ricci flow can be read off the behavior of the scalar curvature function $S(x)$. The Ricci flow is a gradient flow in this case.

The point $x = 1$ (the normal metric) is a local minimum and the second Einstein metric (which is the Kähler — Einstein for W_6) is a local maximum. It is well-known, that when starting with the normal homogeneous metric and shrinking the fibre (i. e. $x < 1$), these metrics will have positive sectional curvature, moreover, $S(x) \to \infty$ as $x \to 0$. Note also that the non-normal Einstein metric has positive Ricci curvature but mixed sectional curvature. It is clear also that $S(x) < 0$ for sufficiently large x. This give us qualitative picture of the Ricci flow's behavior on submersion metrics. We illustrate this by Figures 1 and 2 for W_6. More general constructions of the canonical variation for submersion metrics one can find in the Besse book [7, 9.72].

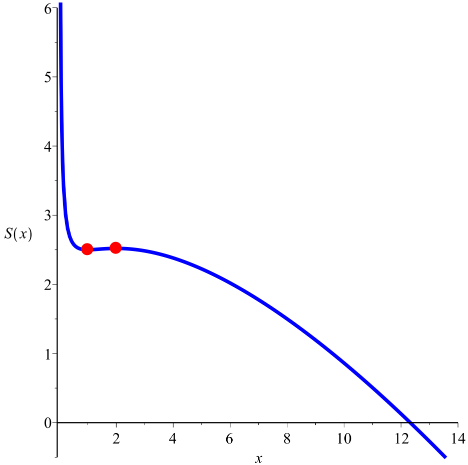

Fig. 1: The scalar curvature of submersion metrics on the Wallach space W_6: the point $x = 1$ (the normal metric) is a local minimum, the point $x = 2$ (the Kähler – Einstein metric) is a local maximum.

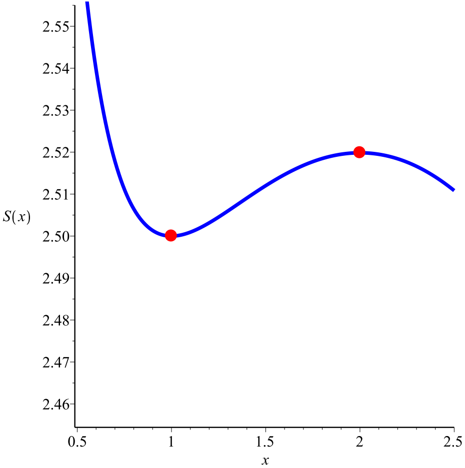

Fig. 2: The scalar curvature of submersion metrics on the Wallach space W_6: the point $x = 1$ (the normal metric) is a local minimum, the point $x = 2$ (the Kähler – Einstein metric) is a local maximum.

Taking into account the above description of the Ricci flow on submersion metrics, we preferably deal with generic invariant metrics on the Wallach spaces. Our first main result is the following

Theorem (Theorem 1)

On the Wallach spaces W_6, W_{12}, and W_{24}, the normalized Ricci flow evolves all generic metrics with positive sectional curvature into metrics with mixed sectional curvature.

Moreover, it is proved that the normalized Ricci flow removes every generic metric from the set of metrics with positive sectional curvature in a finite time and does not return it back to this set. This finite time depends of the initial points and could be as long as we want.

Theorem 1 easily implies the following result by Man-Wai Cheung and N. R. Wallach [13]: on the Wallach spaces W_6, W_{12}, and W_{24}, the normalized Ricci flow evolves some metrics with positive sectional curvature into metrics with mixed sectional curvature.

Our second main result is related to the evolution of metrics with positive Ricci curvature.

Theorem (Theorem 2)

On the Wallach spaces W_{12} and W_{24}, the normalized Ricci flow evolves all generic metrics with positive Ricci curvature into metrics with mixed Ricci curvature.

Moreover, the normalized Ricci flow removes every generic metric from the set of metrics with positive Ricci curvature in a finite time and does not return it back to this set. This finite time depends of the initial points and could be as long as we want.

Note also that the normalized Ricci flow can evolve some metrics with mixed Ricci curvature to metrics with positive Ricci curvature. Moreover, there is a non-extendable integral curve of the normalized Ricci flow with exactly one metric of non-negative Ricci curvature.

In the paper [10], C. Böhm and B. Wilking studied (in particular) some properties of the (normalized) Ricci flow on the Wallach space W_{12}. They proved that the (normalized) Ricci flow on W_{12} evolves certain positively curved metrics into metrics with mixed Ricci curvature (see Theorem 3.1 in [10]).

The same assertion for the space W_{24} obtained by Man-Wai Cheung and N. R. Wallach in [13] (see Theorem 3 in [13]). On the other hand, it was proved in Theorem 8 of [13] that every invariant metric with positive sectional curvature on the space W_6 retains positive Ricci curvature under the Ricci flow. Hence, Theorem 2 fails for W_6.

Note also that for some invariant metrics with positive Ricci curvature on W_6, the Ricci flow can evolve them to metrics with mixed Ricci curvature, see Theorem 3 in [13].

We emphasize that the special status of W_6 follows from Proposition 1 below and the description of the boundary of R, the set of metrics with positive Ricci curvature (13).

The Ricci curvature of invariant Riemannian metrics on a given generalized Wallach space be easily expressed in terms of the variables x_1, x_2, x_3 (that are the multiples of the normal metrics) and special constants a_1, a_2, a_3, that determine many geometric properties of this generalized Wallach space, see details e. g. in [2].

Note that $a_1 = a_2 = a_3 =: a$ for the Wallach spaces W_6, W_{12}, and W_{24}. Moreover, for these spaces, a is equal to $1/6$, $1/8$, $1/9$ respectively.

We deal also with all generalized Wallach spaces with the property $\dim(\mathfrak{p}_1) = \dim(\mathfrak{p}_2) = \dim(\mathfrak{p}_3)$, which is just equivalent to $a_1 = a_2 = a_3 =: a$ (see [2]). We will consider such spaces only for $a \in (0, 1/4) \cup (1/4, 1/2)$, because every generalized Wallach space with $a = 1/4$ has very special properties, e. g. it admits a unique Einstein metric up to a homothety, see [2] for detailed discussion.

Theorem 2 can be extended to some other generalized Wallach spaces.

Theorem (Theorem 3)

Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 =: a$, where $a \in (0, 1/4) \cup (1/4, 1/2)$. If $a < 1/6$, then the normalized Ricci flow evolves all generic metrics with positive Ricci curvature into metrics with mixed Ricci curvature. If $a \in (1/6, 1/4) \cup (1/4, 1/2)$, then the normalized Ricci flow evolves all generic metrics into metrics with positive Ricci curvature.

For instance, the spaces $Sp(3k)/Sp(k) \times Sp(k) \times Sp(k)$ correspond to the case $a = \frac{k}{6k+2} < 1/6$, whereas the spaces $SO(3k)/SO(k) \times SO(k) \times SO(k)$, $k > 2$, correspond to the case $1/6 < a = \frac{k}{6k-4} < 1/4$. Note also that $SO(6)/SO(2) \times SO(2) \times SO(2)$ corresponds to $a = 1/4$, that is a very special case of generalized Wallach spaces with a unique Einstein invariant metric up to a homothety, and $SO(3)$ correspond to $a = 1/2$, the maximal possible value for $a = a_1 = a_2 = a_3$, see details in [2] and [3]. It is interesting also that $1/9$ is the minimal possible value for $a = a_1 = a_2 = a_3$ among non-symmetric generalized Wallach spaces, see [22].

It should also be noted that there are many generalized Wallach spaces with $a = 1/6$, for example, the spaces $SU(3k)/S(U(k) \times U(k) \times U(k))$. All these spaces are Kähler C-spaces, see [22]. We state the following result, that generalizes Theorem 8 of [13].

Theorem (Theorem 4)

Let G/H be a generalized Wallach space with $a_1 = a_2 = a_3 = 1/6$. Suppose that it is supplied with the invariant Riemannian metric (3) *such that $x_k < x_i + x_j$ for all indices with $\{i, j, k\} = \{1, 2, 3\}$, then the normalized Ricci flow on G/H with this metric as the initial point, preserves the positivity of the Ricci curvature.*

It should be noted that $x_k = x_i + x_j$ is just the unstable manifold of the Kähler – Einstein metric for all generalized Wallach spaces with $a = 1/6$.

Recall that the Ricci operator Ric of the metric (3) is given by

$$\mathrm{Ric} = \mathbf{r_1}\ \mathrm{Id}|_{\mathfrak{p}_1} + \mathbf{r_2}\ \mathrm{Id}|_{\mathfrak{p}_2} + \mathbf{r_3}\ \mathrm{Id}|_{\mathfrak{p}_3},$$

where

$$\mathbf{r_i} := \frac{x_j x_k + a(x_i^2 - x_j^2 - x_k^2)}{2x_1x_2x_3}$$

are the principal Ricci curvatures, $\{i,j,k\} = \{1,2,3\}$. Hence, the scalar curvature of this metric is

$$S = \mathbf{d} \cdot \frac{x_1x_2 + x_1x_3 + x_2x_3 - a(x_1^2 + x_2^2 + x_3^2)}{2x_1x_2x_3}.$$

By using the above equalities, the (volume) normalized Ricci flow equation (2) on the Wallach spaces can be reduced to a system of ODE's of the following form:

$$\frac{dx_i}{dt} = -2x_i(t)\left(\mathbf{r_i} - \frac{S}{n}\right), \quad i = 1,2,3, \qquad (4)$$

where $n = \dim(\mathfrak{p}_1) + \dim(\mathfrak{p}_2) + \dim(\mathfrak{p}_3)$.

Note that the metric (3) has the same volume as the standard metric if and only if $x_1x_2x_3 = 1$. It suffices to prove Theorems 1, 2, 3, and 4 only for invariant metrics with

$$\mathrm{Vol} = x_1x_2x_3 \equiv 1. \qquad (5)$$

Indeed, the case of general volume is reduced to this one by a suitable homothety. This observation is the main argument to apply the normalized Ricci flow instead of the non-normalized Ricci flow in the case of the Wallach spaces, as far as in the case of generalized Wallach spaces, see details in [2] and [3].

It is easy to check that $\mathrm{Vol} = x_1x_2x_3$ is a first integral of the system (4). Therefore, we can reduce (4) to the following system of two differential equations on the surface (5):

$$\begin{aligned} \frac{dx_1}{dt} &= \left(x_1x_2^{-1} + x_1^2x_2 - 2\right) - 2ax_1\left(2x_1^2 - x_2^2 - x_1^{-2}x_2^{-2}\right), \\ \frac{dx_2}{dt} &= \left(x_2x_1^{-1} + x_1x_2^2 - 2\right) - 2ax_2\left(2x_2^2 - x_1^2 - x_1^{-2}x_2^{-2}\right). \end{aligned} \qquad (6)$$

We consider also a system of ODE's obtaining in scale invariant variables

$$w_1 := \frac{x_3}{x_1}, \quad w_2 := \frac{x_3}{x_2}. \tag{7}$$

Since (4) is autonomous and

$$\frac{1}{w_i}\frac{dw_i}{dt} = \frac{1}{x_3}\frac{dx_3}{dt} - \frac{1}{x_i}\frac{dx_i}{dt} = -2(\mathbf{r_3} - \mathbf{r_i}),$$

for $i = 1, 2$, then (4) can be reduced to the following system for $w_1 > 0$ and $w_2 > 0$:

$$\begin{aligned} \tfrac{dw_1}{dt} &= f(w_1, w_2) := (w_1 - 1)(w_1 - 2aw_1w_2 - 2aw_2), \\ \tfrac{dw_2}{dt} &= g(w_1, w_2) := (w_2 - 1)(w_2 - 2aw_1w_2 - 2aw_1), \end{aligned} \tag{8}$$

where $t := tx_3$ is a new time-parameter not changing integral curves and their orientation $(x_3 > 0)$.

One can prove the above mentioned theorems with using either the system (6) or the system (8) as the main tool. Comparisons show that the system (8) in the scale invariant variables (w_1, w_2) is more convenient. On the other hand, we prefer to give visual interpretations of the results in both coordinate systems (w_1, w_2) and (x_1, x_2).

Lemma

Let $w_1 > 0$ *and* $w_2 > 0$. *Then*
1) The curves c_1, c_2 *and* c_3 *determined by the equations*

$$w_2 = 1, \quad w_1 = 1 \quad \text{and} \quad w_2 = w_1$$

respectively are invariant sets of the system (8);
2) At $a \neq 1/4$, *the system* (8) *has exactly four different singular points* $E_0 = (1,1)$, $E_1 = (q,1)$, $E_2 = (1,q)$, $E_3 = \left(q^{-1}, q^{-1}\right)$, *where* $q := 2a(1-2a)^{-1}$. *Moreover,* E_1, E_2 *and* E_3 *are hyperbolic saddles and* E_0 *is a hyperbolic unstable node.*

The curves c_1, c_2 and c_3 have the common point E_0 and separate the domain $(0,\infty)^2$ into 6 connected invariant components (see Figure 3). The study of normalized Ricci flow in each pair of these components are equivalent due to the following property of the Wallach spaces: there is a finite group of isometries fixing the isotropy and permuting the modules $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_3$. Therefore, it suffices to study solutions of (8) with initial points given only in the following set

$$\Omega := \left\{(w_1, w_2) \in \mathbb{R}^2 \mid w_2 > w_1 > 1\right\}. \tag{9}$$

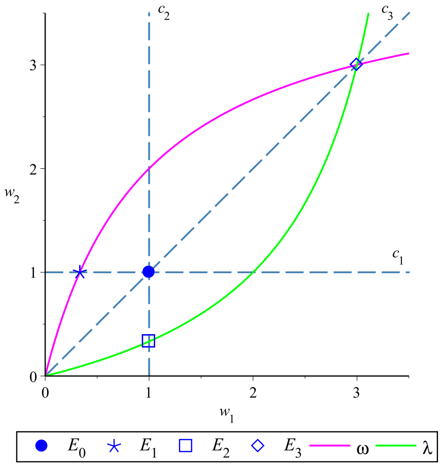

Fig. 3: The curves c_1, c_2, c_3 and the singular points E_0, E_1, E_2, E_3 corresponding to the system (8) for $a = 1/8$.

A simple analysis of the right hand sides of the system (8) provides elementary tools for studying the behavior of its integral curves. For instance, we can predict the slope of integral curves of (8) in Ω and interpret them geometrically etc.

According to this observations, let us consider the curves

$$\begin{aligned}\omega &:= \left\{(w_1, w_2) \in \mathbb{R}^2_+ \mid w_1 - 2aw_1w_2 - 2aw_2 = 0\right\},\\ \lambda &:= \left\{(w_1, w_2) \in \mathbb{R}^2_+ \mid w_2 - 2aw_1w_2 - 2aw_1 = 0\right\}.\end{aligned}$$

Note that the curves ω and λ consist of invariant metrics with the equality $\mathbf{r_3} = \mathbf{r_1}$ and $\mathbf{r_3} = \mathbf{r_2}$ for the principal Ricci curvatures, see Figure 4.

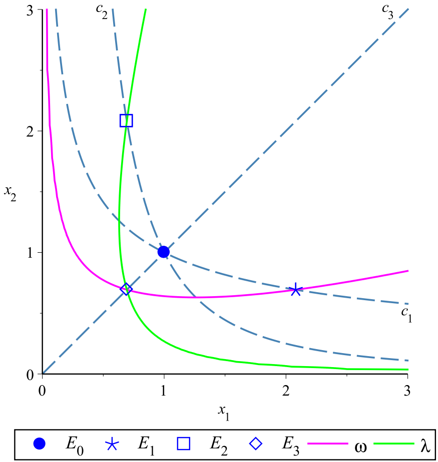

Fig. 4: The curves c_1, c_2, c_3 and the singular points E_0, E_1, E_2, E_3 corresponding to the system (6) for $a = 1/8$.

The domain of positive sectional curvature $D \setminus \{(1,1)\}$

A detailed description of invariant metrics of positive sectional curvature on the Wallach spaces (1) was given by F. M. Valiev in [28]. We reformulate his results in our notation. It should be noted that this description is universal for all Wallach spaces.
Recall that we deal with only positive x_i. Let us consider the functions

$$\gamma_i = \gamma_i(x_1, x_2, x_3) := (x_j - x_k)^2 + 2x_i(x_j + x_k) - 3x_i^2,$$

where $\{i, j, k\} = \{1, 2, 3\}$. Note that under the restrictions $x_i > 0$, the equations $\gamma_i = 0$, $i = 1, 2, 3$, determine cones congruent each to other under the permutation $i \to j \to k \to i$. Note also that these cones have the empty intersections pairwise.

According to results of [28] and the symmetry in γ_1, γ_2, and γ_3 under permutations of x_1, x_2, and x_3, the set of metrics with non-negative sectional curvature is the following:

$$\left\{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \mid \gamma_1 \geq 0,\ \gamma_2 \geq 0,\ \gamma_3 \geq 0\right\}. \qquad (10)$$

The domain of positive sectional curvature $D \setminus \{(1,1)\}$

By Theorem 3 in [28] and the above mentioned symmetry, the set of metrics with positive sectional curvature is the following:

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3_+ \mid \gamma_1 > 0,\ \gamma_2 > 0,\ \gamma_3 > 0\} \setminus \{(t,t,t) \in \mathbb{R}^3 \mid t > 0\}. \quad (11)$$

Let us describe the domain of positive sectional curvature in the coordinates (w_1,w_2). Denote by s_i curves on the plane (w_1,w_2) determined by the equations $\gamma_i\left(\frac{1}{w_1},\frac{1}{w_2},1\right)=0$ (see Figure 5 and also Figure 6 for other coordinates). For $w_1>0$ and $w_2>0$, these equations are respectively equivalent to

$$\begin{aligned} l_1 &:= \ \ w_1^2w_2^2 - 2w_1^2w_2 + 2w_1w_2^2 + w_1^2 + 2w_1w_2 - 3w_2^2 = 0,\\ l_2 &:= \ \ w_1^2w_2^2 + 2w_1^2w_2 - 2w_1w_2^2 - 3w_1^2 + 2w_1w_2 + w_2^2 = 0,\\ l_3 &:= -3w_1^2w_2^2 + 2w_1^2w_2 + 2w_1w_2^2 + w_1^2 - 2w_1w_2 + w_2^2 = 0. \end{aligned} \quad (12)$$

It is easy to check that (10) is a connected set with a boundary consisting of the union of the cones $\gamma_1=0, \gamma_2=0$ and $\gamma_3=0$. Therefore, solving the system of inequalities $\gamma_i\left(\frac{1}{w_1},\frac{1}{w_2},1\right)>0,\ i=1,2,3,$ we get a connected domain on the plane (w_1,w_2) bounded by the curves s_1, s_2 and s_3. Let us denote it by D.

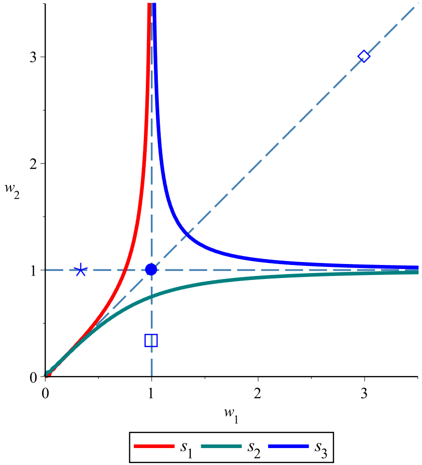

Fig. 5: The curves s_1, s_2, s_3 corresponding to the system (8).

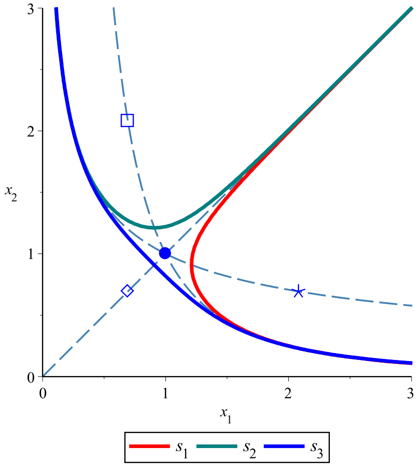

Fig. 6: The curves s_1, s_2, s_3 corresponding to the system (6).

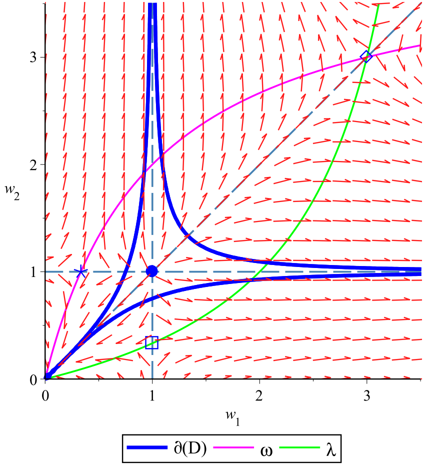

Fig. 7: The case $a = 1/8$: The phase portraits of the system (8).

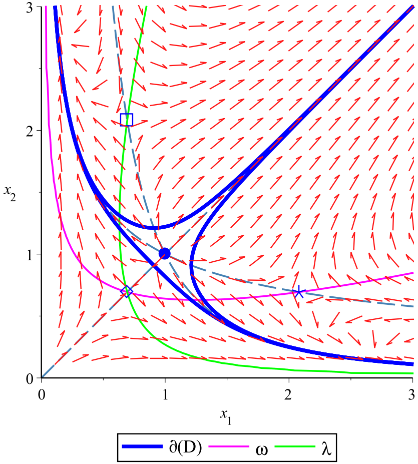

Fig. 8: The case $a = 1/8$: The phase portraits of the system (6).

The domain of positive Ricci curvature R

Let us describe the set R of invariant metrics with positive Ricci curvature on the given Wallach space. Since the principal Ricci curvatures $\mathbf{r_i}$ are expressed as $\frac{x_j x_k + a(x_i^2 - x_j^2 - x_k^2)}{2x_1x_2x_3}$, we consider the functions

$$k_i := x_j x_k + a(x_i^2 - x_j^2 - x_k^2),$$

where $x_i > 0, \quad i \neq j \neq k \neq i, \quad i, j, k \in \{1, 2, 3\}$.
Now, consider the description of the domain R in the coordinates (w_1, w_2). Denote by r_i curves determined by the equations $k_i\left(\frac{1}{w_1}, \frac{1}{w_2}, 1\right) = 0$ respectively (see Figure 9). For $w_1 > 0$ and $w_2 > 0$, these equations are respectively equivalent to

$$\begin{aligned}
\rho_1 &:= -aw_1^2w_2^2 - aw_1^2 + aw_2^2 + w_1^2w_2 = 0,\\
\rho_2 &:= -aw_1^2w_2^2 + aw_1^2 - aw_2^2 + w_1w_2^2 = 0,\\
\rho_3 &:= \quad aw_1^2w_2^2 - aw_1^2 - aw_2^2 + w_1w_2 = 0.
\end{aligned} \tag{13}$$

We easily get on the plane (w_1, w_2) a connected domain R bounded by the curves r_1, r_2 and r_3 solving the system of inequalities $k_i\left(\frac{1}{w_1}, \frac{1}{w_2}, 1\right) > 0$, $i = 1, 2, 3$.

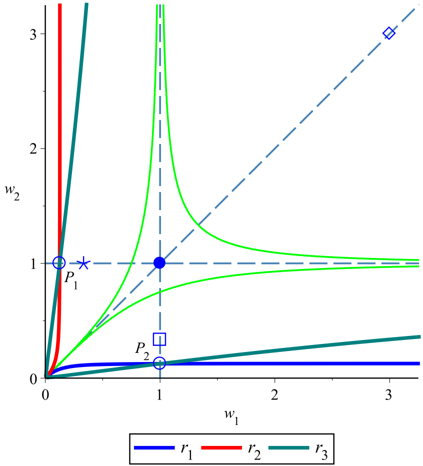

Fig. 9: The case $a = 1/8$: The curves r_1, r_2, r_3 and the points P_1, P_2, P_3 corresponding to the system (8).

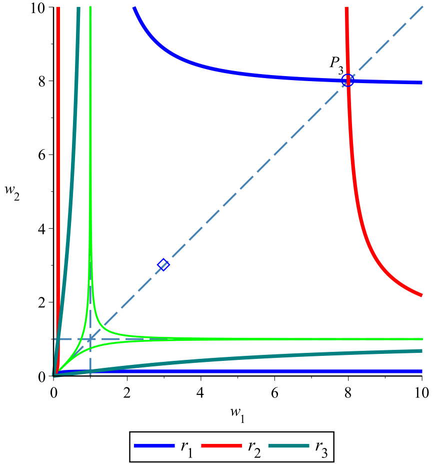

Fig. 10: The case $a = 1/8$: The curves r_1, r_2, r_3 and the points P_1, P_2, P_3 corresponding to the system (8).

In the Figure 11 we can see the domain R together with the phase portraits of the systems (8) for $a = 1/8$.

The point Q is a unique point on the "necessary" part of the boundary of R, such that the normalized Ricci flow is tangent to $\partial(R)$.

The corresponding picture for the the systems (6) is in the Figure 12.

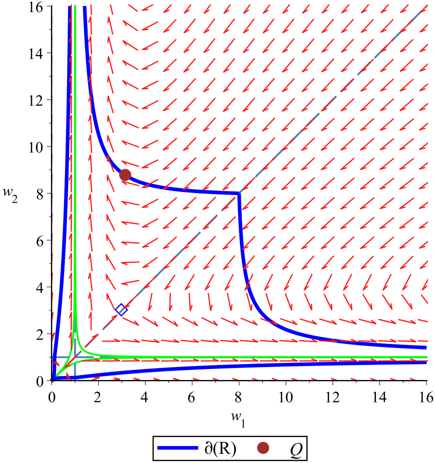

Fig. 11: The case $a = 1/8$: The phase portraits of the system (8).

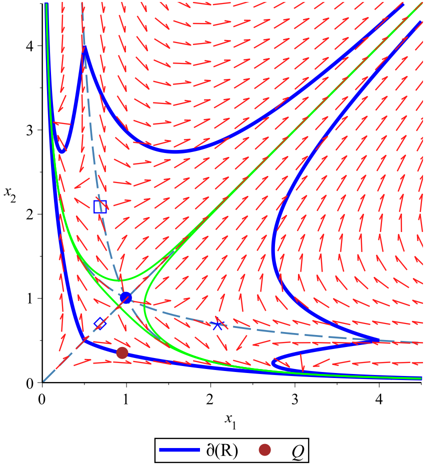

Fig. 12: The case $a = 1/8$: The phase portraits of the system (6).

The main idea

The main idea is to compare the growth rates of trajectories of the normilazed Ricci flow and the corresponding parts of the boundaries $\partial(D)$ and $\partial(R)$ in the coordinate system w_1, w_2. It is useful to deal with the set $\Omega = \{(w_1, w_2) \in \mathbb{R}^2 \mid w_2 > w_1 > 1\}$ and with the asymptotic presentations (for curves) of the type $w_2 \sim C \cdot (w_1 - 1)^{-\alpha}$ as $w_1 \to 1 + 0$ for some positive C and some real α.

Proposition (Proposition 1)

Suppose that a curve γ given in Ω satisfies the asymptotic equality

$$w_2 \sim C \cdot (w_1 - 1)^{-\alpha} \quad \text{as} \quad w_1 \to 1 + 0,$$

where $\alpha > 0$, $C > 0$. Then the following assertion holds: If $\frac{1-2a}{4a} < \alpha$ (respectively, $\frac{1-2a}{4a} > \alpha$), then every integral curve $(w_1(t), w_2(t))$ of (8) in Ω lies under (respectively, over) γ for sufficiently large t.

Sketch of the proof of Theorem 1

Sketch of the proof of Theorem 1. Without loss of generality consider only the part $D \cap \Omega$ of D. Consider any trajectory $\big(w_1(t), w_2(t)\big)$ of (8) initiated at $(w_1^0, w_2^0) \in D \cap \Omega$.

The equation of s_3 (see (12)) has an unique positive solution

$$w_2 \sim \frac{1}{2}(w_1 - 1)^{-1/2} \quad \text{as} \quad w_1 \to 1 + 0.$$

It describes the "upper" part of the boundary of $D \cap \Omega$. Therefore, we have $\alpha = 1/2$ in Proposition 1. Since $\frac{1-2a}{4a} > \alpha = 1/2$ whenever $0 < a < 1/4$ the trajectory $\big(w_1(t), w_2(t)\big)$ lies over the curve s_3 for $w_1 \to 1 + 0$ (corresponding to $t \to +\infty$).

Hence, there exists a point on the curve $s_3 \cap \Omega$ at which $\big(w_1(t), w_2(t)\big)$ intersects $s_3 \cap \Omega$ and leaves the set D.

Sketch of the proof of Theorem 2 and Theorem 3

Sketch of the proof of Theorem 2 and Theorem 3. It is sufficient to consider only the set $R \cap \Omega$, where Ω given by (9).
The equation $\rho_1 = 0$ for the curve r_1 (see (13)) has the solution

$$w_2 \sim \frac{1}{2a}(w_1 - 1)^{-1} \quad \text{as} \quad w_1 \to 1 + 0\,,$$

corresponding to the "upper" part γ of the curve r_1, which is the "upper" part of the boundary of $R \cap \Omega$, the set of metric with positive Ricci curvature in Ω.

Consider the case $a \in (0, 1/6)$ and any trajectory $\big(w_1(t), w_2(t)\big)$ of the system (8) initiated at a point of $R \cap \Omega$. Note that $\frac{1-2a}{4a} > 1$ for all $0 < a < 1/6$. Then according to Proposition 1 the trajectory $\big(w_1(t), w_2(t)\big)$ lies over γ for $w_1 \to 1 + 0$ (corresponding to $t \to +\infty$).

Now, consider the case $a \in (1/6, 1/4) \cup (1/4, 1/2)$. Clearly, $\frac{1-2a}{4a} < 1$ for all $a \in (1/6, 1/2)$. Proposition 1 implies that the normalized Ricci flow evolves every initial metric in Ω into metrics with positive Ricci curvature.

Sketch of the proof of Theorem 4

Sketch of the proof of Theorem 4. It is easy to check that the set of metrics with the property $x_i = x_j + x_k$ is an invariant set of the system (4) with right hand sides $F_i := -2x_i(t)\left(\mathbf{r_i} - \frac{S}{n}\right)$ for $a = 1/6$.
Hence, in the scale invariant coordinates (w_1, w_2) we have an invariant curve $w_1^{-1} + w_2^{-1} = 1$ of the system (8) passing through the point $E_3 = (2, 2)$. Since E_3 is a saddle of the system (8), the curve $w_1^{-1} + w_2^{-1} = 1$ is necessarily one of the separatrices (more exactly, the unstable manifold) of this point E_3 (obviously the line $w_2 = w_1$ is the second separatrix).
We may suppose that the initial metric is in Ω. By the above discussion, the set $\left\{(w_1, w_2) \,|\, w_2 < \frac{w_1}{w_1-1}\right\} \cap \Omega$ is an invariant set of the system (8). Simple calculations show that the curve $\left\{(w_1, w_2) \,|\, w_2 = \frac{w_1}{w_1-1}\right\} \cap \Omega$ lies under the curve $r_1 \cap \Omega \subset \partial(R)$. Hence, every trajectory of (8) initiated in the set $\left\{(w_1, w_2) \,|\, w_2 < \frac{w_1}{w_1-1}\right\} \cap \Omega$ remains in the domain $R \cap \Omega$, that proves the theorem.

It should be noted that the metrics (3) with $x_i = x_j + x_k$ constitute the set of Kähler invariant metrics for W_6.

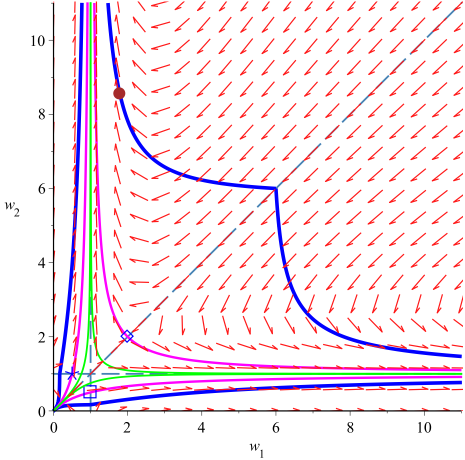

Fig. 13: The case $a = 1/6$: The domains of positive sectional and positive Ricci curvatures, Kähler metrics, the phase portraits of the system (8).

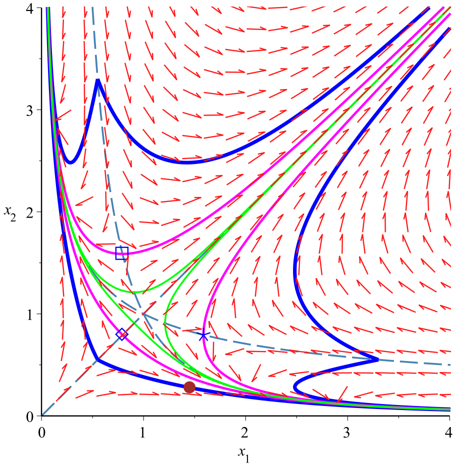

Fig. 14: The case $a = 1/6$: The domains of positive sectional and positive Ricci curvatures, Kähler metrics, the phase portraits of the system (6).

The evolution of the scalar curvature

For completeness of the exposition, we discuss shortly the evolution of the scalar curvature under normalized Ricci flow. We have the following general result related to the evolution of G-invariant metrics on a homogeneous space G/H under the normalized Ricci flow.

Proposition (Proposition 2, R. S. Hamilton [16], J. Lauret [18])

Let $(M = G/H, g_0)$ be a Riemannian homogeneous space. Consider the solution of the normalized Ricci flow (2) on M with $\mathbf{g}(0) = g_0$. Then

$$\frac{\partial S}{\partial t} = 2\,\|\mathrm{Ric}_{\mathbf{g}}\|^2 - \frac{2}{n} \cdot S^2,$$

where $S = S(t)$ is the scalar curvature of metrics $\mathbf{g}(t)$ and $n = \dim(M)$. In particular, the scalar curvature $t \mapsto S(t)$ increases unless g_0 is Einstein.

Therefore, we see that the normalized Ricci flow (on every compact homogeneous space) with an invariant Riemannian metric of positive scalar curvature as the initial point, do not leave the set of the metrics with positive scalar curvature.

For the Wallach space W_{12}, we reproduce an illustration for this observation in Figure 15 (the curve s is the boundary of the set of metrics with positive scalar curvature). Note, that the curve s satisfies the equation

$$a\left(w_1^2w_2^2+w_1^2+w_2^2\right)-w_1^2w_2-w_1w_2^2-w_1w_2=0.$$

The corresponding picture for the coordinate system x_1, x_2, we reproduce in Figure 16.

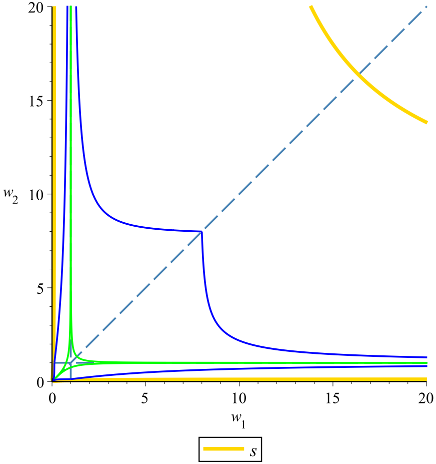

Fig. 15: The case $a = 1/8$: The curve s for the system (8).

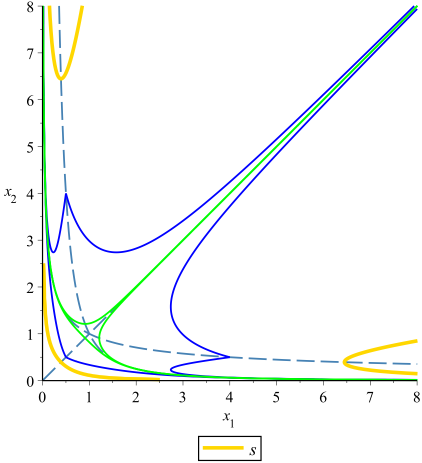

Fig. 16: The case $a = 1/8$: The curve s for the system (6).

Finally, we reproduce additional illustrations suggested us by Wolfgang Ziller. We draw our pictures for the system (4) in the plane $x_1 + x_2 + x_3 = 1$. These pictures preserves the dihedral symmetry of the initial problem.

We reproduce in Figure 17 the domains of positive sectional, positive Ricci, and positive scalar curvatures (we denote them by D, R, and S respectively) of the system (4) in the plane $x_1 + x_2 + x_3 = 1$ for $a = 1/8$.

We also reproduce the phase portrait (of the tangent component) for the system (4) in Figure 18. Note that Riemannian metrics constitute a triangle and the set S is bounded by a circle.

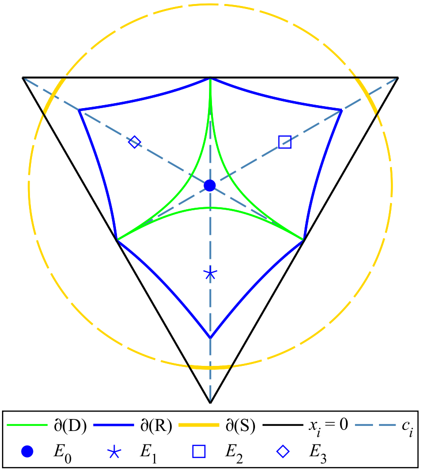

Fig. 17: The case $a = 1/8$: The domains of positive sectional, positive Ricci, and positive scalar curvatures in the plane $x_1 + x_2 + x_3 = 1$.

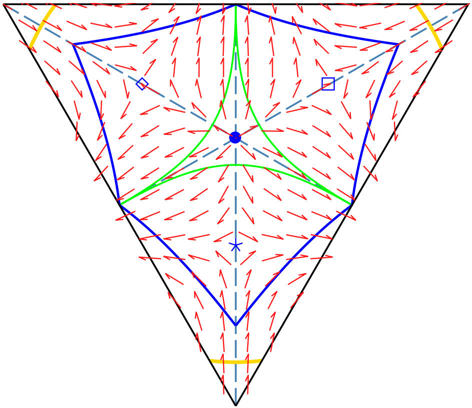

Fig. 18: The case $a = 1/8$: The domains of positive sectional, positive Ricci, and positive scalar curvatures, the phase portrait of the system (4) in the plane $x_1 + x_2 + x_3 = 1$.

Similar pictures could be produced for $a = 1/9$ and $a = 1/6$. We reproduce here only Figure 19 for $a = 1/6$, because the space W_6 admits Kähler invariant metrics, that constitute a small triangle in Figure 19.

Note also that three non-normal Einstein metrics in this case are Kähler – Einstein and one can easily get main properties of the Kähler – Ricci flow on the space W_6 using this picture.

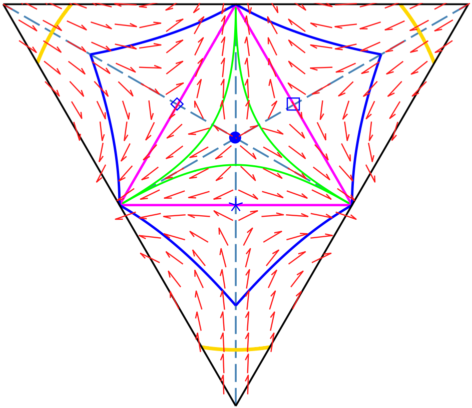

Fig. 19: The case $a = 1/6$: The domains of positive sectional, positive Ricci, and positive scalar curvatures, Kähler metrics, the phase portrait of the system (4) in the plane $x_1 + x_2 + x_3 = 1$.

Thank you for your time and attention!

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