Lightlike manifolds and Conformal Geometry

Francisco J. Palomo
Department of Applied Mathematics

UNIVERSIDAD
DE MÁLAGA

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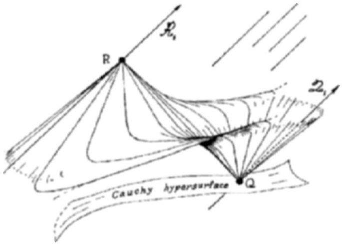
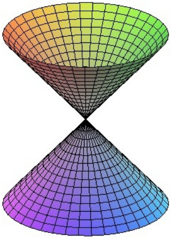
INTRODUCTION

Let (M, g) be a $(n+2)$-dimensional Lorentz manifold (a pseudo-Riemannian manifold with signature $(-, +, ..., +)$).

Lightlike hypersurface

A lightlike hypersurface in M is a smooth co-dimension one embedded submanifold $\psi : \mathcal{L} \to M$ such that the pullback of the metric g to $\mathcal{L}$ is degenerate at every point: $\mathrm{Rad}(T_p\mathcal{L}) := (T_p\mathcal{L})^{\perp} \cap T_p\mathcal{L} \neq \{0\}$ for all $p \in \mathcal{L}$.





The first picture is taken from R. Penrose: Rev. Mod. Phys. 37, 215 (1965)

Let $\psi : \mathcal{L} \to M$ be a lightlike hypersurface of a Lorentz manifold.

$$\mathrm{Rad}(T_p\mathcal{L}) := (T_p\mathcal{L})^\perp \cap T_p\mathcal{L} \neq \{0\}, \quad p \in \mathcal{L}$$

$\mathcal{R}_p = \mathrm{Rad}(T_p\mathcal{L})$ defines a 1-dimensional distribution on $\mathcal{L}$.

$$\Downarrow$$

- Intrinsical geometry: there is no distinguished linear connection on $\mathcal{L}$, in general.
- Extrinsical geometry: the *normal vector fiber bundle* $(T\mathcal{L})^\perp = \mathcal{R} \subset T\mathcal{L}$ is not transverse to $\mathcal{L}$.

$$TM_{|\mathcal{L}} \neq T\mathcal{L} \oplus (T\mathcal{L})^\perp$$

Several motivations...

1) Cauchy Horizons $H^+(S)$

For an acronal spacelike hypersurface S in a Lorentz manifold, the Cauchy horizon $H^+(S)$ marks the limit of the spacetime region controled by S.

2) Degenerate orbits of Lorentz isometric actions.

Let G be a Lie group acting isometrically on a Lorentz manifold (M, g). Any orbit which is lightlike at a point is lightlike everywhere and hence yields a lightlike submanifold of M.

3) Lightlike cones on Lorentz manifolds.

The Gauss lemma implies that for every $p \in M$, the exponential map exp_p applies a portion of the lightlike cone in T_pM on a lightlike hypersurface in M.

4) Lorentz manifolds foliated by lightlike hypersurfaces.

- Cahen-Wallach spaces.
- 2-symmetric Lorentzian spaces (which are not 1-symmetric).
- Plane-fronted waves.

A quotient construction for the extrinsical study of lightlike hypersurfaces.

It was introduced by Kupeli (1987) and developed by Galloway (2000)...

For a lightlike hypersurface $\psi : \mathcal{L} \to (M, g)$ which admits a (global non vanishing) lightlike vector field $\mathcal{Z} \in \mathfrak{X}(\mathcal{L})$ and radical distribution $\mathcal{R}$.
The quotient vector fiber bundle $T\mathcal{L}/\mathcal{R}$ inherits a Riemannian metric

$$\bar{g}([x],[y]) = g(x,y), \quad [x],[y] \in T_p\mathcal{L}/\mathcal{R}_p.$$

We can introduce (with respect to $\mathcal{Z}$):

- The null Wiengarten operator (∇^g the Levi-Civita connection of M)

$$A : T_p\mathcal{L}/\mathcal{R}_p \to T_p\mathcal{L}/\mathcal{R}_p, \quad A[x] = [\nabla^g_x \mathcal{Z}]$$

- The null second fundamental form

$$\mathrm{II} : T_p\mathcal{L}/\mathcal{R}_p \times T_p\mathcal{L}/\mathcal{R}_p \to \mathbb{R}, \quad \mathrm{II}([x],[y]) = \bar{g}(A[x],[y]).$$

This technique seems to provide an accurate method to study the extrinsical geometry of $\mathcal{L}$.

An ad hoc technique for the intrinsical study of lightlike hypersurfaces.

Introduced by Duggal and Bejancu in 1996.

Let $\psi : \mathcal{L} \to (M, g)$ be a lightlike hypersurface.

Fix an arbitrary n-distribution $S(\mathcal{L})$ (the Screen distribution) on $\mathcal{L}$ such that $T\mathcal{L} = \mathcal{R} \oplus S(\mathcal{L})$. Then,

1. $S(\mathcal{L})$ inherits a Riemannian metric.
2. There exists a unique lightlike transverse vector fiber bundle $tr(\mathcal{L})$ orthogonal to $S(\mathcal{L})$ such that

$$\boxed{TM\,|_{\mathcal{L}} = T\mathcal{L} \oplus tr(\mathcal{L}).}$$

For $X, Y \in \mathfrak{X}(\mathcal{L})$, the following decompositions strongly depend on $S(\mathcal{L})$.

$$\nabla^g_X Y = \boxed{\nabla_X Y} + \sigma_{S(\mathcal{L})}(X, Y), \qquad \text{Gauss equation}$$

- The induced linear connection ∇ on $\mathcal{L}$ depends on $S(\mathcal{L})$.
- $\nabla g = 0 \quad \Leftrightarrow \quad \sigma_{S(\mathcal{L})} = 0 \quad \Leftrightarrow \quad \mathcal{L}$ is totally geodesic.

In the above case, the connection depended on an arbitrary choice of a screen distribution.

Is there some way of constructing a torsion free metric linear connection on $\mathcal{L}$?: NOO!

Duggal-Jin, 2007

A torsion free linear connection on a lightlike hypersurface $\mathcal{L}$ compatible with g exists if and only if $\mathcal{R}$ is a Killing distribution.
Even in this case, there is an infinitude of connections with none distinguished.

For a lightlike hypersurface $\psi : \mathcal{L} \to (M, g)$, the distribution $\mathcal{R}$ is said to be Killing when every vector field $\mathcal{Z} \in \mathcal{R}$ is Killing ($L_{\mathcal{Z}} g = 0$.)

$$\Updownarrow$$

There are coordinates systems $(r, x_1, ..., x_n)$ such that $\frac{\partial}{\partial r}$ spans $\mathcal{R}$ and

$$\frac{\partial g_{ij}}{\partial r} = 0.$$

Summing up... from my point of view:

1. The Screen distribution construction is not a good approach to study intrinsic geometric properties of $\mathcal{L}$.
2. The lightlike hypersurfaces are conformal invariants. It would be desirable certain conformal invariance.

Natural questions

- Is it possible to construct an *intrinsic geometric structure* on $\mathcal{L}$?
- This *intrinsic geometric structure* should be independent of any arbitrary election...
- ... and should provide local invariants which permit to distinguish locally two lightlike manifolds.

Although I do not have a definitive answer to the above questions, let me introduce you the tangle of ideas I have developed hoping to find these geometric structures.

A new suitable definition

Let us start with an intrinsic definition of lightlike manifold $\mathcal{L}$ of signature (p, q)

Lightlike manifolds

A lightlike manifold of signature (p, q) is a pair $(\mathcal{L}^{p+q+1}, h)$ where

- $h \in \mathcal{T}_{0,2}(\mathcal{L})$ is a symmetric tensor (the degenerate metric tensor).
- $\mathrm{Rad}(h) := \mathcal{R}$ defines a 1-dimensional distribution on $\mathcal{L}$ (i.e., the radical is the smallest possible).
- The quotient vector fiber bundle $T\mathcal{L}/\mathcal{R}$ inherits a pseudo-Riemannian metric $\bar{h}$ of signature (p, q)

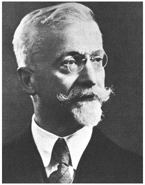
$$\bar{h}([u], [v]) = h(u, v), \quad [u], [v] \in T_x\mathcal{L}/\mathcal{R}_x$$

  for $x \in \mathcal{L}$.

The main ideas to study this kind of *intrinsic geometric structure* on $\mathcal{L}$ will come from the notion of Cartan geometry.

KLEIN AND CARTAN GEOMETRIES

- In the early 1920s, Elie Cartan found a common generalization for the Klein's Erlangen program and Riemann geometry. He called *Espaces généralizés* and now we call Cartan Geometries.



Elie Cartan (1869-1951) (Wikimedia Commons)

Charles Ehresmann (1905-1979)

- Around 1950, Charles Ehresmann gave for the first time a rigorous global definition of a Cartan connection as a particular case of a more general notion now called Ehresmann connection (principal connections).

  This is the point of view of the influential book *Foundations of Differential Geometry, Volumes I and II* by S. Kobayashi and K. Nomizu.

First step: Klein Geometry. The Homogeneous model G/H

1. G is a Lie group and H a closed subgroup of G such that G/H is connected and is considered with a *geometric structure* such that:
2. The left translations, ℓ_g for $g \in G$, are all the automorphisms of the geometric structure (even locally).

Second step: Cartan connection on M modeled on (G, H)

The Cartan connection permits to associate a differential geometric structure to M and so M may be thought as a curved analog of the homogeneous space G/H.

Third step: The equivalence problem

Starting from the differential geometric structure on M, is it possible to construct a (unique) Cartan connection on M modeled on (G, H) such that the related geometric structure from second step is the original one?

First step:

The Klein Geometry: looking for the model of lightlike manifolds $\mathcal{L} = G/H$

1. A homogeneous $(p+q+1)$-dimensional manifold $\mathcal{L} = G/H$ endowed with a degenerate metric tensor h of signature (p,q).
2. The isometries of h should be exactly the left translations by elements of G. Even locally...
3. ... suppose that $\mathcal{L} = G/H$ is connected. Then any isometry between two connected open subsets of $\mathcal{L}$ uniquely globalizes to a left translation by an element of G (Liouville Theorem).

What could it be the model for lightlike manifolds?

Consider $\mathbb{R}^{p+q+2}$ with basis $(\ell, e_1, ..., e_p, t_1, ..., t_q, \eta)$ and endowed with scalar product $\langle\ ,\ \rangle$ of signature $(p+1, q+1)$ corresponding to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{I}_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{where } \mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix}.$$

The $(p+q+1)$-dimensional isotropic (lightlike) cone is given by

$$C := \left\{ v \in \mathbb{R}^{p+q+2} : \langle v, v\rangle = 0, \quad v \neq 0 \right\}.$$

$C = C^{p+q+1}$ inherits from $\langle\ ,\ \rangle$ a degenerate metric tensor of signature (p, q) with radical distribution $\mathcal{R}_v = \mathbb{R}\cdot v$ for any $v \in C$.

(The cone C is almost our model)

The antipodal map $x \mapsto -x$ preserves the degenerate metric tensor.

Our candidate to homogeneous model:

$$\left(\mathcal{L}^{p+q+1} := C/\mathbb{Z}_2, \quad h := \langle\ ,\ \rangle \right)$$

For later use, let us denote by $\tau : C \to \mathcal{L}$ the projection.

$\mathcal{L}$ as homogeneous space

- The action, $O(p+1,q+1)\times C\to C$ is transitive and preserves $\langle\ ,\ \rangle$.
- Consider the Möbius group

$$G := PO(p+1,q+1) = O(p+1,q+1)/\{\pm \mathrm{Id}\}.$$

- The induced action

$$G\times\mathcal{L}\to\mathcal{L},\quad [g]\cdot\tau(v) := \tau(g\cdot v)$$

is still transitive and preserves h.

- Thus, we can identify $\mathcal{L}$ with G/H, where $H\subset G$ is the isotropy group of the class $\tau(\ell)=\{\pm\ell\}\in\mathcal{L}$ ($\ell\in C$ the first vector of the above basis). Moreover,

$$G\subset \mathrm{Iso}(h).$$

$$G=\mathrm{Iso}(h)??$$

Looking for a geometric description of the model of lightlike manifolds $\mathcal{L} = G/H$

- Denote by $\pi : \mathbb{R}^{p+q+2} \setminus \{0\} \to \mathbb{R}P^{p+q+1}$ the natural projection.
- Let us consider the space of lines in C (i.e., the Möbius space of signature (p, q))

$$\mathbb{S}^{(p,q)} := \pi(C).$$

- $G = PO(p+1, q+1)$ acts naturally on $\mathbb{S}^{(p,q)}$.
- The Möbius space $\mathbb{S}^{(p,q)}$ carries a conformal structure $[c]$ of signature (p, q) (inherited from π).
- For $p + q \geq 2$, the Lie group $\mathrm{Conf}(\mathbb{S}^{(p,q)})$ of global conformal transformations of $\mathbb{S}^{(p,q)}$ satisfies

$$\boxed{G = \mathrm{Conf}(\mathbb{S}^{(p,q)}).}$$

- An explicit description of the Möbius space:

$$\boxed{(\mathbb{S}^p \times \mathbb{S}^q)/\mathbb{Z}_2 = \mathbb{S}^{(p,q)}, \quad [x^+, x^-] \mapsto \pi(x^+, x^-)}$$

and $[c]$ corresponds to the conformal class of the metric tensor c: the product of the two round metrics of radius one with opposite signs.

The manifold $\mathbb{S}^{(p,q)} \times \mathbb{R}_{>0}$ admits the degenerate metric tensor $h := t^2 \cdot c \oplus 0.$

Denoting every element of $(\mathbb{S}^p \times \mathbb{S}^q)/\mathbb{Z}_2 = \mathbb{S}^{(p,q)}$ by $[x^+, x^-]$ for $x^+ \in \mathbb{S}^p$ and $x^- \in \mathbb{S}^q$, we have the following isometry

$$\mathbb{S}^{(p,q)} \times \mathbb{R}_{>0} \to \mathcal{L}^{p+q+1}, \quad ([x^+, x^-], t) \mapsto \tau(t \cdot (x^+, x^-))$$

Theorem [1]

- For $p + q \geq 2$, the group $\mathrm{Iso}(\mathcal{L})$ is the Lie group G.
- For $p + q \geq 3$, every isometry betweeen two connected open subsets of $\mathcal{L}$ is the restriction of the left translation by an element of $G = \mathrm{Iso}(\mathcal{L})$.
- If $p = 2$ and $q = 0$, the (global) isometry group of $\mathcal{L} \subset \mathbb{L}^3$ is also isomorphic to $G = \mathrm{Conf}(\mathbb{S}^2)$ but the group of local isometries of $\mathcal{L}$ is the group of local conformal transformations of $\mathbb{S}^2$.

$\Rightarrow$ First step satisfied!!

[1] Bekkara, Frances and Zeghib (2009) for Lorentzian signature $(p+1, 1)$.

Taking a look at the model $\mathcal{L}^{p+q+1} = G/H$ at Lie groups level

The Möbius sphere $(\mathbb{S}^{(p,q)}, [c])$ as a Klein Geometry

Recall, the Lie group $G = \mathrm{Conf}(\mathbb{S}^{(p,q)})$ acts transitively by conformal transformations on $\mathbb{S}^{(p,q)}$.
The isotropy group of $\pi(\ell) := \mathbb{R} \cdot \ell \in \mathbb{S}^{(p,q)}$ is

$$P = \left\{ \begin{bmatrix} \lambda & -\lambda w^t C & -\frac{\lambda}{2}\langle w, w\rangle \\ 0 & C & w \\ 0 & 0 & \lambda^{-1} \end{bmatrix} : \lambda \in \mathbb{R} \setminus \{0\}, w \in \mathbb{R}^{p+q}, C \in O(p,q) \right\}$$

Thus, $\mathbb{S}^{(p,q)} = G/P$. (P is called the Poincaré conformal group)

The Klein Geometry (G, P) is the model of conformal geometry.

On the other hand, $\mathcal{L}^{p+q+1} = G/H$ where H is the isotropy group of $\tau(\ell) = \{\pm\ell\}$.

$$H = \{[g] \in P : \lambda = \pm 1\} \cong \mathbb{R}^{p+q} \rtimes O(p,q) = \mathrm{Iso}(\mathbb{R}^{p,q})$$

We also have a natural projection

$$\mathbb{P} : \mathcal{L} \to \mathbb{S}^{(p,q)}, \quad \tau(v) \mapsto \pi(v)$$

that corresponds with the projection $\mathbb{P}$

$$\mathbb{P} : \mathcal{L} = G/H \longrightarrow \mathbb{S}^{(p,q)} = G/P, \quad g\,H \mapsto g\,P.$$

$\mathbb{P}$ is a fiber bundle with fiber the homogeneous space $P/H \simeq \mathbb{R}_{>0}$.
Our identification of $\mathcal{L}$ gives another interpretation for $\mathbb{P}$.

$$\mathbb{P} : \mathbb{S}^{(p,q)} \times \mathbb{R}_{>0} \longrightarrow \mathbb{S}^{(p,q)}, \quad (\pi(v), t) \mapsto \pi(v).$$

Thus, every section of $\mathbb{P}$ corresponds to an election of a metric tensor in the conformal class $[c]$.

Taking a look at the model $\mathcal{L}^{p+q+1} = G/H$ at Lie algebras level

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & z & 0 \\ x & A & -z^t \\ 0 & -x^t & -a \end{pmatrix} : a \in \mathbb{R}, x \in \mathbb{R}^{p+q}, z \in (\mathbb{R}^{p+q})^*, A \in \mathfrak{o}(p,q) \right\}$$

$$\boxed{\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1} \quad \text{and} \quad \boxed{\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1}$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & z & 0 \\ 0 & A & -z^t \\ 0 & 0 & 0 \end{pmatrix} : z \in (\mathbb{R}^n)^*, A \in \mathfrak{o}(p,q) \right\} = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{g}_1 \leq \mathfrak{p} \leq \mathfrak{g}$$

$$(a, x) \in \mathbb{R} \oplus \mathbb{R}^{p+q} \simeq \mathfrak{g}/\mathfrak{h} \simeq T_{\tau(\ell)}\mathcal{L}$$

An arbitrary Klein Geometry (G, H) is said to be...

- First order when the representation of H given by

$$\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h}.$$

  is injective.
  In this case, $G \subset L(G/H)$ (a fiber bundle of frames over G/H).

- Reductive (with complement fixed $\mathfrak{m}$)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \text{ as vector spaces and}$$

$$\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}.$$

- $\mid k \mid$-graded ($k \geq 1$ and the grading is assumed to be fixed)

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k,$$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k.$$

  where $\mathfrak{g}_{-k} \neq \{0\}$, $\mathfrak{g}_k \neq \{0\}$ and $\mathfrak{g}_i = \{0\}$ for $\mid i \mid > k$.

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{conformal model}$$

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{h} = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{g}_1 \leq \mathfrak{p} \quad \text{lightlike model}$$

The model of conformal geometry $\mathbb{S}^{(p,q)} = G/P$ is a $\mid 1 \mid$-graded Klein Geometry.

The lightlike Klein Geometry $\mathcal{L} = G/H$ is of first order but is not reductive.

$$\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(a, x) = (a - \langle C^{-1}x, w\rangle, Cx),$$

where $h \simeq (w, C) \in \mathbb{R}^{p+q} \rtimes O(p, q)$.

$$\mathbb{P} : G/H \to G/P$$

applies a first order non reductive geometry to a 1-graded geometry.

Cartan Geometry of type (G, H) on M

- A principal fiber bundle $\pi : \mathcal{P} \to M$ with structure group H.
- A $\mathfrak{g}$-valuated one form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called the Cartan connection such that:
  1. $\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.
  2. For ξ_X, the fundamental vector field corresponding to $X \in \mathfrak{h}$,
  $$\omega(\xi_X) = X \qquad \left(\text{where } \xi_X(u) := \frac{d}{dt}\,|_0\, (u \cdot \exp(tX))\right)$$
  3. For every $h \in H$, let r^h be the corresponding right multiplication on $\mathcal{P}$. Then
  $$(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$$
  That is, the following diagram commutes.
  $$\begin{array}{ccc} T_u\mathcal{P} & \xrightarrow{\ \omega(u)\ } & \mathfrak{g} \\ {\scriptstyle T_u r^h}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}(h^{-1})} \\ T_{uh}\mathcal{P} & \xrightarrow[\ \omega(uh)\ ]{} & \mathfrak{g} \end{array}$$

$$\boxed{\dim(\mathcal{P}) = \dim(G) \quad \Rightarrow \quad \dim(M) = \dim(G/H)}$$

The tangent bundle of a Cartan geometry of type (G, H)

Consider the representation of H given by

$$\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h}.$$

For each $u \in \mathcal{P}$ with $\pi(u) = x \in M$, there is a canonical linear isomorphism ϕ_u such that the following diagram conmutes

$$\begin{array}{ccc} T_u\mathcal{P} & \xrightarrow{\omega(u)} & \mathfrak{g} \\ {\scriptstyle T_u\pi}\downarrow & & \downarrow{\scriptstyle \rho} \\ T_xM & \xrightarrow[\phi_u \cong]{} & \mathfrak{g}/\mathfrak{h} \\ & & \Updownarrow \\ & & T_o(G/H) \end{array} \quad \text{with } \phi_{uh} = \underline{\mathrm{Ad}}(h^{-1})\phi_u \text{ for all } h \in H.$$

From now on, we return to the homogeneous model for lightlike manifolds $\mathcal{L} = G/H$.

Second step: Cartan Geometry on $\mathcal{M}$ modeled on (G, H)

Proposition

Let $(\pi : \mathcal{P} \to \mathcal{M}, \omega)$ be a Cartan geometry with model (G, H).

Then, $\mathcal{M}$ can be endowed with a lightlike manifold structure with degenerate metric tensor h. Moreover, there is a vector field $\mathcal{Z} \in \mathfrak{X}(\mathcal{M})$ which globally spans the radical distribution $\mathcal{R} = \mathrm{Rad}(h)$.

1. For every $u \in \mathcal{P}$ with $\pi(u) = x$, consider

$$\phi_u : T_x\mathcal{M} \to \mathfrak{g}/\mathfrak{h} \cong \mathbb{R} \oplus \mathbb{R}^{p+q} \cong T_{\tau(\ell)}\mathcal{L}$$

   and then we introduce a degenerate metric product h_u on each $T_x\mathcal{M}$
2. h_u does not depend on the election of $u \in \mathcal{P}$ with $\pi(u) = x$.

The *natural* hyperplane of $\mathbb{R} \oplus \mathbb{R}^{p+q}$ is not invariant by $\underline{\mathrm{Ad}}(H)$. There is no screen distribution.

$\Rightarrow$ Second step satisfied!!

Third step: The equivalence problem for lightlike manifolds

Work in progress...

Correspondence spaces

Let $(\pi : \mathcal{P} \to M, \omega)$ be a Cartan geometry with arbitrary model (G, P).

We define the correspondence space $\mathcal{C}(M)$ of M for $H \subset P$ to be the quotient space $\mathcal{P}/H$:

$$\begin{array}{ccc} & \mathcal{P} & \\ & \downarrow \quad \searrow & \\ \mathcal{P}/H = \mathcal{C}(M) & \xrightarrow{\mathbb{P}} & M = \mathcal{P}/P \end{array}$$

The projection $\mathbb{P} : \mathcal{C}(M) \to M$ is a fiber bundle with fiber the homogeneous space P/H and

$$(\Pi : \mathcal{P} \to \mathcal{C}(M), \omega) \text{ is a Cartan geometry of type } (G, H).$$

Return to ours fixed Lie groups $H \subset P \subset G$...

Which are the correspondence spaces for $H \subset P \subset G$?

Conformal Riemannian structures on M

$\Updownarrow$ 1-1 correspondence

Cartan connections on M with model (G, P)
(satisfying certain *curvature* properties)

Proposition

Let $(M, [g])$ be a conformal pseudo-Riemannian manifold of signature (p, q) and $(\pi : \mathcal{P} \to M, \omega)$ the corresponding Cartan geometry with model (G, P).

Then the correspondence space for $H \subset P$ is

$$\mathcal{C}(M) = M \times \mathbb{R}_{>0}$$

endowed with the degenerate metric tensor of signature (p, q)

$$h = t^2 \cdot g \oplus 0.$$

In particular, $\mathcal{E}(M)$ admits a distinguished Cartan connection of type (G, H) and $\mathcal{C}(M)$ can be view as the bundle of scales associated to $(M, [g])$.

Let $(\mathcal{M}, h)$ be a lightlike manifold with radical distribution $\mathcal{R} = \mathrm{Span}(\mathcal{Z})$.

Under which conditions is the lightlike manifold $\mathcal{M}$
the bundle of scales of pseudo-Riemann conformal manifold?
(these will admit a distinguished Cartan connection of type (G, H))

We hope to find an answer from the next two approaches:

1. By analyzing the orbit space $\mathcal{V} = \mathcal{M}/\mathcal{Z}$...
2. By constructing a Cartan connection on $\mathcal{M}$ of type (G, H) using dual connections.

A dual connection on $\mathcal{M}$ is an $\mathbb{R}$-bilinear map

$$\Box : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \Omega^1(\mathcal{M})$$

such that $\Box_{f\,X} Y = f \Box_X Y$ and $\Box_X(f\,Y) = X(f) h(Y, -) + f \Box_X Y$.

The torsion tensor is $\mathrm{T}(X, Y, Z) = \Box_X Y(Z) - \Box_Y X(Z) - h([X, Y], Z)$ and $\Box$ is compatible with h whenever $X\,h(Y, Z) = \Box_X Y(Z) + \Box_X Z(Y)$.

For every election of a torsion tensor T, there is a unique dual connection $\Box$ on $\mathcal{M}$ such that $\Box$ is compatible with h and has torsion T.