Hyperkähler Implosion and Nahm's Equations

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Symplectic Reduction

Symplectic Reduction

- (M^{2n}, ω) symplectic, i.e. $\omega \in \Omega^2(M)$, $\omega^n \neq 0$, $d\omega = 0$
- e.g. cotangent bundles T^*N, coadjoint orbits, Kähler manifolds
- local model $\omega = \sum_{i=1}^n dx^i \wedge dy^i$ on $\mathbb{R}^{2n}$.
- K compact connected Lie group, acts on M preserving ω,
- moment map $\mu : M \to \mathfrak{k}^*$,($\mathfrak{k} = \mathrm{Lie}(K)$) K-equivariant,

$$d\mu(X)(\xi) = \omega(X^\xi, X) \qquad \forall X \in TM, \xi \in \mathfrak{k}.$$

Symplectic reduction at level $\tau \in \mathfrak{k}^*$:

$$M /\!\!/_\tau K := \mu^{-1}(\tau)/C(\tau),$$

$C(\tau) = \mathrm{Stab}(\tau) =$ centraliser of τ (w.r.t. coadjoint action)

Symplectic Reduction, Example 1

- $M = \mathbb{C}^2$, $\omega = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i$
- $\mathrm{U}(1)$-action

$$e^{i\theta} : (z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2).$$

- Moment map $\mu : \mathbb{C}^2 \to \mathbb{R} \cong \mathfrak{u}(1)$,

$$\mu(z) = \frac{1}{2}(|z_1|^2 + |z_2|^2)$$

- $$M \mathbin{/\!\!/}_\tau U(1) = \begin{cases} S^3_{2\tau}/_{U(1)} \cong \mathbb{CP}^1 & \tau > 0. \\ \{\mathrm{pt}\} & \tau = 0. \end{cases}$$

Symplectic Reduction, Example 2

- $M = T^*K \cong K \times \mathfrak{k}^*$, $T_{(1,\tau)}M \cong \mathfrak{k} \times \mathfrak{k}^*$,

$$\omega_{(1,\tau)}((\xi,\alpha),(\eta,\beta)) = \alpha(\eta) - \beta(\xi) + \tau([\xi,\eta])$$

- K acts on itself by right-translation, induces action on $K \times \mathfrak{k}^*$ by

$$g : (k,\tau) \mapsto (kg^{-1}, g\tau g^{-1})$$

- Moment map $\mu : M = T^*K \to \mathfrak{k}^*$,

$$\mu(k,\tau) = \tau$$

- $M /\!\!/_\tau K = \mu^{-1}(\tau)/C(\tau) = K/C(\tau) \times \{\tau\} \cong (\mathcal{O}_\tau, \omega^{KKS})$.
- $K = \mathrm{SU}(2)$, then for $\tau \in \mathfrak{t} \cong \mathbb{R}$

$$M /\!\!/_\tau K = (\mathcal{O}_\tau, \omega^{KKS}) \cong \begin{cases} \mathbb{CP}^1 & \tau > 0. \\ \{\mathrm{pt}\} & \tau = 0. \end{cases}$$

Symplectic Reduction, infinite-dimensional Example

- Fix $\tau \in \mathfrak{t} \subset \mathfrak{k}$.
- $M = \{(T_0, T_1) : [0, \infty) \to \mathfrak{k} \otimes \mathbb{R}^2 : \; T_0(\infty) = 0, T_1(\infty) = \tau\}$
- Tangent vectors: $(X_0, X_1) : [0, \infty) \to \mathfrak{k} \otimes \mathbb{R}^2, X_i(\infty) = 0.$
- $$\omega(X, Y) = \int_0^\infty \mathrm{Tr}(X_0 Y_1 - X_1 Y_0) dt,$$
  compatible Kähler metric $\|X\|^2 = \int_0^\infty \|X_0\|^2 + \|X_1\|^2 dt$
- Gauge transformations:
  $\mathcal{G} = \{u : [0, \infty) \to K : \; u(0) = 1, u(\infty) \in C(\tau)\}$
  $$(T_0, T_1) \mapsto (u T_0 u^{-1} - \dot{u} u^{-1}, u T_1 u^{-1})$$
- Moment map: $\mu(T_0, T_1) = \dot{T}_1 + [T_0, T_1]$
- Moduli Space: $M /\!\!/_0 \mathcal{G} = \{\dot{T}_1 + [T_0, T_1] = 0\}/\mathcal{G}$

Symplectic Reduction, infinite-dimensional Example

- Identify $M /\!\!/_0 \mathcal{G} = \{\dot{T}_1 + [T_0, T_1] = 0\}/\mathcal{G}$ as a symplectic manifold:
- Given solution (T_0, T_1) find $u : [0, \infty) \to K$,

$$(T_0, T_1) = u.(0, \tau) = (-\dot{u}u^{-1}, u\tau u^{-1}), \qquad u(\infty) \in C(\tau)$$

- u unique up to constant in $C(\tau)$.

Get an iso:

$$\begin{array}{rcl} M /\!\!/_0 \mathcal{G} & \cong & (\mathcal{O}_\tau, \omega^{KKS}), \\ (T_0, T_1) & \mapsto & T_1(0) = u(0)\tau u(0)^{-1} \end{array}$$

Symplectic Implosion

Symplectic Implosion

Ingredients:

- $T \subset K$ max. torus, $\mathfrak{t} = \mathrm{Lie}(T)$
- $\mathfrak{t}_+^*$ closed positive Weyl chamber in $\mathfrak{t}^*$
- (M, ω) Hamiltonian K-manifold

Guillemin-Jeffrey-Sjamaar (2001): Imploded cross-section

M_{impl} = stratified symplectic space, Hamiltonian T-action

$$M_{impl} /\!\!/_\tau T = M /\!\!/_\tau K \qquad \forall \tau \in \mathfrak{t}_+^*$$

Abelianisation of symplectic reduction.

Symplectic Implosion, Example

- $K = \mathrm{SU}(2)$, $M = T^*K$ with K acting by right-translation,
- $T = \mathrm{U}(1)$, $\mathfrak{t}^*_+ \cong [0, \infty)$
- M_{impl} should come with $\mathrm{U}(1)$-action, such that $M_{impl} /\!\!/ \mathrm{U}(1) = \mathcal{O}_\tau$, coadjoint orbit.
- First Example says that $\mathbb{C}^2$ works. How to get $M_{impl} = \mathbb{C}^2$ from $M = T^*K$?

$$
\begin{aligned}
\mathbb{C}^2 &= \{pt\} \sqcup \mathbb{C}^2 \setminus \{0\} \\
&\cong \{pt\} \sqcup (S^3 \times (0, \infty)) \\
&\cong (\mathrm{SU}(2) \times \mathfrak{t}^*_+)/(\text{collapse origin to a point})
\end{aligned}
$$

The Universal Implosion

In general:

- K compact, semi-simple, simply connected
- $(\mathfrak{t}^*_+)_C$ face of $\mathfrak{t}^*_+$, i.e. elements with centraliser C
- $T^*K \cong K \times \mathfrak{k}^*$
- Hamiltonian K-action induced by right-translation with moment map

$$\mu(k, \tau) = \tau$$

$$\Rightarrow \mu^{-1}(\mathfrak{t}^*_+) = K \times \mathfrak{t}^*_+.$$

-

$$\begin{aligned}(T^*K)_{impl} &= \coprod_C (K \times (\mathfrak{t}^*_+)_C)/[C, C] \\ &= K \times (\text{interior of } \mathfrak{t}^*_+) \sqcup (\text{lower-dim strata})\end{aligned}$$

- i.e., take $K \times \mathfrak{t}^*_+$, collapse by commutator subgroups of centraliser of faces.

Properties of the Universal Implosion T^*K_{impl}

- stratified symplectic space.
- Top stratum: $K \times (\text{interior of } \mathfrak{t}^*_+) \subset (T^*K)_{impl}$,
- Hamiltonian $K \times T$-action
- $(T^*K)_{impl} /\!\!/_\tau T = \mathcal{O}_\tau$, the coadjoint orbit through τ
- T^*K_{impl} is universal in the sense that in general

$$M_{impl} = (M \times (T^*K)_{impl}) /\!\!/_0 K_{diagonal}$$

Link with algebraic geometry

Consider again $K = \mathrm{SU}(2)$, $T^*K_{impl} \cong \mathbb{C}^2$.

- $N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{SL}(2,\mathbb{C}) = K_{\mathbb{C}}$ acts on $\mathrm{SL}(2,\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b+na \\ c & d+nc \end{pmatrix}$$

- Invariants given by a, c. Therefore

$$\mathbb{C}^2 \cong \mathrm{SL}(2,\mathbb{C}) /\!\!/_{GIT} N := \mathrm{Spec}(\mathcal{O}(K_{\mathbb{C}})^N)$$

- In fact, $\mathbb{C}^2 = \overline{\mathbb{C}^2 \setminus \{0\}} = \overline{\mathrm{SL}(2,\mathbb{C}).\binom{1}{0}}$
- In general $N \subset K_{\mathbb{C}}$ max unipotent subgroup

$$T^*K_{impl} \cong K_{\mathbb{C}} /\!\!/_{GIT} N := \mathrm{Spec}(\mathcal{O}(K_{\mathbb{C}})^N),$$

- $T^*K_{impl} \cong \overline{K_{\mathbb{C}}.v} \subset E = \bigoplus_\rho V_\rho$,
  V_ρ fundamental rep., $v = \sum_\rho v_\rho$ sum of highest weight vectors

Analogue in Hyperkähler Geometry?

Hyperkähler manifolds

- (M, g, I, J, K) hyperkähler manifold, $IJ = -JI = K$ Kähler forms $\omega_I = g(I-,-)$ etc.
- e.g. $T^*K_{\mathbb{C}}$, complex coadjoint orbits, many moduli spaces
- K compact, simply connected acts preserving HK structure
- Hyperkähler moment map $\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{k}^* \otimes \mathbb{R}^3$,
- Hyperkähler quotient

$$M /\!\!/\!\!/ K = \mu^{-1}(0)/K \cong (\mu_J + i\mu_K)^{-1}(0) /\!\!/ K_{\mathbb{C}}$$

$$M_{hkimpl} = ???$$

Idea: Find "universal implosion" $\mathcal{Q}$ first, should carry $K \times T$-action, then define

$$M_{hkimpl} = (M \times \mathcal{Q}) /\!\!/\!\!/ K$$

The case $K = \mathrm{SU}(n)$

Dancer-Kirwan-Swann ('13): $K = \mathrm{SU}(n)$,

$$\mathcal{Q} = \{\mathbb{C} \rightleftarrows \mathbb{C}^2 \rightleftarrows \cdots \rightleftarrows \mathbb{C}^n\} /\!\!/\!\!/_{\prod_{k=1}^{n-1} \mathrm{SU}(k)},$$ quiver variety

- $\mathcal{Q}$ is stratified hyperkähler
- Top stratum contains $K_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}^{reg}$
- hyperkähler $K \times T$-action,
- $\mathcal{Q} /\!\!/\!\!/_{(\tau_1,\tau_2,\tau_3)} T \cong$ Kostant variety, i.e. closure of regular complex coadjoint orbit,
  e.g. $\mathcal{O}_{\tau_2 + i\tau_3}$ if $\tau_2 + i\tau_3$ is regular, nilpotent variety if $\tau = 0$.
- Action of $\mathrm{SU}(2)$ rotating the complex structures
- $\mathcal{Q} \cong (K_{\mathbb{C}} \times \mathfrak{b}) /\!\!/_{GIT} N$, complex symplectic GIT quotient of $T^*K_{\mathbb{C}}$ by N
- Contains universal symplectic implosion $K_{\mathbb{C}} /\!\!/_{GIT} N$ as fixed point set of a $\mathbb{C}^*$-action (scaling in fibres of $T^*K_{\mathbb{C}}$)

More general K?

Nahm's Equations

Nahm's Equations

- $(T_0, T_1, T_2, T_3) : U \to \mathfrak{k} \otimes \mathbb{R}^4$, $U \subset \mathbb{R}$ interval

$$\dot{T}_i + [T_0, T_i] = [T_j, T_k],$$

- gauge transformations $u : U \to K$

$$u.T_0 = uT_0u^{-1} - \dot{u}u^{-1}, \quad u.T_i = uT_iu^{-1}.$$

- model solutions:

$$T_{\tau,\sigma} : \qquad T_0 = 0, \quad T_i = \tau_i + \frac{\sigma_i}{2(t+1)}.$$

$\tau = (\tau_1, \tau_2, \tau_3) \in \mathfrak{t} \otimes \mathbb{R}^3$, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ $\mathfrak{su}(2)$-rep in $\mathrm{Lie}(C(\tau))$, i.e. $[\sigma_i, \sigma_j] = -2\sigma_k$ etc., $[\tau_i, \sigma_j] = 0$

- Formally, Nahm's equations = hyperkähler moment map.
  expect Nahm moduli spaces to carry hyperkähler structure

Nahm Moduli Spaces and Lie Groups

- $U = [0,1]$ (Kronheimer)

$$T^*K_{\mathbb{C}} \cong \frac{\{\text{Solutions to Nahm equations on } [0,1]\}}{\{u\colon\ u(0) = 1 = u(1)\}}$$

- $U = [0,\infty)$ (Biquard, Kovalev, Kronheimer) Any solution is asymptotic to a model solution $T_{\tau,\sigma}$

$$\mathcal{M}(\tau,\sigma) = \frac{\{\text{Solutions on } [0,\infty) \text{ asymptotic to fixed } T_{\tau,\sigma}\}}{\{u\colon\ u(0) = 1,\ u(\infty) \in C(\tau) \cap C(\sigma)\}}$$

for generic complex structure $\mathcal{M}(\tau,\sigma) \cong$ complex (co)adjoint orbit

Both examples should be interpreted as hyperkähler quotients in an infinite-dimensional setting and the metric is the L^2-metric.

A Candidate for the Universal Hyperkähler Implosion

- Consider the space $\mathcal{A}$ of functions $T = (T_0, T_1, T_2, T_3) : [0, \infty) \to \mathfrak{k}^4$ such that $\exists \tau, \sigma$ with

$$T_i(t) = \tau_i + \frac{\sigma_i}{2(t+1)} + \ldots, \quad i = 1, 2, 3$$

- Take the subset of solutions to the Nahm equations, stratify by centraliser of τ and collapse by gauge transformations with a limit in $[C, C]$ to get a space

$$\mathcal{Q}^{Nahm} = \coprod_C \mathcal{Q}_C,$$

where

$$\mathcal{Q}_C = \frac{\{\text{solutions } T \text{ asymptotic to } T_{\tau,\sigma},\ C(\tau) = C\}}{\{u : u(0) = 1, u(\infty) \in [C, C]\}}$$

A Candidate for the Universal Hyperkähler Implosion

- Use $[C, C]$-action to move σ's into normal form to get refined stratification:
$$\mathcal{Q}_C = \coprod_{[\sigma] \subset \mathfrak{c}} \mathcal{Q}_{C,\sigma},$$
$$\mathcal{Q}_{C,\sigma} = \frac{\{\text{solutions } T \text{ asymptotic to } T_{\tau,\sigma},\ C(\tau) = C,\ \sigma \text{ fixed}\}}{\{u : u(0) = 1, u(\infty) \in [C, C] \cap C(\sigma)\}},$$

- So altogether:
$$\mathcal{Q}^{Nahm} = \coprod_{C} \coprod_{[\sigma] \subset \mathfrak{c}} \mathcal{Q}_{C,\sigma}.$$

- The refined strata $\mathcal{Q}_{C,\sigma}$ arise as hyperkähler quotients

The metric

- Tangent vectors to $\mathcal{Q}_C$:

$$X_i = \delta_i + \frac{\epsilon_i}{2(t+1)} + \dots$$

$X_i(\infty) = \delta_i \neq 0$, tangent vectors do not lie in L^2.

- Use Bielawski's metric ($b > 0$):

$$||X||_b^2 := b \sum_{i=0}^{3} |X_i(\infty)|^2 + \int_0^\infty (|X_i(t)|^2 - |X_i(\infty)|^2) dt.$$

- Finite on strata $\mathcal{Q}_C$, invariant under gauge transformations
- Gives moment map interpretation to Nahm equations so that $\mathcal{Q}_{C,\sigma}$ = hyperkähler quotient
- Reduces to L^2-metric on subvarieties where the limit τ is fixed
- non-degenerate?

Properties of $\mathcal{Q}^{Nahm}$

- $\mathcal{Q}^{Nahm}$ is stratified hyperkähler
- Top stratum contains $K_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}^{reg}$
- Hyperkähler $K \times T$-action given by gauge transformations $u(0) \in K, u(\infty) \in T$
- $\mathcal{Q}_{C,\sigma} /\!\!/\!\!/_\tau T \cong \mathcal{M}(\tau, \sigma) \Rightarrow \mathcal{Q}^{Nahm} /\!\!/\!\!/ T =$ Kostant variety
- Action of $\mathrm{SU}(2)$ rotating the complex structures
- Contains universal symplectic implosion T^*K_{impl} as fixed point set of circle action $(T_2 + iT_3) \mapsto e^{i\theta}(T_2 + iT_3)$, i.e. solutions with $T_2 = 0 = T_3$.
- $\mathcal{Q}^{Nahm} \cong (T^*K_{\mathbb{C}} \times \mathcal{Q}^{Nahm}) /\!\!/\!\!/ K$
- However: $\mathcal{Q}^{Nahm}$ not iso to $(K_{\mathbb{C}} \times \mathfrak{b}) /\!\!/_{GIT} N$, so does not agree with quiver model in $\mathrm{SU}(n)$-case.