Invariant Einstein Metrics on Stiefel Manifolds

Marina Statha

Ph.D. Student

(Joint work with A. Arvanitoyeorgos and Y. Sakane)

ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΑΤΡΩΝ
UNIVERSITY OF PATRAS

Stiefel manifolds

Stiefel manifolds $V_k\mathbb{F}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ are the set of all orthonormal k-frames in $\mathbb{F}^n$. It can be shown that $V_k\mathbb{F}^n$ is diffeomorphic to a homogeneous space G/H. In particular:

- In case $\mathbb{F} = \mathbb{R}$ ✓

$$V_k\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-k)$$

- In case $\mathbb{F} = \mathbb{C}$

$$V_k\mathbb{C}^n \cong \mathrm{SU}(n)/\mathrm{SU}(n-k)$$

- In case $\mathbb{F} = \mathbb{H}$ ✓

$$V_k\mathbb{H}^n \cong \mathrm{Sp}(n)/\mathrm{Sp}(n-k)$$

In all cases the Stiefel manifolds are *reductive homogeneous spaces*, with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ and $\mathfrak{m} \cong T_o(G/H)$, with respect to negative of Killing form of $\mathfrak{g}$.

If H is connected then $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m} \Leftrightarrow [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

G-invariant metrics on G/H

A G-invariant metric g on homogeneous space G/H is the metric for which the diffeomorphism $\tau_\alpha : G/H \to G/H, gH \mapsto \alpha gH$ is an isometry. It can be shown that

Proposition 1

There exists a one-to-one correspondence between:

1. G-invariant metrics g on G/H
2. $\mathrm{Ad}^{G/H}$-invariant inner products $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}$, that is

$$\langle \mathrm{Ad}^{G/H}(h)X,\, \mathrm{Ad}^{G/H}(h)Y\rangle = \langle X,\, Y\rangle \quad \text{for all } X, Y \in \mathfrak{m}, h \in H$$

3. (if H is compact and $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to the negative of the Killing form B of G) $\mathrm{Ad}^{G/H}$-equivariant, B-symmetric and positive definite operators $A : \mathfrak{m} \to \mathfrak{m}$ such that $\langle X, Y\rangle = B(A(X), Y)$.

We call such an inner product $\mathrm{Ad}^G(H)$-invariant, or simply $\mathrm{Ad}(H)$-invariant

Isotropy irreducible homogeneous space

In the case where the isotropy representation of a reductive homogeneous space G/H

$$\begin{array}{rcl} \mathrm{Ad}^{G/H} : H & \longrightarrow & \mathrm{Aut}(\mathfrak{m}) \\ h & \longmapsto & (d\tau_h)_o : \mathfrak{m} \to \mathfrak{m} \end{array}$$

is **irreducible**, then G/H admits a unique (up to scalar) G-invariant metric g, which is also Einstein $\rightarrow \mathrm{Ric}_g = \lambda \cdot g$.

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- These spaces have been studied in 1968 by J. Wolf.

Some examples of such spaces are the following:

- $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$
- $\mathrm{Spin}(7)/\mathrm{G}_2 \cong S^7$
- $\mathrm{G}_2/\mathrm{SU}(3) \cong S^6$
- $\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) \cong \mathbb{C}P^n$.

Isotropy reducible homogeneous space

In the case where the isotropy representation is a direct sum of irreducible representations $\varphi_i : H \to \mathrm{Aut}(\mathfrak{m}_i)$, $i = 1, 2, \ldots s$, that is

$$\mathrm{Ad}^{G/H} \cong \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_s \to \mathrm{Aut}(\mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s),$$

then we have the following two cases:

(A)

- The representations φ_i are **non equivalent**.

In 2004 Böhm-Wang-Ziller conjectured the following: Let G/H be a compact homogeneous space whose isotropy representation splits into a finite sum of non-equivalent and irreducible, subrepresentations. Then the number of G-invariant Einstein metrics on G/H is finite.

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(B)

- Some of the representations φ_i are **equivalent**, that is $\varphi_i \approx \varphi_j$ $(i \neq j)$.

Isotropy reducible homogeneous space, case (A)

When the representations φ_i are **non equivalent** then the decomposition of $\mathfrak{m}$

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s$$

is unique and $\mathfrak{m}_i, \mathfrak{m}_j$ $i \neq j$ are perpendicular.

- In this case all $\mathrm{Ad}(H)$- invariant inner products on $\mathfrak{m}$ are described as follows:

$$\langle \cdot, \cdot \rangle = x_1(-B)|_{\mathfrak{m}_1} + x_2(-B)|_{\mathfrak{m}_2} + \cdots + x_s(-B)|_{\mathfrak{m}_s} \; x_i \in \mathbb{R}^+, i = 1, 2, \ldots, s$$

- The matrix of the operator $A : \mathfrak{m} \to \mathfrak{m}$ with respect to $(-B)$-orthonormal basis is:

$$\begin{pmatrix} x_1 \mathrm{Id}_{\mathfrak{m}_1} & & 0 \\ & \ddots & \\ 0 & & x_s \mathrm{Id}_{\mathfrak{m}_s} \end{pmatrix}.$$

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The G-invariant metrics that correspond to these inner products are called diagonal.

Ricci tensor for diagonal metrics

Now for the Ricci tensor of diagonal G-invariant metrics we have the following:

We set $d_i := \dim \mathfrak{m}_i$ and let $\{e^i_\alpha\}_{\alpha=1}^{d_i}$ be a $(-B)$-orthonormal basis adapted to the above decomposition of $\mathfrak{m}$, i.e. $e^i_\alpha \in \mathfrak{m}_i$ $i = 1, 2, \ldots, s$.

Consider the numbers $A^\gamma_{\alpha\beta} = (-B)([e^i_\alpha, e^j_\beta], e^k_\gamma)$ such that

$$[e^i_\alpha, e^j_\beta] = \sum_\gamma A^\gamma_{\alpha\beta} e^k_\gamma$$

and set

$$A_{ijk} := \begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$$

where the sum taken over all three indices α, β, γ with $e^i_\alpha \in \mathfrak{m}_i$, $e^j_\beta \in \mathfrak{m}_j$, $e^k_\gamma \in \mathfrak{m}_k$.

The numbers A_{ijk} are non-negative, independent of the $(-B)$-orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j, \mathfrak{m}_k$, and are symmetric in all three indices:

$$A_{ijk} = A_{jik} = A_{kij}.$$

Ricci tensor for diagonal metrics

- The Ricci tensor $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$ of a G-invariant Riemannian metric on G/H has also a diagonal form, i.e. $\mathrm{Ric}_{\langle\cdot,\cdot\rangle} = \sum_{k=0}^{s} r_k x_k (-B)|_{\mathfrak{m}_k}$. We have the following proposition due to Park and Sakane (1997).

Proposition 2

The components $r_1, \dots, r_q$ of the Ricci tensor $\mathrm{Ric}_{\langle\cdot,\cdot\rangle}$ on G/H are given by

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{j,i}\frac{x_k}{x_j x_i}\begin{bmatrix}k\\ ji\end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{x_j}{x_k x_i}\begin{bmatrix}j\\ ki\end{bmatrix} \quad (k = 1, \dots, q), \quad (1)$$

where the sum is taken over $i, j = 1, \dots, q$. In particular for each k it holds that

$$\sum_{i,j}^{s}\begin{bmatrix}j\\ ki\end{bmatrix} = \sum_{i,j} A_{kij} = d_k := \dim \mathfrak{m}_k. \quad (2)$$

Isotropy reducible homogeneous space, case (B)

When some of the φ_i, φ_j in the isotropy representation of G/H are **equivalent**, then

- the diagonal G-nvariant metrics is not unique, and
- the submodules $\mathfrak{m}_i, \mathfrak{m}_j$ does not necessarily perpendicular.

In this case the matrix of the operator $(\cdot,\cdot) = \langle A\cdot,\cdot\rangle$ has some non zero non diagonal elements.

▶ **Also the Ricci tensor is not easy to describe**

Isotropy reducible homogeneous space, case (B)--Examples

- For the real Stiefel manifolds $V_k\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-k)$ the isotropy representation is given as follows:

$$\mathrm{Ad}^{\mathrm{SO}(n)}\big|_{\mathrm{SO}(n-k)} = \cdots = \underbrace{\wedge^2\lambda_{n-k}}_{\mathrm{Ad}^{\mathrm{SO}(n-k)}} \oplus \underbrace{1\oplus\cdots\oplus 1}_{\binom{k}{2}-\text{times}} \oplus \underbrace{\lambda_{n-k}\oplus\cdots\oplus\lambda_{n-k}}_{k-\text{times}}$$

Isotropy reducible homogeneous space, case (B)--Examples

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For $n = 4$ and $k = 2$ the matrix of the operator $A : \mathfrak{m} \to \mathfrak{m}$ has the following form:

$$\begin{pmatrix} x_0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & \lambda & 0 \\ 0 & 0 & x_1 & 0 & \lambda \\ 0 & \lambda & 0 & x_2 & 0 \\ 0 & 0 & \lambda & 0 & x_2 \end{pmatrix} \qquad \lambda\in\mathbb{R}, x_i\in\mathbb{R}^+ i = 0,1,2.$$

- For the quaternionic Stiefel manifolds $V_k\mathbb{H}^n$ the isotropy representation is given as follows:

$$\mathrm{Ad}^{Sp(n)}\otimes\mathbb{C}\big|_{\mathrm{Sp}(n-k)} \quad = \quad \ldots = \underbrace{S^2\nu_{n-k}}_{\mathrm{Ad}^{\mathrm{Sp}(n-k)}} \oplus \underbrace{1\oplus\cdots\oplus 1}_{\binom{2+2k-1}{2}-\text{times}} \oplus \underbrace{\nu_{n-k}\oplus\cdots\oplus\nu_{n-k}}_{2k-\text{times}}.$$

Some history

- Kobayashi (1963): Proved the existence of an $\mathrm{SO}(n)$-invariant Einstein metric on the unit tangent bundle $T_1S^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-2)$.
- Sagle (1970) - Jensen (1973): Proved the existence of $\mathrm{SO}(n)$-invariant Einstein metrics on the Stiefel manifolds $V_k\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-k)$, for $k \geq 3$

$$\text{metrics of the form: } \leftrightarrow \langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & a & 1 \\ a & a & 1 \\ 1 & 1 & * \end{pmatrix}.$$

- Back - Hsiang (1987) and Kerr (1998): Proved that for $n \geq 5$ the Stiefel manifolds $V_2\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-2)$ admit exactly one (diagonal) $\mathrm{SO}(n)$-invariant Einstein metric.
- Arvanitoyeorgos-Dzhepko-Nikonorov (2009): Showed that for $s > 1$ and $l > k > 3$ the Stiefel manifolds $V_{sk}\mathbb{F}^{sk+l} \cong G(sk+l)/G(l)$ admit at least four $G(sk+l)$-invariant Einstein metrics which are also $\mathrm{Ad}\,(G(k)^s \times G(l))$-invariant (two of these are Jensen's metrics) where $G(\ell) \in \{\mathrm{SO}(\ell), \mathrm{Sp}(\ell)\}$.

$$\text{metrics of the form: } \leftrightarrow \langle \cdot, \cdot \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & * \end{pmatrix}.$$

General construction

As seen before, the G-invariant metrics $\mathcal{M}^G$ on $G/H \cong V_k\mathbb{F}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$ are not only diagonal. For this reason the complete description of G-invariant Einstein metrics is difficult, because the Ricci tensor is not easy to describe. So we search for a subset of these metrics which are diagonal.

General construction

Let G/H a homogeneous spaces with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We consider the operator

$$\mathrm{Ad}(n) : \mathfrak{g} \to \mathfrak{g}$$

where $n \in N_G(H) = \{g \in G : gHg^{-1} = H\}$. Then

Proposition 3

The operator $\mathrm{Ad}(n)|_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{g}$ takes values in $\mathfrak{m}$, that is $\varphi = \mathrm{Ad}(n)|_{\mathfrak{m}} \in \mathrm{Aut}(\mathfrak{m})$. Also, $(\mathrm{Ad}(n)|_{\mathfrak{m}})^{-1} = (\mathrm{Ad}(n)|_{\mathfrak{m}})^t$.

General construction

We define the isometric action

$$\Phi \times \mathcal{M}^G \to \mathcal{M}^G, \quad (\varphi, A) \mapsto \varphi \circ A \circ \varphi^{-1} \equiv \tilde{A},$$

where Φ is the set $\{\varphi = \mathrm{Ad}(n)|_{\mathfrak{m}} : n \in N_G(H)\} \subset \mathrm{Aut}(\mathfrak{m})$.

Proposition 4

The action of Φ on $\mathcal{M}^G$ is well defined, i.e. $\tilde{A}$ is $\mathrm{Ad}(H)$-equivariant, symmetric and positive definite.

Remark: Metrics corresponding to the operator A are equivalent, up to automorphism $\mathrm{Ad}(n) : \mathfrak{m} \to \mathfrak{m}$, to the metrics corresponding to the operator $\tilde{A}$.

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From the above action we consider the set of all fixed points (subset of $\mathcal{M}^G$):

$$(\mathcal{M}^G)^\Phi = \{A \in \mathcal{M}^G \ : \ \varphi \circ A \circ \varphi^{-1} = A \text{ for all } \varphi \in \Phi\}$$

- Any element of $(\mathcal{M}^G)^\Phi$ parametrizes all $\mathrm{Ad}(N_G(H))$-invariant inner products of $\mathfrak{m}$ and thus it defines a subset of all inner products on $\mathfrak{m}$.

General construction

- Since $H \subset N_G(H)$ we have:

Proposition 5

Let G/H be a homogeneous space. Then there exists a one-to-one correspondence between:

(1) G-invariant metrics on G/H,

(2) $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$,

(3) Fixed points

$$(\mathcal{M}^G)^{\Phi_H} = \{A \in \mathcal{M}^G : \psi \circ A \circ \psi^{-1} = A, \text{ for all } \psi \in \Phi_H\}$$

of the action $\Phi_H = \{\phi = \mathrm{Ad}(h)|_{\mathfrak{m}} : h \in H\} \subset \Phi$ on $\mathcal{M}^G$.

- $(\mathcal{M}^G)^{\Phi} \subset (\mathcal{M}^G)^{\Phi_H}$.

K closed subgroup of G

- We work with some closed subgroup K of G such that

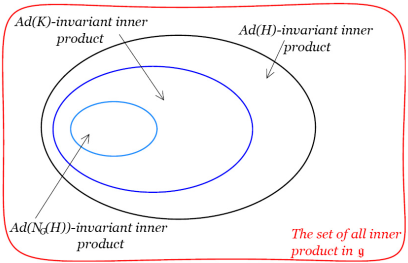
$$H \subset K \subset N_G(H) \subset G.$$

Then the fixed point set of the non trivial action of

$$\Phi_K = \{\varphi = \mathrm{Ad}(k)|_{\mathfrak{m}} : k \in K\} \subset \Phi \text{ on } \mathcal{M}^G \text{ is}$$

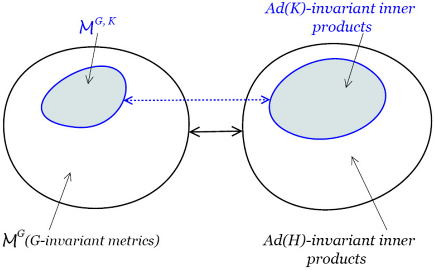
$$(\mathcal{M}^G)^{\Phi_K} = \{A \in \mathcal{M}^G \ : \ \varphi \circ A \circ \varphi^{-1} = A \text{ for all } \varphi \in \Phi_K\},$$

and this set determines a subset of all $\mathrm{Ad}(K)$-invariant inner products of $\mathfrak{m}$.
We have the inclusions $(\mathcal{M}^G)^{\Phi} \subset (\mathcal{M}^G)^{\Phi_K} \subset (\mathcal{M}^G)^{\Phi_H}$.



K closed subgroup of G

By Proposition 5 the subset $(\mathcal{M}^G)^{\Phi_K}$ is in one-to-one correspondence with a subset $\mathcal{M}^{G,K}$ of all G-invariant metrics, call it $\mathrm{Ad}(K)$-invariant, as shown in the following figure:



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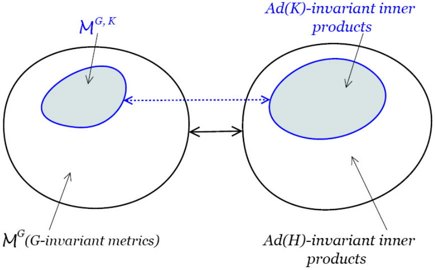

Proposition 6

Let K be a subgroup of G with $H \subset K \subset G$ and such that $K = L \times H$, for some subgroup L of G. Then K is contained in $N_G(H)$.

K closed subgroup of G

- We apply the previous proposition for the Stiefel manifolds

$$V_{k_1+k_2}\mathbb{F}^{k_1+k_2+k_3} \cong G_{k_1+k_2+k_3}/G_3,$$

$G_{k_1+k_2+k_3} \in \{\mathrm{SO}(k_1+k_2+k_3), \mathrm{Sp}(k_1+k_2+k_3)\}$, $G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$ ($i = 1, 2, 3$) and $\mathbb{F} \in \{\mathbb{R}, \mathbb{H}\}$, where we take the following two cases for the subgroup $K = L \times G_3$:

(A) $K = \big(G_1 \times G_2\big) \times G_3$, and search for

$$\mathrm{Ad}(K) \equiv \mathrm{Ad}\left(\big(G_1 \times G_2\big) \times G_3\right) \text{-invariant metrics.}$$

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(B) $K = \mathrm{U}(k_1+k_2) \times \mathrm{Sp}(k_3)$, and search for

$$\mathrm{Ad}(K) \equiv \mathrm{Ad}(\mathrm{U}(k_1+k_2) \times \mathrm{Sp}(k_3))\text{-invariant metrics.}$$

The benefit for such metrics is that they are diagonal metrics on the homogeneous space.

We study the case (A)

$$K=(G_1\times G_2)\times G_3 \text{ where } G_i\in\{\mathrm{SO}(k_i),\mathrm{Sp}(k_i)\},\ i=1,2,3$$

that is

$$K=\mathrm{SO}(k_1)\times \mathrm{SO}(k_2)\times \mathrm{SO}(k_3)\longrightarrow V_{k_1+k_2}\mathbb{R}^n$$

$$K=\mathrm{Sp}(k_1)\times \mathrm{Sp}(k_2)\times \mathrm{Sp}(k_3)\longrightarrow V_{k_1+k_2}\mathbb{H}^n$$

$K = (G_1 \times G_2) \times G_3, \ \ G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$

We view the Stiefel manifold $V_{k_1+k_2}\mathbb{F}^n$, where $n = k_1 + k_2 + k_3$ as total space over the generalized Wallach space, i.e:

$$\frac{G_1 \times G_2 \times G_3}{G_3} \longrightarrow \frac{G_n}{G_3} \longrightarrow \frac{G_n}{G_1 \times G_2 \times G_3}$$

▶ The tangent space $\mathfrak{p}$ of the generalized Wallach space has three non equivalent $\mathrm{Ad}(K)$-invariant, irreducible isotropy summands, that is

$$\mathfrak{p} = \mathfrak{p}_{12} \oplus \mathfrak{p}_{13} \oplus \mathfrak{p}_{23},$$

and the tangent space of the fiber is the Lie algebra

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{where} \quad \mathfrak{g}_i \in \{\mathfrak{so}(k_i), \mathfrak{sp}(k_i)\}, \ i = 1, 2.$$

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- Therefore, the tangent space $\mathfrak{m}$ of the total space can be written as a direct sum of five **non equivalent** $\mathrm{Ad}(K)$-invariant, irreducible components:

$$\begin{aligned} \mathfrak{m} &= \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{p}_{12} \oplus \mathfrak{p}_{13} \oplus \mathfrak{p}_{23} \\ &= \begin{pmatrix} \mathfrak{g}_1 & \mathfrak{p}_{12} & \mathfrak{p}_{13} \\ -{}^t\mathfrak{p}_{12} & \mathfrak{g}_2 & \mathfrak{p}_{23} \\ -{}^t\mathfrak{p}_{13} & -{}^t\mathfrak{p}_{23} & 0 \end{pmatrix} \end{aligned}$$

$K = (G_1 \times G_2) \times G_3, \quad G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$

From the previous decomposition any $\mathrm{Ad}(K)$-invariant metric is diagonal and is determined by $\mathrm{Ad}(K)$-invariant inner products of the form:

$$\begin{aligned} \langle \cdot, \cdot \rangle &= x_1 \, (-B)|_{\mathfrak{g}_1} + x_2 \, (-B)|_{\mathfrak{g}_2} \\ &\quad + x_{12} \, (-B)|_{\mathfrak{p}_{12}} + x_{13} \, (-B)|_{\mathfrak{p}_{13}} + x_{23} \, (-B)|_{\mathfrak{p}_{23}} \end{aligned} \tag{3}$$

$$\leftrightarrow$$

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} x_1 & x_{12} & x_{13} \\ x_{12} & x_2 & x_{23} \\ x_{13} & x_{23} & * \end{pmatrix}.$$ Here $k_1 \geq 2, k_2 \geq 2$ and $k_3 \geq 1.$

In the case where we have $k_1 = 1$, then for the real Stiefel manifold $V_{1+k_2}\mathbb{R}^{1+k_2+k_3}$ the above inner products take the form

$$\langle \cdot, \cdot \rangle = x_2 \, (-B)|_{\mathfrak{so}(k_2)} + x_{12} \, (-B)|_{\mathfrak{m}_{12}} + x_{13} \, (-B)|_{\mathfrak{m}_{13}} + x_{23} \, (-B)|_{\mathfrak{m}_{23}} \tag{4}$$

$$\leftrightarrow$$

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_2 & x_{23} \\ x_{13} & x_{23} & * \end{pmatrix}.$$ Here $k_1 = 1, k_2 \geq 2$ and $k_3 \geq 1.$

$K = (G_1 \times G_2) \times G_3, \ \ G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$

We need to determine the Ricci components r_1, r_2, r_{ij} ($1 \leq i < j \leq 3$ for the metric that correspond to the inner products (3) and (4). We first need to identify the non zero numbers $A_{ijk} := \begin{bmatrix} k \\ ij \end{bmatrix}$. From some Lie bracket relations of $\mathfrak{g}_i$ and $\mathfrak{p}_{ij}$ we have:

$A_{111}, \ \ A_{222}, \ \ A_{1(12)(12)}, \ \ A_{1(13)(13)}, \ \ A_{2(12)(12)}, \ \ A_{2(23)(23)}, \ \ A_{(12)(23)(13)}.$

From the Lemma below (due to Arvanitoyeorgos, Dzhepko and Nikonorov) we have,

Lemma 5

For $a, b, c = 1, 2, 3$ and $(a-b)(b-c)(c-a) \neq 0$ the following relations hold:

real case	quaternionic case
$A_{aaa} = \frac{k_a(k_a-1)(k_a-2)}{2(n-2)}$	$A_{aaa} = \frac{k_a(k_a+1)(2k_a+1)}{n+1}$
$A_{(ab)(ab)a} = \frac{k_a k_b(k_a-1)}{2(n-2)}$	$A_{(ab)(ab)a} = \frac{k_a k_b(2k_a+1)}{(n+1)}$
$A_{(ab)(bc)(ac)} = \frac{k_a k_b k_c}{2(n-2)}$	$A_{(ab)(bc)(ac)} = \frac{2k_a k_b k_c}{n+1}.$

$K = (G_1 \times G_2) \times G_3, \quad G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$

Lemma 6

The components of the Ricci tensor for the $\mathrm{Ad}(K)$-invariant metric determined by (3) for the **real case** are given as follows:

$$
\begin{aligned}
r_1 &= \frac{k_1 - 2}{4(n-2)x_1} + \frac{1}{4(n-2)}\left(k_2 \frac{x_1}{{x_{12}}^2} + k_3 \frac{x_1}{{x_{13}}^2}\right),\\
r_2 &= \frac{k_2 - 2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(k_1 \frac{x_2}{{x_{12}}^2} + k_3 \frac{x_2}{{x_{23}}^2}\right),\\
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&\quad - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{12}}^2} + (k_2-1)\frac{x_2}{{x_{12}}^2}\right),\\
r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_1-1)\frac{x_1}{{x_{13}}^2}\right)\\
r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2}\right)
\end{aligned}
$$

where $n = k_1 + k_2 + k_3$.

$K = (G_1 \times G_2) \times G_3, \ \ G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$

Lemma 7

The components of the Ricci tensor for the $\mathrm{Ad}(K)$-invariant metric determined by (3) for the **quaternionic case** are given as follows:

$$\begin{aligned}
r_1 &= \frac{k_1+1}{4(n+1)x_1} + \frac{k_2}{4(n+1)}\frac{x_1}{{x_{12}}^2} + \frac{k_3}{4(n+1)}\frac{x_1}{{x_{13}}^2},\\
r_2 &= \frac{k_2+1}{4(n+1)x_2} + \frac{k_1}{4(n+1)}\frac{x_2}{{x_{12}}^2} + \frac{k_3}{4(n+1)}\frac{x_2}{{x_{23}}^2},\\
r_{12} &= \frac{1}{2x_{12}} + \frac{k_3}{4(n+1)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right)\\
&- \frac{2k_1+1}{8(n+1)}\frac{x_1}{{x_{12}}^2} - \frac{2k_2+1}{8(n+1)}\frac{x_2}{{x_{12}}^2},\\
r_{13} &= \frac{1}{2x_{13}} + \frac{k_2}{4(n+1)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{2k_1+1}{8(n+1)}\frac{x_1}{{x_{13}}^2}\\
r_{23} &= \frac{1}{2x_{23}} + \frac{k_1}{4(n+1)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{2k_2+1}{8(n+1)}\frac{x_2}{{x_{23}}^2}.
\end{aligned}$$

$K = (G_1 \times G_2) \times G_3, \;\; G_i \in \{\mathrm{SO}(k_i), \mathrm{Sp}(k_i)\}$

Lemma 8

The components of the Ricci tensor for the $\mathrm{Ad}(K)$-invariant metric determined by (4) **(real case only)**, are given as follows:

$$r_2 = \frac{k_2 - 2}{4(n-2)x_2} + \frac{1}{4(n-2)}\left(\frac{x_2}{{x_{12}}^2} + k_3\frac{x_2}{{x_{23}}^2}\right),$$

$$r_{12} = \frac{1}{2x_{12}} + \frac{k_3}{4(n-2)}\left(\frac{x_{12}}{x_{13}x_{23}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{12}}^2}\right),$$

$$r_{23} = \frac{1}{2x_{23}} + \frac{1}{4(n-2)}\left(\frac{x_{23}}{x_{13}x_{12}} - \frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{23}x_{13}}\right) - \frac{1}{4(n-2)}\left((k_2-1)\frac{x_2}{{x_{23}}^2}\right),$$

$$r_{13} = \frac{1}{2x_{13}} + \frac{k_2}{4(n-2)}\left(\frac{x_{13}}{x_{12}x_{23}} - \frac{x_{12}}{x_{13}x_{23}} - \frac{x_{23}}{x_{12}x_{13}}\right),$$

where $n = 1 + k_2 + k_3$.

Einstein metrics on $V_{1+k_2}\mathbb{R}^n$

- For the Stiefel manifolds $V_4\mathbb{R}^n \cong \mathrm{SO}(n)/\mathrm{SO}(n-4)$, where $k_2 = 3$ and $k_3 = n-4$, the

$$\mathrm{Ad}(\mathrm{SO}(3)\times\mathrm{SO}(n-4))\text{-invariant Einstein metrics}$$

are the solutions of the system

$$r_2 = r_{12}, \quad r_{12} = r_{13}, \quad r_{13} = r_{23},$$

and we set $x_{23} = 1$. Then we have

$$\begin{aligned}
f_1 &= -(n-4){x_{12}}^3 x_2 + (n-4){x_{12}}^2 x_{13} x_2^2 + (n-4) x_{12} {x_{13}}^2 x_2 \\
&\quad -2(n-2)x_{12}x_{13}x_2 + (n-4)x_{12}x_2 + {x_{12}}^2 x_{13} + 3x_{13}x_2^2 = 0,\\
f_2 &= (n-3){x_{12}}^3 - 2(n-2){x_{12}}^2 x_{13} - (n-5)x_{12}{x_{13}}^2 \\
&\quad +2(n-2)x_{12}x_{13} + (3-n)x_{12} + 2{x_{12}}^2 x_{13}x_2 - 2x_{13}x_2 = 0,\\
f_3 &= (n-2)x_{12}x_{13} - (n-2)x_{12} + {x_{12}}^2 - x_{12}x_{13}x_2 \\
&\quad -2{x_{13}}^2 + 2 = 0.
\end{aligned} \tag{5}$$

We take a Gröbner basis for the ideal I of the polynomial ring $R = \mathbb{Q}[z, x_2, x_{12}, x_{13}]$ which is generated by $\{f_1,\ f_2,\ f_3,\ z\,x_2\,x_{12}\,x_{13} - 1\}$, to find non zero solutions of the above system.

Einstein metrics on Real Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^n$

By the aid of computer programs for symbolic computations we obtain the following results:

Theorem 1 (A. Arvanitoyeorgos-Y. Sakane-M.S.)

The Stiefel manifolds $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$ $(n \geq 6)$ admit at least four invariant Einstein metrics. Two of them are Jensen's metrics and the other two are given by the $\mathrm{Ad}(\mathrm{SO}(3)\times \mathrm{SO}(n-4))$-invariant inner products of the form (4).

In the same way, for the Stiefel manifolds $V_5\mathbb{R}^7$, we consider the cases

$$k_1 = 2, k_2 = 3, k_3 = 2 \qquad k_1 = 1, k_2 = 4, k_3 = 2$$

Then we have:

Theorem 2 (A. Arvanitoyeorgos-Y. Sakane-M.S.)

The Stiefel manifold $V_5\mathbb{R}^7 = \mathrm{SO}(7)/\mathrm{SO}(2)$ admits at least six invariant Einstein metrics. Two of them are Jensen's metrics, the other two are given by $\mathrm{Ad}(\mathrm{SO}(2)\times \mathrm{SO}(3)\times \mathrm{SO}(2))$-invariant inner products of the form (3), and the rest two are given by $\mathrm{Ad}(\mathrm{SO}(4)\times \mathrm{SO}(2))$-invariant inner products of the form (4).

Einstein metrics on quaternionic Stiefel manifolds $V_{k_1+k_2}\mathbb{H}^n$

For the quaternionic Stiefel manifolds we solve the system $r_1 = r_2,\ r_2 = r_{12},\ r_{12} = r_{13},\ r_{13} = r_{23}$ and we obtain the following results:

- For the case $k_1 = 1,\ k_2 = 1,\ k_3 = 1$ the $\mathrm{Ad}(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))$-invariant Einstein metrics on $V_2\mathbb{H}^3$ are

$$
\begin{aligned}
(x_1, x_2, x_{12}, x_{13}, x_{23}) &\approx (0.276281,\ 0.251266,\ 0.460887,\ 0.568722,\ 1)\\
&\approx (1.112249,\ 0.417937,\ 1.598741,\ 0.595776,\ 1)\\
&\approx (0.701500,\ 1.866891,\ 2.683459,\ 1.678482,\ 1)\\
&\approx (0.441809,\ 0.485793,\ 0.810389,\ 1.758325,\ 1).
\end{aligned}
$$

Two are Jensen's metrics:

$$
\begin{aligned}
(x_1, x_2, x_{12}, x_{13}, x_{23}) &\approx (0.472797,\ 047.2797,\ 0.472797,\ 1,\ 1)\\
&\approx (1.812916,\ 1.812916,\ 1.812916,\ 1,\ 1),
\end{aligned}
$$

and the other two are Arvanitoyeorgos-Dzhepko-Nikonorov metrics:

$$
\begin{aligned}
(x_1, x_2, x_{12}, x_{13}, x_{23}) &\approx (0.3448897,\ 0.3448897,\ 0.800199,\ 1,\ 1)\\
&\approx (0.483972,\ 0.483972,\ 2.585187,\ 1,\ 1).
\end{aligned}
$$

Einstein metrics on Quaternionic Stiefel manifolds $V_{k_1+k_2}\mathbb{H}^n$

- In the same way for $k_1 = n-2,\ k_2 = 1,\ k_3 = 1$ the $\mathrm{Ad}(\mathrm{Sp}(n-2) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1))$-invariant Einstein metrics on $V_{n-1}\mathbb{H}^n$ are
  1. $3 < n < 8$ there are 8 metrics, 2 of Jensen's metrics and 6 are new.
  2. $7 < n < 30$ there are 10 metrics, 2 of Jensen's and 8 are new.
  3. $n > 29$ there are 12 metrics, 2 Jensen's and the rest 10 are new.

Einstein metrics on Quaternionic Stiefel manifolds $V_{k_1+k_2}\mathbb{H}^n$

- In the same way for $k_1 = n-2,\ k_2 = 1,\ k_3 = 1$ the $\mathrm{Ad}(\mathrm{Sp}(n-2)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))$-invariant Einstein metrics on $V_{n-1}\mathbb{H}^n$ are
  1. $3 < n < 8$ there are 8 metrics, 2 of Jensen's metrics and 6 are new.
  2. $7 < n < 30$ there are 10 metrics, 2 of Jensen's and 8 are new.
  3. $n > 29$ there are 12 metrics, 2 Jensen's and the rest 10 are new.

▶ In case where $k_1 = n-3,\ k_2 = 1,\ k_3 = 2$ the $\mathrm{Ad}(\mathrm{Sp}(n-3)\times\mathrm{Sp}(1)\times\mathrm{Sp}(2))$-invariant Einstein metrics on $V_{n-2}\mathbb{H}^n$ are

1. $n = 4$ there are 8 metrics, 2 Jensen's, two Nikonorov-Arvanitoyeorgos-Dzhepko and 4 are new.
2. $4 < n < 10$ there are 8 metrics, 2 Jensen's and 6 new.
3. $n = 10$ there are 10 metrics, 2 Jensen's and 8 new.
4. $11 < n < 28$ there are 8 metrics, 2 Jensen's and 6 new.
5. $27 < n < 41$ there are 10 metrics, 2 Jensen's and 8 new.
6. $n > 40$ there are 12 metrics, 2 Jensen's and 10 are new.

We now study the case (B)

$$\boldsymbol{K = \mathrm{U}(k_1 + k_2) \times \mathrm{Sp}(k_3)}$$

for the quaternionic Stiefel manifolds $V_{k_1+k_2}\mathbb{H}^n$, where $n = k_1 + k_2 + k_3$.

We set $p = k_1 + k_2$, so $k_3 = n - p$.

$K = \mathrm{U}(p) \times \mathrm{Sp}(n-p)$

In this case we view the Stiefel manifold $V_p\mathbb{H}^n$, where $n = k_1 + k_2 + k_3$, as a total space over the flag manifold with two isotropy summands i.e:

$$\frac{\mathrm{U}(p) \times \mathrm{Sp}(n-p)}{\mathrm{Sp}(n-p)} \longrightarrow \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-p)} \longrightarrow \frac{\mathrm{Sp}(n)}{\mathrm{U}(p) \times \mathrm{Sp}(n-p)}$$

▶ The tangent space $\mathfrak{m}$ of the base space is written as a direct sum of two non equivalent $\mathrm{Ad}(K)$-invariant irreducible isotropy summands $\mathfrak{m}_1$, $\mathfrak{m}_2$ of dimension $d_2 = \dim(\mathfrak{m}_1) = 4p(n-p)$ and $d_3 = \dim(\mathfrak{m}_2) = p(p+1)$.

Also, the tanent space of the fiber $\mathrm{U}(p) \cong \mathrm{U}(1) \times \mathrm{SU}(p)$ is the Lie algebra $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ where $\mathfrak{h}_0$ is the center of $\mathfrak{u}(p)$ and $\mathfrak{h}_1 = \mathfrak{su}(p)$, with $d_0 = \dim(\mathfrak{h}_0) = 1$ and $d_1 = \dim(\mathfrak{h}_1) = p^2 - 1$.

▶ Therefore the tangent space $\mathfrak{p}$ of Stiefel manifold can be written as direct sum of four non equivalent $\mathrm{Ad}(K)$-invariant irreducible submodules:

$$\mathfrak{p} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2.$$

$K = \mathrm{U}(p) \times \mathrm{Sp}(n-p)$

The diagonal $\mathrm{Ad}(K)$-invariant metrics on $V_p\mathbb{H}^n$ are determined by the following $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{p}$

$$\langle\cdot,\cdot\rangle = u_0(-B)|_{\mathfrak{h}_0} + u_1(-B)|_{\mathfrak{h}_1} + x_1(-B)|_{\mathfrak{m}_1} + x_2(-B)|_{\mathfrak{m}_2}. \qquad (6)$$

We know that $[\mathfrak{m}_1,\mathfrak{m}_1] \subset \mathfrak{h} \oplus \mathfrak{m}_2$, $[\mathfrak{m}_2,\mathfrak{m}_2] \subset \mathfrak{h}$, $[\mathfrak{m}_1,\mathfrak{m}_2] \subset \mathfrak{m}_1$, hence the only non zero numbers $A_{ijk} = \begin{bmatrix} k \\ ij \end{bmatrix}$ are

$$A_{220},\ A_{330},\ A_{111},\ A_{122},\ A_{133},\ A_{322}.$$

From Arvanitoyeorgos-Mori-Sakane we obtain the following:

Lemma 9

For the metric $\langle\cdot,\cdot\rangle$ on $\mathrm{Sp}(n)/\mathrm{Sp}(n-p)$, the non-zero numbers A_{ijk} are given as follows:

$$A_{220} = \frac{d_2}{d_2+4d_3} \quad A_{330} = \frac{4d_3}{d_2+4d_3} \quad A_{111} = \frac{2d_3(2d_1+2-d_3)}{d_2+4d_3}$$
$$A_{122} = \frac{d_1 d_2}{d_2+4d_3} \quad A_{133} = \frac{2d_3(d_3-2)}{d_2+4d_3} \quad A_{322} = \frac{d_2 d_3}{d_2+4d_3}$$

$K = \mathrm{U}(p) \times \mathrm{Sp}(n-p)$

Lemma 10

The components of the Ricci tensor for the $\mathrm{Ad}(K)$-invariant metric determined by (6) are given as follows:

$$r_0 = \frac{u_0}{4x_1^2}\frac{d_2}{(d_2+4d_3)} + \frac{u_0}{4x_2^2}\frac{4d_3}{(d_2+4d_3)}$$

$$r_1 = \frac{1}{4d_1u_1}\frac{2d_3(2d_1+2-d_3)}{(d_2+4d_3)} + \frac{u_1}{4x_1^2}\frac{d_2}{(d_2+4d_3)} + \frac{u_1}{2d_1x_2^2}\frac{d_3(d_3-2)}{(d_2+4d_3)}$$

$$r_2 = \frac{1}{2x_1} - \frac{x_2}{2x_1^2}\frac{d_3}{(d_2+4d_3)} - \frac{1}{2x_1^2}\left(u_0\frac{1}{(d_2+4d_3)} + u_1\frac{d_1}{(d_2+4d_3)}\right)$$

$$r_3 = \frac{1}{x_2}\left(\frac{1}{2} - \frac{1}{2}\frac{d_2}{(d_2+4d_3)}\right) + \frac{x_2}{4x_1^2}\frac{d_2}{(d_2+4d_3)} - \frac{1}{x_2^2}\left(u_0\frac{2}{(d_2+4d_3)} + u_1\frac{d_3-2}{(d_2+4d_3)}\right)$$

where $d_1 = p^2-1$, $d_2 = 4p(n-p)$, $d_3 = p(p+1)$.

Next, we solve the Einstein equation for the Stiefel manifold $V_2\mathbb{H}^n$. In this case we have $d_0 = 1$, $d_1 = 3$, $d_2 = 8(n-2)$, $d_3 = 6$.

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$

Theorem 3 (A. Arvanitoyeorgos-Y. Sakane-M.S.)

The Stiefel manifold $V_2\mathbb{H}^n \cong \mathrm{Sp}(n)/\mathrm{Sp}(n-2)$ admits four invariant Einstein metrics. Two of them are Jensen's metrics and the other two are given by the $\mathrm{Ad}(\mathrm{U}(2) \times \mathrm{Sp}(n-2))$-invariant inner products of the form (6).

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$

Theorem 3 (A. Arvanitoyeorgos-Y. Sakane-M.S.)

The Stiefel manifold $V_2\mathbb{H}^n \cong \mathrm{Sp}(n)/\mathrm{Sp}(n-2)$ admits four invariant Einstein metrics. Two of them are Jensen's metrics and the other two are given by the $\mathrm{Ad}(\mathrm{U}(2) \times \mathrm{Sp}(n-2))$-invariant inner products of the form (6).

Proof

We consider the system of equation

$$r_0 = r_1, \quad r_1 = r_2, \quad r_2 = r_3. \tag{7}$$

We set $x_2 = 1$ and then system (7) reduces to

$$\begin{aligned} f_1 &= 2nu_0u_1 - 2nu_1^2 + 6u_0u_1x_1^2 - 4u_0u_1 - 4u_1^2x_1^2 + 4u_1^2 - 2x_1^2 = 0 \\ f_2 &= 4nu_1^2 - 8nu_1x_1 + u_0u_1 + 8u_1^2x_1^2 - 5u_1^2 - 8u_1x_1 + 6u_1 + 4x_1^2 = 0 \\ f_3 &= 8nx_1 - 4n + 4u_0x_1^2 - u_0 + 8u_1x_1^2 - 3u_1 - 24x_1^2 + 8x_1 + 2 = 0. \end{aligned} \tag{8}$$

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$

We consider a polynomial ring $R = \mathbb{Q}[z, u_0, u_1, x_1]$ and an ideal I generated by $\{f_1,\ f_2,\ f_3,\ z\, u_0\, u_1\, x_1\ - 1\}$ to find non zero solutions for the system (8). We take a lexicographic order $>$ with $z > u_0 > x_1 > u_1$ for a monomial ordering on R. Then, the Gröbner basis for the ideal I contains the polynomial $(u_1 - 1)U_1(u_1)$ where U_1 is a given by:

$$U_1(u_1) = (4n-1)^4 {u_1}^8 - 2(4n-55)(4n-1)^3 {u_1}^7 \\ +(4n-1)^2(512n^3 - 48n^2 - 2040n + 2903){u_1}^6 - 4(4n-1)(288n^4 \\ -3224n^3 + 216n^2 + 10419n - 6076){u_1}^5 + (14336n^6 - 5120n^5 \\ -103168n^4 + 78208n^3 + 104608n^2 - 104280n + 30583){u_1}^4 \\ -2(2048n^6 - 1536n^5 + 3840n^4 - 11408n^3 - 28320n^2 \\ +59088n - 22489){u_1}^3 + (2048n^5 + 832n^4 - 10848n^3 + 17924n^2 \\ -23472n + 13237){u_1}^2 - 4(n-1)(64n^4 - 96n^3 + 336n^2 \\ -374n + 205)u_1 + 4(n-1)^2(4n-1)^2$$

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$ $\quad - - -(u_1 - 1)U_1(u_1) - - -$

Case A: $u_1 \neq 1$

We prove that the equation $U_1(u_1) = 0$ has two positive solutions. Observe that

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$ $\qquad ---(u_1 - 1)U_1(u_1)---$

Case A: $u_1 \neq 1$

We prove that the equation $U_1(u_1) = 0$ has two positive solutions. Observe that

- For $u_1 = 0$

$$U_1(0) = 68112 - 133344n + 73744n^2 + 47360n^3 - 61696n^4 + 3328n^5 + 10240n^6$$ is positive for all $n \geq 3$,

- For $u_1 = 1/5$

$$U_1(1/5) = 1098.64 - 2511.49n + 1988.33n^2 - 639.029n^3 + 15.3295n^4 + 46.1537n^5 - 9.8304n^6$$ is negative for $n \geq 3$,

so we have one solution $u_1 = \alpha_1$ between $0 < \alpha_1 < 1/5$.

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$ $\qquad - - -(u_1 - 1)U_1(u_1) - - -$

Case A: $u_1 \neq 1$

We prove that the equation $U_1(u_1) = 0$ has two positive solutions. Observe that

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- For $u_1 = 1/5$

$$U_1(1/5) = 1098.64 - 2511.49n + 1988.33n^2 - 639.029n^3$$
$$+15.3295n^4 + 46.1537n^5 - 9.8304n^6$$ is negative for $n \geq 3$,

so we have one solution $u_1 = \alpha_1$ between $0 < \alpha_1 < 1/5$.

- For $u_1 = 1$

$$U_1(1) = 68112 - 133344n + 73744n^2 + 47360n^3 - 61696n^4$$
$$+3328n^5 + 10240n^6$$ is always positive for $n \geq 3$,

hence we have a second solution $u_1 = \beta_1$ between $1/5 < \beta_1 < 1$.

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2)$ $\qquad ---(u_1-1)U_1(u_1)---$

Next, we consider the ideal J generated by the polynomials

$$\{f_1,\, f_2,\, f_3,\, z\, u_0\, u_1\, x_1\, (u_1 - 1) - 1\}.$$

We take the lexigographic orders $>$ with

1. $z > u_0 > x_1 > u_1$. Then the Gröbner basis of J contains the polynomial $U_1(u_1)$ and the polynomial

$$a_1(n)\, x_1 + W_1(u_1, n)$$

2. $z > x_1 > u_0 > u_1$. Then the Gröbner basis of J contains the polynomial $U_1(u_1)$ and the polynomial

$$a_2(n)\, u_0 + W_2(u_1, n)$$

where $a_i(n)$ $i = 1, 2$ is a polynomial of n of degree 17 for $i = 1$, and of degree 16 for $i = 2$. For $n \geq 3$ the polynomial $a_i(n)$ $i = 1, 2$ is positive. Thus for positive values $u_1 = \alpha_1, \beta_1$ found above we obtain real values $x_1 = \gamma_1, \gamma_2$ and $u_0 = \alpha_0, \beta_0$ as solutions of system (8).

$K = \mathrm{U}(2)\times \mathrm{Sp}(n-2)$ $\qquad ---(u_1-1)U_1(u_1)---$

Now we prove that the solutions $x_1 = \gamma_1, \gamma_2$ and $u_0 = \alpha_0, \beta_0$ are positive.
We consider the ideal J with the lexicographic order $>$ with

1. $z > u_0 > u_1 > x_1$ then the Gröbner basis of J contains the $U_1(u_1)$ and the polynomial
$$X_1(x_1) = \sum_{k=0}^{8} b_k(n)x_1^k$$
2. $z > x_1 > u_1 > u_0$ then the Gröbner basis of J contains the $U_1(u_1)$ and the polynomial
$$U_0(u_0) = \sum_{k=0}^{8} c_k(n)u_0^k$$

for $n \geq 3$ the coefficients of the polynomials $b_k(n),\ \ c_k(n)$ are positive when the k is even degree and negative for odd degree. Thus if the equations $X_1(x_1) = 0$ and $U_0(u_0) = 0$ has real solutions, then these are all positive. So the solutions $x_1 = \gamma_1, \gamma_2$ and $u_0 = \alpha_0, \beta_0$ are positive.

$K = \mathrm{U}(2) \times \mathrm{Sp}(n-2) \qquad - - -(u_1 - 1)U_1(u_1) - - -$

Case B: $u_1 = 1$

Then from the system (8) we get the solutions:

$$\{u_0 = 1,\, u_1 = 1,\, x_1 = \frac{2 + 2n - \sqrt{-2 - 4n + 4n^2}}{6},\, x_2 = 1\}$$

and

$$\{u_0 = 1,\, u_1 = 1,\, x_1 = \frac{2 + 2n + \sqrt{-2 - 4n + 4n^2}}{6},\, x_2 = 1\}$$

which are Jensen′s metrics.

So the new Einstein metrics on $V_2\mathbb{H}^n$ are of the form

$$\{u_0 = \alpha_0,\, u_1 = \alpha_1,\, x_1 = \gamma_1,\, x_2 = 1\}$$

$$\{u_0 = \beta_0,\, u_1 = \beta_1,\, x_1 = \gamma_2,\, x_2 = 1\}$$

Comparison of the metrics on $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

- Jensen's metrics on Stiefel manifold $V_4\mathbb{R}^n = \mathrm{SO}(n)/\mathrm{SO}(n-4)$

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & a & 1 \\ a & a & 1 \\ 1 & 1 & * \end{pmatrix}, \quad \mathrm{Ad}(\mathrm{SO}(4) \times \mathrm{SO}(n-4))\text{-invariant.}$$

- Our Einstein metrics

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 0 & \beta & \gamma \\ \beta & \alpha & 1 \\ \gamma & 1 & * \end{pmatrix}, \quad \mathrm{Ad}(\mathrm{SO}(3) \times \mathrm{SO}(n-4))\text{-invariant}$$

($\alpha, \beta, \gamma \neq 1$ are all different).

- For the Stiefel manifolds $V_\ell\mathbb{R}^{k+k+\ell} = \mathrm{SO}(2k+\ell)/\mathrm{SO}(\ell)$ ($\ell > k \geq 3$)

Einstein metrics of Arvanitoyeorgos, Dzhepko and Nikonorov

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} \alpha & \beta & 1 \\ \beta & \alpha & 1 \\ 1 & 1 & * \end{pmatrix} \quad (\ \alpha, \beta \text{ are different }).$$

New Einstein metrics on complex Stiefel manifold $V_3\mathbb{C}^{n+3}$

Theorem

On a complex Stiefel manifold $V_3\mathbb{C}^{n+3} \cong \mathrm{SU}(n+3)/\,\mathrm{SU}(n)$ for $n \geq 2$, there exist new invariant Einstein metrics which are different from Jensen's metrics.

- In this case we view the Stiefel manifold $V_3\mathbb{C}^{n+3}$ as a total space over the generalized flag manifold

$$\mathrm{SU}(1+2+n)/\,\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2)\times\mathrm{U}(n)) \quad n \geq 2$$

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