Strictly nearly pseudo-Kähler manifolds with large symmetry groups

Henrik Winther

University of Tromsø
Joint work with B.Kruglikov.

March 10, 2016

Problem

Our goal is to produce examples of highly symmetric almost complex structures J, meaning that the symmetry group G is of higher dimension than the base manifold M, which will be $\dim M = 6$, corresponding to 3D complex space. This is particularly interesting if G also preserves some other structure compatible with J, such as almost symplectic and almost pseudo-Hermitian structures (g, J, ω).

Symmetries of ACM in general

The main invariant of an almost complex structure J is the Nijenhuis tensor

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

This is a complete obstruction to local integrability. When $N_J = 0$, the (local) symmetry algebra will be infinite dimensional, consisting of the holomorphic vector fields.

Symmetries of ACM in general

The main invariant of an almost complex structure J is the Nijenhuis tensor

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

This is a complete obstruction to local integrability. When $N_J = 0$, the (local) symmetry algebra will be infinite dimensional, consisting of the holomorphic vector fields.
In general, one can take any operator J st. $J^2 = -1$ on a Lie algebra $\mathfrak{g}$ with $\dim(\mathfrak{g}) = 6$, and extend this to a left invariant structure on a Lie group G. However, for generic algebras $\mathfrak{g}$ the symmetry group of such J will be only G itself. Determining which $\mathfrak{g}$ yields more symmetries is an intractable problem.
The space of all almost complex homogeneous spaces is even larger, thus some restrictions are necessary to get good results.

Non-Degenerate Nijenhuis Tensors

Definition

The Nijenhuis tensor N_J of an almost complex structure J is called non-degenerate (NDG) if

$$N_J : \Lambda^2_{\mathbb{C}} T_x M \to T_x M$$

is an isomorphism of real vector spaces.

R.Bryant and M.Verbitsky separately discovered Hitchin-type volume-functionals in dim 6 relating complex and NDG structures. The critical points are always either complex or NDG.

Theorem

If J has NDG N_J, then it has finite dimensional symmetry algebra.

Nearly Kähler and SN(P)K

An almost Hermitian structure (g, J, ω) is called nearly Kähler if

$$\nabla\omega \in \Omega^3 M,$$

and strictly nearly Kähler (SNK) if $d\omega \neq 0$. SNK structures are always NDG. Thomas Friedrich and Ralf Grunewald proved that a 6-dimensional Riemannian manifold admits a Riemannian (real) Killing spinor if and only if it is nearly Kähler.

Nearly Kähler and SN(P)K

An almost Hermitian structure (g, J, ω) is called nearly Kähler if

$$\nabla\omega \in \Omega^3 M,$$

and strictly nearly Kähler (SNK) if $d\omega \neq 0$. SNK structures are always NDG. Thomas Friedrich and Ralf Grunewald proved that a 6-dimensional Riemannian manifold admits a Riemannian (real) Killing spinor if and only if it is nearly Kähler. J.B. Butruille classified homogeneous SNK structures in dim 6, which also proved a conjecture by Wolf and Gray: All such spaces are 3-symmetric.

1. $S^6 = G_2^c/SU(3)$
2. $\mathbb{C}P^3 = Sp(2)/U(2)$
3. $S^3 \times S^3 = SU(2)^3/SU(2)_\Delta$
4. $F(1,2) = SU(3)/\mathbb{T}^2$

ACM with irreducible isotropy

The Homogeneous spaces with irreducible isotropy representation were classified for arbitrary dimension by J.A.Wolf. Irreducible isotropy is a strong condition, and there are not many almost complex examples from their classification in dim 6.
The list consists of 6 (pseudo-) Hermitian symmetric spaces (note that these automatically get $N_J = 0$), and also the spaces

1. $SL(2, \mathbb{C})$ acting on itself as a real Lie group equipped with its natural structure $J = i$.
2. $S^6 = G_2^c/SU(3)$, the sphere S^6, and its non-compact version $G_2^*/SU(2,1)$ AKA $S^{(2,4)}$

Intermediate case

Wolf had only a few examples in 6D, and with one exception they were complex. We consider an intermediate case between Wolf and the most general setting. The isotropy group of the Calabi sphere S^6 is $SU(3)$, a simple group. Generalizing from this example and with the knowledge that representations of semi-simple groups often preserve interesting geometric objects, we ask:

Problem 1: Alekseevsky, Kruglikov, Winther

What are the (non-flat) 6D almost complex homogeneous spaces $(M = G/H, J)$ with semi-simple isotropy group H? (Aside from the Calabi structure on the sphere $S^6 = G_2^c/SU(3)$, $S^{(2,4)}$, and the Calabi-Eckmann manifolds $S^1 \times S^5, S^3 \times S^3$)

Theorem

The only 6D homogeneous almost complex structures on $M = G/H$ with semi-simple isotropy group H are (up to covering and quotient by discrete central subgroup):

- (I) *Unique up to sign structures on $S^6 = G_2^c/SU(3)$ and $S^{(2,4)} = G_2^*/SU(2,1)$;*
- (II_1) *4-parametric family on $U(3)/SU(2)$, $U(2,1)/SU(2)$;*
- (II_2) *2-parametric family on $U(2,1)/SU(1,1)$, $GL(3)/SU(1,1)$;*
- (III) *left-invariant almost complex structures on a 6D Lie group with H-invariant group operation.*

Theorem

In those cases where N_J is NDG, the symmetry group of J contains only the indicated group.

Corollary

The only compact examples are $S^6, S^1 \times S^5, S^3 \times S^3$.

Observation: The most symmetric model of a type of geometric structure is often unique, and there is a significant gap between the symmetry dimension of the maximal model and the so called sub-maximal model. It is interesting to determine the size of this gap.
The maximal NDG J are $S^6 = G_2^c/SU(3)$ and $S^{(2,4)}$, $\dim(G) = 14$. It is important to note that these are also SNK and SNPK.

Example: The Calabi structure on S^6

Let $S^6 \subset \Im(\mathbb{O})$. The tangent space T_xS^6 is invariant under multiplication by x, and $x^2 = -1$. This defines J.
There exists a complex basis x_1, x_2, x_3 of T_xS^6 such that $N_J(x_1, x_2) = x_3, N_J(x_3, x_1) = x_2, N_J(x_2, x_3) = x_1$, and since N_J is complex anti-linear this means that $Ker(N_J) = 0$ so J is non-degenerate. The Calabi structure is almost-hermitian and in particular nearly Kähler.

Problem: Kruglikov, Winther

What are the sub-maximal models of SNPK, SNK and generally NDG (J, N_J)?

Overview of Results: Local

Theorem

Let J be NDG and not locally G_2-symmetric. Then $\dim \mathfrak{sym}(J) \leq 10$. In the case of equality, the regular orbits of the symmetry algebra $\mathfrak{sym}(J)$ are open (local transitivity) and J is equivalent near regular points to an invariant structure on one of the homogeneous spaces

- $Sp(2)/U(2)$, *which is SNK;*
- $Sp(1,1)/U(2)$, *which is SNPK of signature (4,2);*
- $Sp(4,\mathbb{R})/U(1,1)$, *which is SNPK of signature (4,2).*

Corollary

The gap between maximal and sub-maximal symmetry dimensions of $\mathfrak{sym}(J)$ for $\dim M = 6$ is the same for non-degenerate almost complex structures as for SNK and SNPK.

Overview of Results: Global

We investigate the possibility of singular orbits, with the conclusion that there are none. For simplicity we formulate the global version.

Theorem

Let (M, J) be a connected non-degenerate almost complex manifold with $\dim \mathrm{Aut}(J) = 10$*. Then M is equal to the regular orbit of its automorphism group, and hence it is a global homogeneous space of one of three types indicated in Theorem 1.*

sub-sub-maximal models of NDG AC

Having found the sub-maximal models to have symmetry dimension 10, we may investigate the sub-sub-maximal models of NDG almost complex structures, which have symmetry dimension 9. As Lie algebras of dimension 3 are either solvable or simple, we investigate whether there are any sub-sub-maximal models with solvable isotropy. It turns out that there are none.

Theorem

Let (M, J) be a homogeneous non-degenerate almost complex manifold with $\dim \mathrm{Aut}(J) = 9$. Then the isotropy subgroup of $\mathrm{Aut}(J)$ is simple.

sub-sub-maximal models of SNPK

We observe that in the previous cases, the SNPK examples have been the SNK examples up to changes of real forms of G/H. One might wonder if this is true generally. But it is not:

Theorem

The SNK and SNPK structures with symmetry algebra of dimension 9 are

- $S^3 \times S^3$, *which is SNK*
- $SU(1,1) \times SU(1,1)$, *which is SNPK of signature (4,2)*
- *Left invariant str. on solvable group related to split-quaternions, which is SNPK of signature (4,2)*

The last example does not have a Riemannian analogue.

Sketch of Proof

We begin by constructing extra almost Hermitian structures from the Nijenhuis tensor, and observing that the symmetry will necessarily have to preserve these as well as (J, N_J).

Sketch of Proof

We begin by constructing extra almost Hermitian structures from the Nijenhuis tensor, and observing that the symmetry will necessarily have to preserve these as well as (J, N_J).
Then we determine the possible isotropy algebras. It will indeed be smaller than for a general AC str. because more objects need to be stabilized. We show that the sub-maximal model must have open regular orbits, and perform algebraic reconstruction from representation theoretic data. This yields $\mathfrak{g}$ and the local theorem.
Finally, consider geometric structures which exist on submanifolds of NDG almost complex manifolds, and show that the possible lower dimensional orbits of G can not admit such structures. This yields the global theorem.

Theorem

If (J, N_J) is NDG, then $\mathfrak{sym}(J)$ is fully determined by its 1st jets, ie the isotropy representation of $\mathfrak{h} \subset \mathfrak{sym}(J)$ is faithful.

Theorem

If (J, N_J) is NDG, then $\mathfrak{sym}(J)$ is fully determined by its 1st jets, ie the isotropy representation of $\mathfrak{h} \subset \mathfrak{sym}(J)$ is faithful.

This means that one approach is to find the maximal linear symmetries of N_J, and attempt to reconstruct the homogeneous space from algebraic data.

Theorem

NDG Nijenhuis tensors can be classified algebraically into 4 types (think of them as normal forms)

1. $N(X_1, X_2) = X_2, N(X_1, X_3) = \lambda X_3, N(X_2, X_3) = e^{i\phi} X_1$
2. $N(X_1, X_2) = X_2, N(X_1, X_3) = X_2 + X_3, N(X_2, X_3) = e^{i\phi} X_1$
3. $N(X_1, X_2) = e^{-i\psi} X_3, N(X_1, X_3) = -e^{i\psi} X_2, N(X_2, X_3) = e^{i\phi} X_1$
4. $N(X_1, X_2) = X_1, N(X_1, X_3) = X_2, N(X_2, X_3) = X_2 + X_3$

When the Nijenhuis tensor is NDG we associate a bilinear (1,1)-form

$$h(v,w) = \mathrm{Tr}[N_J(v, N_J(w,\cdot)) + N_J(w, N_J(v,\cdot))]$$

and a holomorphic 3-form

$$\zeta(u,v,w) = \mathrm{alt}[h(N_J(u,v),w) - i\,h(N_J(u,v),Jw)]$$

(alt is the total skew-symmetrizer). When both are non-degenerate the symmetries of (J, N_J) must preserve the (pseudo-)Hermitian metric and the holomorphic volume form. This means that the possible isotropy algebras of the symmetry algebras of NDG ACS in 6D of dimension ≥ 3 are special unitary algebras $\mathfrak{su}(3)$, $\mathfrak{su}(1,2)$, or a subalgebra of these.

Returning to almost Hermitian geometry

Theorem

The forms h, ζ are non-degenerate except for isolated exceptional parameters for all 4 types of NDG Nijenhuis tensor.

This means that generally we need to consider almost (pseudo-) Hermitian structures whenever we are dealing with NDG N_J.

Returning to almost Hermitian geometry

Theorem

The forms h, ζ are non-degenerate except for isolated exceptional parameters for all 4 types of NDG Nijenhuis tensor.

This means that generally we need to consider almost (pseudo-) Hermitian structures whenever we are dealing with NDG N_J. Computing linear symmetry algebras of the types yields

Theorem

1. $\mathfrak{sym}(N_1) = \mathfrak{su}(1,2)$ *for generic paremeters. For exceptional paremeters,* $\mathfrak{sym}(N_1) = \mathfrak{u}(1,1)$
2. $\mathfrak{sym}(N_2) \subset \mathfrak{su}(1,2)$, *except for one 2D non-unitary algebra.*
3. $\mathfrak{sym}(N_3) = \mathfrak{su}(3)$ *or* $\mathfrak{sym}(N_3) = \mathfrak{su}(1,2)$ *for generic paremeters. Exceptionals yield subalgebras* $\mathfrak{u}(2), \mathfrak{u}(1,1)$.
4. $\mathfrak{sym}(N_4) \subset \mathfrak{su}(1,2)$

Possible isotropy subalgebras

From the previous computation, it is clear that the isotropy subalgebra will be a subalgebra of $\mathfrak{su}(3)$ or $\mathfrak{su}(1,2)$. We also know from Butruille's list that $\mathbb{C}P^3 = SP(2)/U(2)$ has an invariant NDG Nijenhuis tensor. Therefore we only need to consider subalgebras which have dimension greater or equal $\dim \mathfrak{u}(2) = 4$. The list is as follows:

- $\mathfrak{u}(2) \subset \mathfrak{su}(3)$.
- $\mathfrak{u}(2) \subset \mathfrak{su}(1,2)$.
- $\mathfrak{u}(1,1) \subset \mathfrak{su}(1,2)$.
- The maximal solvable parabolic subalgebra $P \subset \mathfrak{su}(1,2)$ of dim 5.
- The 4D maximal subalgebras of P.

The regular orbits are open

We must also consider the possibility that the sub-maximal model is not a homogeneous space. However, by our earlier theorem, the isotropy representation still needs to be one of those we listed. Constructing an inhomogeneous symmetry group from such a representation creates rather strong constraints:

Theorem

The isotropy representation $\mathfrak{m}$ must have an invariant submodule $\mathfrak{n}$ with the dimension of the orbit. The quotient representation $\mathfrak{m}/\mathfrak{n}$ must be trivial.

The regular orbits are open

We must also consider the possibility that the sub-maximal model is not a homogeneous space. However, by our earlier theorem, the isotropy representation still needs to be one of those we listed. Constructing an inhomogeneous symmetry group from such a representation creates rather strong constraints:

Theorem

The isotropy representation $\mathfrak{m}$ must have an invariant submodule $\mathfrak{n}$ with the dimension of the orbit. The quotient representation $\mathfrak{m}/\mathfrak{n}$ must be trivial.

Parabolic subalgebras are usually defined as stabilizers of flags. In particular, P and its subalgebras preserve a 4D submodule. This could at most yield a 9D symmetry algebra. However, the condition on the quotient means that we have to drop to a smaller algebra, the subalgebra of P which acts trivially on $\mathfrak{m}/\mathfrak{n}$. This has subalgebra has dimension 3, yielding only $\dim G = 7$ which is too small. Thus the sub-maximal model is locally transitive.

Obstructions to singular orbits

A singular orbit $\mathcal{O}$ is in particular a homogeneous space of G. Thus, it is also a submanifold of M and will inherit some geometry from the almost complex structure and Nijenhuis tensor. Note that the maximal subgroups of greatest dimension are 7D, so $\dim \mathcal{O} \geq 3$ unless $\dim \mathcal{O} = 0$.

- $\dim \mathcal{O} = 3$: invariant real 2D distribution L with an invariant complex structure, or
- $\mathcal{O}$ is totally real, in which case it has an invariant map $\mathfrak{h}$-invariant map $\Lambda^2 T\mathcal{O} \to T\mathcal{O}$.
- $\dim \mathcal{O} = 4$: Distribution L of dimension 2 or 4 with invariant complex str. (4 is almost complex).
- $\dim \mathcal{O} = 5$: Distribution L of dimension 4 with invariant complex str., and $T\mathcal{O} = L \oplus \mathbb{R}$ or
- L-valued 2-form $\theta \in \Lambda^2 L^* \otimes L$.