On Besse orbifolds

M. Amann, Chr. Lange

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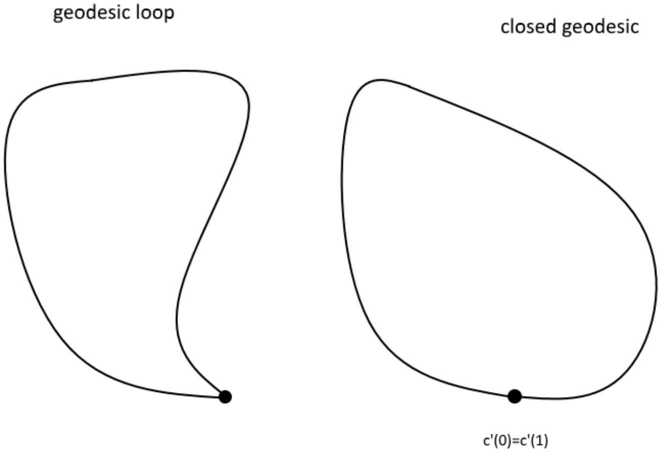
Closed geodesics



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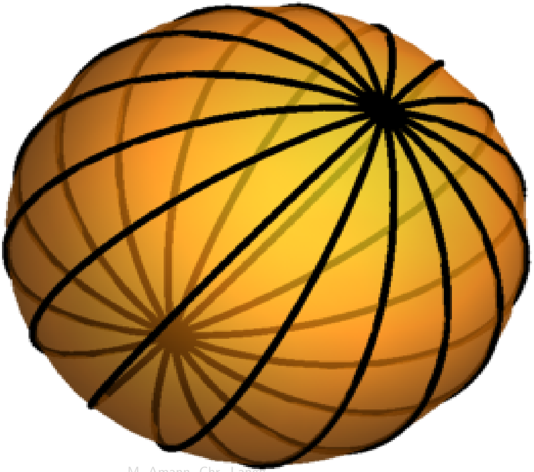

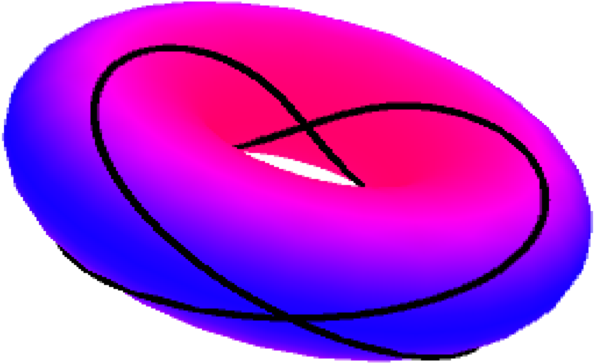

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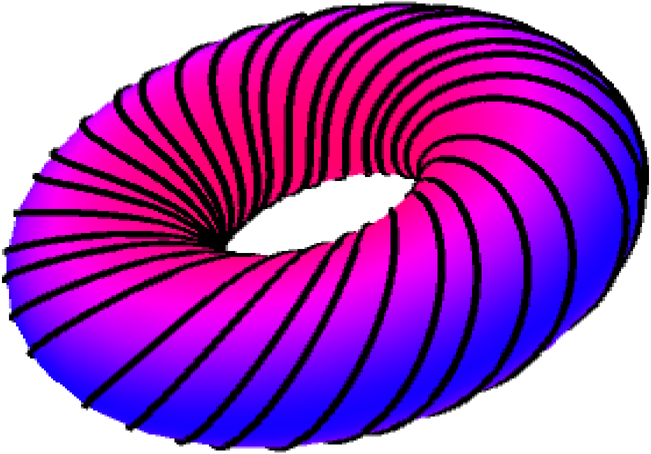

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- This is a metric-dependent problem. **"Lyusternik-Fet"**: Every closed Riemannian manifold has a closed geodesic. **"Theorem of the three geodesics"**: Every metric on a homeomorphism sphere possesses three closed (simple) geodesics.
- How is the topology of a manifold/orbifold related to the existence of infinitely many closed geodesics?

Existence of closed geodesics

Theorem (Sullivan–Vigué-Poirrier, Gromoll–Meyer, McCleary, Saneblidze)

If M is a compact simply-connected Riemannian manifold whose $\mathbb{Q}$- or $\mathbb{Z}_p$-cohomology algebra requires at least two generators, then M has infinitely many geometrically distinct closed geodesics in any metric.

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Note: Any metric on $\mathbb{S}^2$ has infinitely many closed geodesics (Bangert–Franks). Maybe this is true on every manifold (Klingenberg)?

What if **all** geodesics are closed?

Besse manifolds

Definition

A **Besse metric** g *on the manifold* M *satisfies that all of its geodesics are closed. A* **Besse manifold** *is a manifold endowed with a Besse metric.*

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Theorem (Bott–Samelson)

Let M *be a Besse manifold, then so is its universal cover* $\tilde{M}$ *and*

$$H^*(\tilde{M};\mathbb{Z}) \cong H^*(X;\mathbb{Z})$$

with X *a CROSS as above.*
In particular, if $n := \dim \tilde{M}$ *is odd, then* $\tilde{M}$ *is homeomorphic to* $\mathbb{S}^n$*.*

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A curve $\gamma\colon [0,l] \to \mathcal{O}$ is a **(orbifold) geodesic** *if it locally lifts to a geodesic on a manifold chart. A closed (orbifold) geodesic is a continuous loop which is an (orbifold) geodesic on each subinterval. An orbifold is called* **Besse** *if all its geodesics are closed.*

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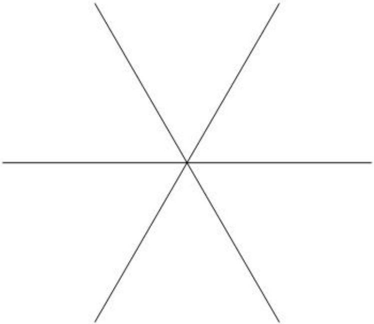
4. ... or already now: Although every "known closed orbifold" possesses a closed geodesic, it is not known if this is true in general.

Examples



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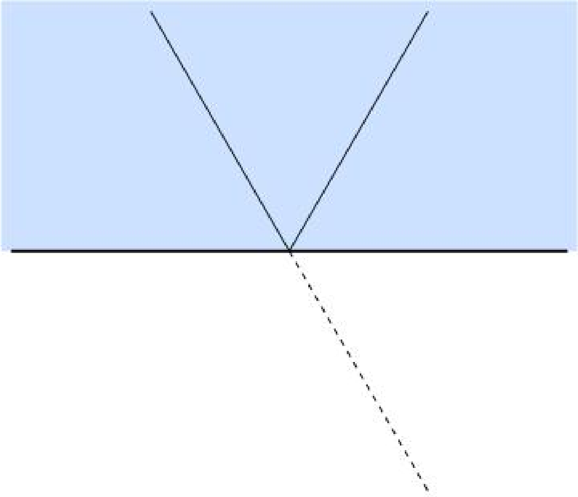

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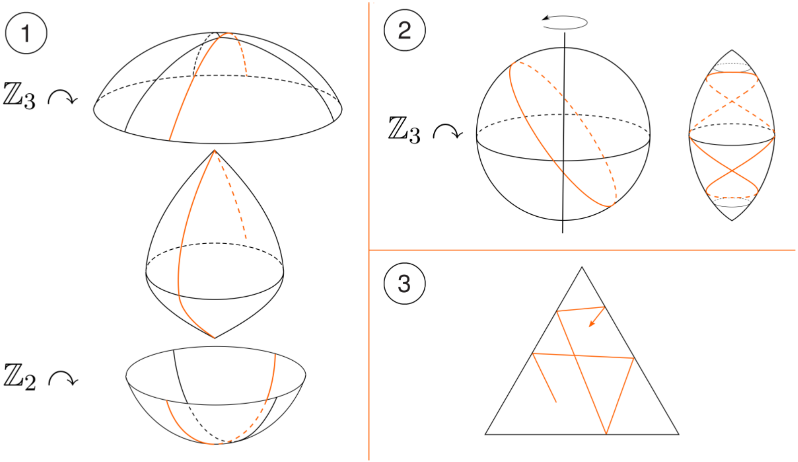
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Results on 2-orbifolds

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Problems:

1. How does the moduli space of Besse metrics on 2-orbifolds look like?
2. What can be said about Besse orbifolds in higher dimensions?

Consider $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ with the non-standard action of $\mathbb{C} \supseteq \mathbb{S}^1 \subseteq \mathbb{S}^{2n+1}$ given by

$$w \cdot (z_0, \ldots, z_n) := (w^{a_0} z_0, \ldots, w^{a_n} z_n)$$

with rotation numbers $a_i \in \mathbb{Z} \setminus \{0\}$. Set $a := (a_0, \ldots, a_n)$. This action is almost free (clearly, not necessarily free) and the quotient

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The projection π induces a quotient orbifold metric on $\mathbb{C}\mathbf{P}^n_a$ such that π is a Riemannian submersion.
Since $\mathbb{S}^{2n+1}$ is a Besse manifold, every geodesic lifts to a closed geodesic and projects downwards to a closed geodesic. Hence $\mathbb{C}\mathbf{P}^n_a$ is a Besse orbifold.

Replace now $\mathbb{C}$ by $\mathbb{H}$, the hypersphere $\mathbb{S}^{2n+1}$ by $\mathbb{S}^{4n+3}$, and the sphere $\mathbb{S}^1$ by the unit quaternions $\mathbb{S}^3$. Likewise, this yields a Besse orbifold

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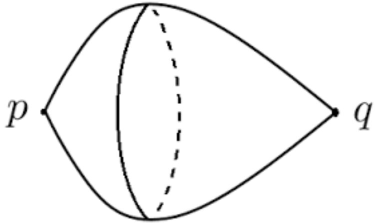

Orbifold coverings

A map $\phi : \mathcal{O}' \to \mathcal{O}$ between Riemannian orbifolds of the same dimension is called *covering* if it is a submetry, i.e. $\phi(B_r(x)) = B_r(\phi(x))$ holds for all $x \in \mathcal{O}'$ and all $r \geq 0$.

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If $\phi : \mathcal{O}' \to \mathcal{O}$ is a covering, then every point $x \in \mathcal{O}$ has a neighborhood U isometric to some M/Γ such that each connected component of $\phi^{-1}(U)$ is isometric to M/Γ_i for some subgroup $\Gamma_i < \Gamma$ such that ϕ is compatible with the natural projections $M/\Gamma \to M/\Gamma_i$.

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The *fundamental group* of $\mathcal{O}$ can be defined as the group of deck transformations of $\tilde{\mathcal{O}} \to \mathcal{O}$. An orbifold is called *simply-connected* if it does not admit a non-trivial covering.

Orbifold analogues of the manifold results

Theorem (Guruprasad–Haefliger)

Let $\mathcal{O}$ be a simply-connected closed orbifold. Suppose that $H^*(\mathcal{O};\mathbb{Q})$ as an algebra is not generated by one single element. Then there are infinitely many geometrically distinct closed (orbifold) geodesics on $\mathcal{O}$.

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Theorem (A., L., Radeschi)

An odd dimensional Besse orbifold $\mathcal{O}$ is covered by a manifold. Hence, by a sphere.

Frame bundle

The action of Γ_i extends to the bundle of orthogonal frames over U_i by differentials yielding the **frame bundle** $\mathrm{Fr}(\mathcal{O})$ locally modelled on $\mathrm{Fr}(\mathbb{R}^n)/\Gamma_i$.

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Theorem

For a given orbifold $\mathcal{O}$, its frame bundle $\mathrm{Fr}(\mathcal{O})$ is a smooth manifold with a smooth, effective and almost free $\mathbf{O}(n)$-action, and $\mathcal{O}$ is naturally isomorphic to the resulting quotient orbifold $\mathrm{Fr}(\mathcal{O})/\mathbf{O}(n)$.

Orbifold classifying space

Definition

We call

$$\mathbf{B}\mathcal{O} = \mathrm{Fr}(\mathcal{O}) \times_{\mathbf{O}(n)} \mathbf{EO}(n)$$

the **classifying space** *of the orbifold* $\mathcal{O} = \mathrm{Fr}(\mathcal{O})/\mathbf{O}(n)$.

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One can show that orbifold coverings of $\mathcal{O}$ are in one-to-one correspondence with ordinary coverings of $\mathbf{B}\mathcal{O}$. In particular, we have

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Definition

Orbifold cohomology *is defined as*

$$H^*_{orb}(\mathcal{O}) := H^*(\mathbf{B}\mathcal{O})$$

Remark

- *The orbifold* $\mathcal{O} = \mathrm{Fr}(\mathcal{O})/\mathbf{O}(n)$ *is a manifold if and only if* $\mathbf{O}(n)$ *acts freely on* $\mathrm{Fr}(\mathcal{O})$.

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\mathbf{B}(\mathbb{R}^n/\Gamma_i) &= \mathrm{Fr}(\mathbb{R}^n)/\Gamma_i \times_{\mathbf{O}(n)} \mathbf{EO}(n) \\
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and hence $H^*_{orb}(\mathbb{R}^n/\Gamma_i) = H^*(\mathbf{B}\Gamma_i) = H^*(\Gamma_i)$ *and* $\pi_1^{orb}(\mathbb{R}^n/\Gamma_i) = \Gamma_i$. *Moreover,*

$$H^*_{orb}(\mathcal{O};\mathbb{Q}) \cong H^*(\mathcal{O};\mathbb{Q})$$

A Conjecture

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The integral orbifold cohomology of such a weighted $\mathbb{C}\mathbf{P}^n$ is the integral cohomology of the usual $\mathbb{C}\mathbf{P}^n$ up to its dimension n degenerating to $\mathbb{Z}_k$-torsion in even degrees from degree $n+2$ on, where k is the product of the weights.

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So maybe the best to hope for is

Conjecture

Also in the even-dimensional case integral orbifold cohomology is monicly generated.

Sketching the proof

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As a consequence (for $G = \mathbf{O}(n)$ and $X = \mathrm{Fr}(\mathcal{O})$) we obtain

Theorem

The orbifold $\mathcal{O}$ has the structure of a manifold if and only if $\dim H^*(\mathbf{B}\mathcal{O};\mathbb{Z}_p) < \infty$ for all $p \geq 0$.

Idea for this:
$\mathcal{O}$ is manifold $\iff$ $\mathbf{O}(n)$ acts freely on $F\mathcal{O}$. If so, $H^*(\mathbf{B}\mathcal{O};\mathbb{Z}_p)$ is $H^*(F\mathcal{O}/\mathbf{O}(n);\mathbb{Z}_p)$, which is finite-dimensional, since $F\mathcal{O}/\mathbf{O}(n)$ is a manifold.

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Remains to prove: Whenever orbifold cohomology is finite dimensional, the orbifold is already a manifold.
Suppose that $\mathbf{O}(n)$ acts on $F\mathcal{O}$ with a point of finite isotropy $\Gamma \neq 0$.
Since $\Gamma \neq 0$, we may pick a non-trivial element in there generating a finite cyclic group; without restriction $\mathbb{Z}_p$ for a prime p.

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Since p is prime, $\mathbb{Z}_p$ acts freely if and only if it acts without fixed-points. Thus it remains to show that $\mathbb{Z}_p$ cannot have a fixed-point, contradicting its very construction.

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We compute the equivariant cohomology of the action as

$$H^*_{\mathbb{Z}_p}(F\mathcal{O};\mathbb{Z}_p) = H^*(F\mathcal{O} \times_{\mathbb{Z}_p} \mathbf{EO}(n);\mathbb{Z}_p)$$

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Obtain

$$\begin{aligned} S^{-1}H^*_{\mathbb{Z}_p}((F\mathcal{O})^{\mathbb{Z}_p};\mathbb{Z}_p) &\cong S^{-1}(H^*((F\mathcal{O})^{\mathbb{Z}_p} \times \mathbf{B}\mathbb{Z}_p;\mathbb{Z}_p)) \\ &\cong S^{-1}(H^*(\mathbf{B}\mathbb{Z}_p;\mathbb{Z}_p) \otimes H^*((F\mathcal{O})^{\mathbb{Z}_p};\mathbb{Z}_p)) \end{aligned}$$

and derive that $S^{-1}H^*_{\mathbb{Z}_p}((F\mathcal{O})^{\mathbb{Z}_p};\mathbb{Z}_p)$ vanishes if and only if $\mathbb{Z}_p$ acts without fixed-points. Equivalently, since $H^*(\mathbf{B}\mathbb{Z}_p;\mathbb{Z}_p)$ is infinite dimensional over $\mathbb{Z}_p$, we conclude that

$$\dim_{\mathbb{Z}_p} H^*_{\mathbb{Z}_p}(F\mathcal{O};\mathbb{Z}_p) < \infty \qquad \Longleftrightarrow \qquad \mathbb{Z}_p \text{ acts freely.}$$

Sketching the proof

Now we take profit of the manifold recognition theorem: Via

$$\mathbf{B}\mathcal{O} = \mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)} \xrightarrow{\phi} \mathbf{EO}(n)/\mathbf{O}(n) = \mathbf{BO}(n)$$

$H^*(\mathbf{B}F\mathcal{O};\mathbb{Z}_p)$ is an $H^*(\mathbf{BO}(n);\mathbb{Z}_p)$-module. This yields the **orbifold Stiefel–Whitney and Pontryagin classes**

$$w_i(\mathcal{O}) := \phi^*(w_i)$$
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By a spectral sequence argument we prove

Corollary

If the orbifold Pontryagin and Stiefel–Whitney classes are nilpotent elements, the orbifold is actually a manifold.

We want to show that all orbifold characteristic classes vanish in our case!

Sketching the proof

As a next step we do Morse theory on the loop space in order to better understand the orbifold classifying space $\mathbf{B}\mathcal{O} = \mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)}$.

Sketching the proof

As a next step we do Morse theory on the loop space in order to better understand the orbifold classifying space $\mathbf{B}\mathcal{O} = \mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)}$. For regular points q_1, q_2 and fixed $z \in \pi^{-1}(q_2)$ a suitable version of an orbifold loop space is

$$\Omega^{\mathsf{orb}}_{q_1,q_2}\mathcal{O} := \{\gamma : [0,1] \to \mathrm{Fr}(\mathcal{O}) \mid \gamma \text{ piecew. smooth, } \gamma(0) \in \pi^{-1}(q_1), \gamma(1) = z\}$$

where $\pi\colon\ \mathrm{Fr}(\mathcal{O}) \to \mathcal{O}$. There is a homotopy equivalence

$$\Omega^{\mathsf{orb}}_{q_1,q_2}\mathcal{O} \simeq \Omega(\mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)})$$

The Besse condition yields that this space has **uniformly bounded** Betti numbers in all coefficients.

Sketching the proof

The Besse condition yields that this space has **uniformly bounded** Betti numbers in all coefficients.

- First apply this in the rational case using rational homotopy theory. This yields

$$H^*(\mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)};\mathbb{Q}) \cong H^*(\mathbb{S}^n;\mathbb{Q})$$

  and

$$H^*(\Omega(\mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)});\mathbb{Q}) \cong H^*(\Omega\mathbb{S}^n;\mathbb{Q})$$

This can be seen as follows: Given a minimal Sullivan model $(\Lambda V, \mathrm{d})$ of $(\mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)})$, a model for the loop space $\Omega((\mathrm{Fr}\,\mathcal{O})_{\mathbf{O}(n)})$ is given by $(\Lambda V^{-1}, 0)$, since an H-space has trivial differential.

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There are two even-degree spherical cohomology classes in $H^*(\Omega(\mathrm{Fr}\,\mathcal{O})_{\mathbf{O}(n)})$
$\iff$ $\dim V^{\mathrm{odd}} \geq 2$
$\iff$ there is a polynomial algebra $\mathbb{Q}[u, v] \subseteq H^*(\Omega\,\mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)}; \mathbb{Q})$.

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Since Betti numbers are uniformly bounded, this cannot be the case and $\mathbf{B}\mathcal{O} = (\mathrm{Fr}(\mathcal{O}))_{\mathbf{O}(n)}$ has a monicly generated rational cohomology algebra, in odd dimensions the one of a sphere.

Sketching the proof

Push this down to $\mathbb{Z}_p$-coefficients using work of McCleary drawing on Hopf structures, the Bockstein spectral sequence, etc.

$$H^*(\Omega\mathbf{B}\mathcal{O}) \cong H^*(\Omega(\mathrm{Fr}(\mathcal{O})_{\mathbf{O}(n)})) \cong H^*(\Omega^{\mathsf{orb}}_{q_1,q_2}\mathcal{O};\mathbb{Z}_p) \cong H^*(\Omega\mathbb{S}^n;\mathbb{Z}_p)$$

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Then, up to an application of the Wu formula, by degree reasons the orbifold characteristic classes vanish.

Thank you very much