Aperiodic Structures in group theory, geometry and harmonic analysis

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based on joint work with
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New trends and open problems in Geometry and Global Analysis,
Rauischholzhausen August 27th - 31st, 2018

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But isn't this bias justified?

Many problem simplify dramatically, and in fact only become tractable at all, in the presence of symmetries. Also, nature seems to have a bias towards symmetric structures, just consider how atoms are organized in crystals.

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For limitations of time, space and competence of the speaker, I will focus on classical topics in geometric group theory, Lie theory and harmonic analysis which one usually studies in periodic situations. I will try to convince you that many of these topics can be studied in aperiodic settings.

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Groups as large-scale geometric objects

(X, d) proper geodesic metric space
Γ (discrete) group acting on (X, d) by isometries
(X, d) geometric model for Γ $:\Leftrightarrow$ $\Gamma \curvearrowright (X, d)$ properly & cocompactly

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Lemma (Schwarz–Milnor)

A group Γ admits a geometric model iff it is fin.-gen.
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In this case, any two geometric models are quasi-isometric.

Thus, fin. gen. group $\Gamma \rightsquigarrow$ canonical QI type $[\Gamma]$.
Properties of $[\Gamma]$ are properties of Γ (geometric properties).

Information flows both ways

Geometric properties sometimes imply algebraic properties, e. g.

Theorem(Gromov)

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Example (Encoding QI types by finite presentations)

$\Gamma := \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = e \rangle$ has geometric model $\mathbb{H}^2$

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If $\Gamma \curvearrowright (X, d)$ geometrically, then $[\Gamma] = [X]$. Sometimes the converse holds (QI rigidity), e.g. for Riemannian symmetric space of non-compact type without Euclidean factors (Tukkia, Pansu, Kleiner–Leeb).

A different perspective

(X, d) proper geodesic metric space.

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Geometric action $\Gamma \curvearrowright (X, d)$	$\leftrightarrow$	ρ has finite kernel and $\rho(\Gamma)$ is a uniform lattice in $\mathrm{Is}(X, d)$.

Recall:
$\mathrm{Is}(X, d)$ is a compactly-generated lcsc group w.r.t. compact-open topology (and every cglcsc group is isomorphic to some $\mathrm{Is}(X, d)$).
A uniform lattice in a lcsc group is a discrete, cocompact subgroup.

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The study of geometric actions is the study of finite extensions of lattices in compactly-generated lcsc groups.

Studying lattices

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- Γ is finite, finitely generated, abelian, nilpotent, solvable, amenable, a-T-menable, has Property (T), (FL^p), exponential growth, ... iff G is compact, compactly generated, abelian, nilpotent, ...
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A key tool in transferring information between Γ and G is the associated homogeneous space G/Γ. This has two key properties:

- It is a compact space with a jointly continuous G-action (topological dynamical system).
- It admits a unique G-invariant probability measure with which it is an ergodic measurable G-dynamical system.

These dynamical systems are transitive, hence the orbit structure is trivial.

Yet another perspective

If $\Gamma \curvearrowright (X, d)$ geometrically and $o \in X$, then for every point $x \in \Gamma.o$ define the associated Voronoi cell by

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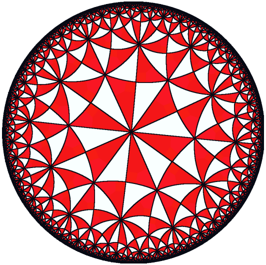
These Voronoi cells form a periodic tiling of X by compact convex tiles, and Γ acts transitively on tiles, i.e. each tile is a fundamental domain.



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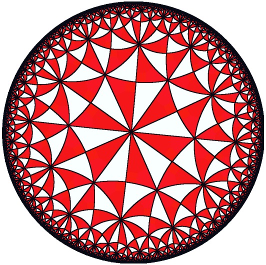

Conversely, the symmetry group of a periodic tiling with compact tiles is a uniform lattice in the isometry group.

Lattices in crystallography and harmonic analysis

Lattices in $\mathbb{R}^n$ serve as mathematical models for crystals. Experimentally, crystals are often studied by diffraction experiments, i. e. by shooting a laser at them and measuring the resulting diffraction picture. To evaluate such experiments one uses Poisson summation formula:

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Poisson summation formula (mathematical version)

For all sufficiently regular and sufficiently fast decaying functions $f : \mathbb{R}^d \to \mathbb{C}$ the following identity holds:

$$\sum_{x \in \Gamma} f(x) = \sum_{\xi \in \Gamma^*} \widehat{f}(\xi).$$

Here, $\Gamma^* \subset \mathbb{R}^n$ denotes the dual lattice

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For lattices in non-commuative groups there exist similar (but much more complicated) formulas (e. g. trace formulas).

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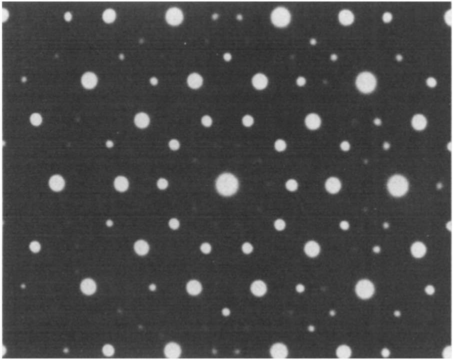
The following picture was produced in 1983 by Dan Shechtman when conducting a diffraction experiment at an aluminum-manganese alloy. It single-handedly ended the period of classical crystallography:



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- '84 Dan Shechtman discovers quasi-crystals experimentally; subsequent joint efforts of mathematicians, physicists and crystallographers show that Meyer's harmonious sets (now called Meyer sets) are the correct mathematical models.
- '94 L. Pauling ("There are no quasi-crystals, only quasi-scientists.") dies.

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We will not follow the historical development, but present approximate groups from our current point of view.

Approximate subgroups

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Definition (T.Tao, 2008)

Let G be a group, $k \in \mathbb{N}$. A subset $\Lambda < G$ is called a *k-approximate subgroup* if:

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We are going to construct infinite approximate subgroups which are not relatively dense in an actual subgroup. These examples will in fact be approximate lattices.

Approximate lattices

If (X, d) is a metric space, then $\Xi \subset X$ is called a Delone set if

$$0 < \inf\{d(x,y) \mid x, y \in \Xi, x \neq y\} < \sup\{d(x, \Xi) \mid x \in X\} < \infty.$$

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Relation to Meyer's original definition (Björklund–H.)

A Delone set $\Lambda \subset G$ is a Meyer set if $(\Lambda^{-1}\Lambda)^k$ is discrete for all $k \in \mathbb{N}$. Uniform approximate lattices are precisely the symmetric Meyer sets.

Meyer's "Cut-and-Project"-construction

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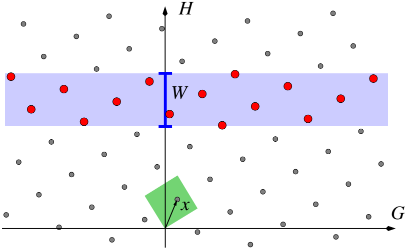
CUT out the strip $(G \times W) \cap \Gamma$ of the lattice ...



(Illustration: G. Keller / C. Richard)

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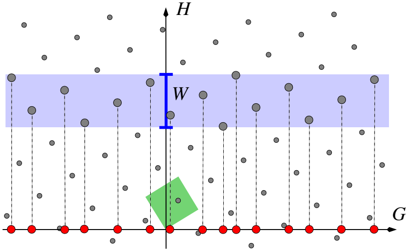

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... and PROJECT the lattice points inside this strip to G.

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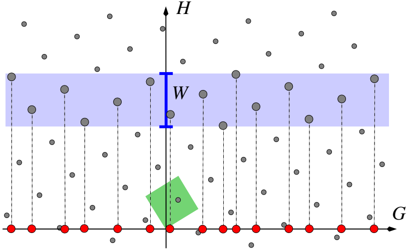

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Outcome

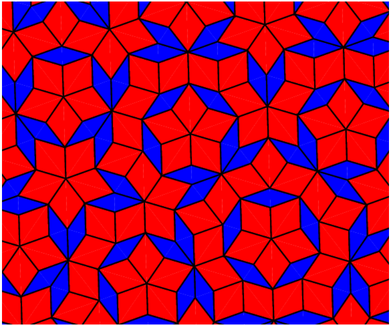
The resulting model set

$$\Lambda := \Lambda(G, H, \Gamma, W) = \pi_G((G \times W) \cap \Gamma) \subset G$$

(and hence every relatively dense subset of a model set) is a Meyer set.

A famous Euclidean example

The Penrose tiling is the Voronoi tiling associated with a model set in $\mathbb{R}^2$ (but Penrose did not construct it in this way, nor was he aware of Meyer's work).



Many other Euclidean tilings constructed in the last 50 years turned out to be Meyer tilings.

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A 1-connected nilpotent Lie group G admits a uniform lattice if its Lie algebra admits a basis with rational structure constants. It admits a Meyer set iff its Lie algebra admits a basis with algebraic structure constants. Thus most nilpotent Lie groups which admit a uniform approximate lattice do not admit an actual uniform lattice.

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Theorem [Björklund–H.]

If Λ_o is a model set in a lcsc group G, then there exists a model set $\Lambda = \Lambda(G, H, \Gamma, W)$ with H a connected Lie group such that Λ_o is a relatively dense subset of a finite enlargement of Λ. In particular, if G does not admit a uniform lattice coupling with a non-trivial connected Lie group, then every model set in G is essentially periodic.

When Meyer was awarded the Abel prize, he was asked by the EMS what was his favourite among the theorems he proved. He named the following:

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Informally, in the abelian world, aperiodic order is just a shadow of periodic order in higher dimensions.

The holy grail of aperiodic order

Is it true that every uniform approximate lattice in a lcsc group is a relatively dense subset of a model set?

We do not know the answer to this question for any non-abelian lcsc group (except those which don't admit uniform approximate lattices).

Towards the holy grail for nilpotent Lie groups

Theorem [Björklund–H.]

$A \to G \xrightarrow{\pi} Q$ abelian extension, $\Lambda \subset G$ uniform approximate lattice. Then the following conditions are equivalent:

1. $\pi(\Lambda)$ is a uniform approximate lattice in Q.
2. Over a relatively dense subset of $\pi(\Lambda)$ the fibers are Meyer sets.

In this case, Λ and A are called adapted; A is universally adapted if it is adapted for every Λ.

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Thus uniform approximate lattices in nilpotent 1-connected Lie groups are iterated extensions of Meyer sets by abelian Meyer sets.

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2. Call $f \in C(G)$ strongly pattern equivariant if for some $R > 0$,

$$B_R(x) \cap \Lambda = B_R(y) \cap \Lambda \quad \Rightarrow \quad f(x) = f(y).$$

Then the uniform closure $C_{PE}(G) \subset C(G)$ of such functions is a commutative unital C^*-algebra and $\Omega_\Lambda = \mathrm{spec}(C_{PE}(G))$.

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- There do, however, always exist stationary measures on the hull, which is e. g. enough to prove that an envelope of a uniform approximate lattice is unimodular (Björklund–H.).
- In the case of model sets, one has the following deep theorem, which in the abelian case goes back to Schlottmann:

Theorem (Björklund–H.–Pogorzelski)

If $\Lambda(G, H, \Gamma, W) \subset G$ is a model set, then there exists a unique G-invariant measure ν_Λ on Ω_Λ. Moreover, $L^2(\Omega_\Lambda, \nu_\Lambda) \cong L^2((G \times H)/\Gamma)$ decomposes discretely as a G-representation.

Application to diffraction of quasi-crystals

The discrete decomposability of $L^2(\Omega_\Lambda, \nu_\Lambda)$ for model sets has the following consequence in the abelian setting:

Theorem (Meyer's diffraction formula)

If $\Lambda = \Lambda(\mathbb{R}^n, \mathbb{R}^k, \Gamma, W) \subset \mathbb{R}^n$ then for all $f \in C_c(\mathbb{R}^n)$,

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This explains Shechtman's diffraction experiment. The Bragg peaks are those k for which $(k, \ell) \in \Gamma^\perp$ and $|\widehat{\chi_W}(\ell)|^2 > \epsilon$, where ϵ is the accuracy of the detector.

Beyond the abelian case

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- On the LHS, f is a bi-K-invariant function on G, balls are replaced by suitable approximation sequences (B_R), and we evaluate f a $f(Kg^{-1}hK)$, where $gK \in \Lambda \cap B_R$ and $hK \in \Lambda$.

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- On the RHS, the spherical Fourier transform $\widehat{f}$ is evaluated at elements of the spherical automorphic spectrum of Γ, and the coefficients are given by a certain integral transform depending on the Gelfand pair and the lattice called the shadow transform. (If K is open in G, then this is just a version of the Hecke correspondence from analytic number theory.)

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Equivalently, it is a homomorphism $\rho_\infty : \Lambda^\infty \to \mathrm{Is}(X, d)$ such that $\ker(\rho_\infty) \cap \Lambda^k$ is finite for all $k \in \mathbb{N}$ and $\rho_\infty(\Lambda)$ is a uniform approximate lattice in $\mathrm{Is}(X, d)$. (Cordes–H.–Tonić)

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So far, everything is as for groups, but there is one major difference: An approximate group need not have a geometric model!

Associating a QI type with an approximate group

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First generalization

$(\Lambda, \Lambda^\infty)$ is geometrically finitely-generated if there exists a large-scale geodesic metric d_o on Λ representing $[\Lambda]_c$. In this case, its canonical QI type is defined as $[\Lambda] := [\Lambda, d_o]$ (independent of d_o).

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Second generalization

$(\Lambda, \Lambda^\infty)$ is algebraically finitely-generated if Λ^∞ is a f.g. group. In this case its external QI type is $[\Lambda]_{\mathrm{ext}} := [\Lambda, d_S|_{\Lambda\times\Lambda}]$ for a word metric d_S on Λ^∞ (independent of S).

Distortion in approximate groups

Let $(\Lambda, \Lambda^\infty)$ be a countable group which is both geometrically and algebraically finitely-generated.

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Theorem (Cordes–H.–Tonić)

A distorted approximate group does not have a geometric model.

QI rigidity

In view of the previous result, we formulate the QI rigidity problem as follows:

Definition

A proper geodesic metric space (X, d) is QI rigid with respect to approximate groups if for every undistorted geometrically and algebraically finitely-generated approximate group $(\Lambda, \Lambda^\infty)$ we have

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The rank one case is currently open. We believe that the real-hyperbolic case should work, except possibly for $\mathbb{H}^2$ (ongoing joint work with T. Dymarz).

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$$\lambda(g) : \quad \begin{tikzcd} \Lambda \arrow[rrrr, bend left, "L_g"] & \Lambda^2 \arrow[l, bend left, "p_1"] & \Lambda^3 \arrow[l, bend left, "p_2"] & \dots \arrow[l, bend left, "p_3"] & \Lambda^{k+1} \arrow[l, bend left, "p_k"] \end{tikzcd} .$$

A key tool

A finitely-generated group Γ admits a geometric action on itself with respect to a word metric. This is not the case for an approximate groups (even an undistorted ones), but it does admit a geometric quasi-action on itself. Rather then defining this formally, let us describe this quasi-action in the undistorted case.

Fix a word metric on Λ^∞. Then the inclusions $\Lambda^k \hookrightarrow \Lambda^{k+1}$ are quasi-isometries, hence admits quasi-inverses $p_k : \Lambda^{k+1} \to \Lambda^k$. Now if $\gamma \in \Lambda^k$, then left-multiplication L_γ by γ maps Λ to Λ^{k+1} and we define $\lambda(g) : \Lambda \to \Lambda$ by

$$\lambda(g) : \quad \begin{tikzcd} \Lambda \arrow[rrrr, bend left, "L_g"] & \Lambda^2 \arrow[l, "p_1"] & \Lambda^3 \arrow[l, "p_2"] & \dots \arrow[l, "p_3"] & \Lambda^{k+1} \arrow[l, "p_k"] \end{tikzcd}.$$

Then $\rho : \Lambda^\infty \to \mathrm{QI}(\Lambda)$ is the left-regular geometric quasi-action.

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