Twistor Deformations and Global Torelli Theorem for Singular Symplectic Varieties

Christian Lehn, TU Chemnitz

31.08.2018

New trends and open problems in Geometry and Global Analysis

Castle Rauischholzhausen

Outline:

§1 Hodge structures

§2 Symplectic manifolds and symplectic varieties

§3 Hyperkähler manifolds and twistor families

§4 Torelli Theorem

§5 Open problems

§1 Hodge structures

Definition Hodge structure of weight $k \in \mathbb{Z}$

$:\Longleftrightarrow$ • finitely generated $\mathbb{Z}$-module H

• decomposition $H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$

where $H^{p,q} \subseteq H_{\mathbb{C}}$ complex subspace

s.th. $H^{p,q} = \overline{H^{q,p}}$ (Hodge symmetry)

Theorem (Hodge, Kodaira)

1) X compact Kähler mfd $\Rightarrow$ $H^k(X,\mathbb{Z})$ carries a HS of wt k

2) $f: X \to Y$ holomorphic $\Rightarrow$ $f^*: H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})$ is a morphism of HS

Variation of Hodge structure

e.g. $f: X \to \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ proper submersion, X Kähler

$$\underset{\text{Ehresmann}}{\Rightarrow} \quad \begin{tikzcd} X \arrow[dr] & \cong & X_0 \times \Delta \arrow[dl] \\ & \Delta & \end{tikzcd} \quad \Rightarrow \quad H^2(X_t, \mathbb{Z}) =: \Lambda \quad \text{indep. of } t \in \Delta$$

$H^2(X_t, \mathbb{Z})$ does not vary, its Hodge structure does!

$$\boxed{\Lambda \otimes \mathbb{C} = H^{2,0}_t \oplus H^{1,1}_t \oplus H^{0,2}_t}$$

Period map $\phi: \Delta \to \mathcal{D}_\Lambda$

period domain

Period domain for K3 surfaces

Suppose S compact Kähler surface $\Rightarrow$ intersection pairing

$H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to \mathbb{Z}$ satisfies:

$$(*) \quad \begin{aligned} &\left(H^{p,q}(S),\ H^{r,s}(S)\right) = 0 \quad \text{unless } (p,q) = (s,r) \\ &\left(H^{2,0}(S) \oplus H^{0,2}(S)\right)^{\perp} = H^{1,1}(S) \end{aligned}$$

Definition Hodge lattice := wt 2 Hodge structure H together with pairing $(\cdot,\cdot) : H \times H \to \mathbb{Z}$ s.th. $(*)$ holds

H is of K3 type $:\Leftrightarrow$ $h^{2,0} = 1$ and $\operatorname{sign}(H) = (3,n)$

Period domain for Hodge lattices of K3 type

Fix lattice Λ of sign $(3,n)$ $\rightsquigarrow$ $\Omega_\Lambda = \{ v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (v,v) = 0,\ (v,\bar{v}) > 0 \}$

$$\begin{array}{l} \Psi \\ \omega \end{array} \rightsquigarrow \Lambda \otimes \mathbb{C} = H^{2,0}_\omega \oplus H^{1,1}_\omega \oplus H^{0,2}_\omega$$

§2 Symplectic manifolds and symplectic varieties

⚠ symplectic = holomorphic symplectic

Definition (X,σ) symplectic manifold

$:\Leftrightarrow$ X complex manifold

$\sigma : T_X \times T_X \to \mathbb{C}$ holomorphic symplectic form

with $d\sigma = 0$

Consequences 1) $\dim_{\mathbb{C}} X = 2n$ 2) $T_X \cong \Omega_X$

3) $c_1(X) = c_1(T_X) = -c_1(\Omega_X)$ $\Rightarrow c_1(X) = 0$

$\overset{2)}{=} c_1(\Omega_X)$

4) $\det \Omega_X \ni \sigma^{\wedge n}$ nowhere vanishing global section

$\Rightarrow \det \Omega_X$ trivial line bundle

Theorem (Bogomolov – Beauville) X compact Kähler mfd, $c_1(X)=0$ in $H^2(X,\mathbb{R})$. Then there is a finite topological cover $\tilde{X} \to X$ s.th.

$$\tilde{X} \cong \underset{\text{complex torus}}{T} \times \prod_i \underset{\text{Calabi-Yau}}{CY_i} \times \prod_j \underset{\text{irreducible symplectic}}{S_j}$$

Recall Y is CY $:\Leftrightarrow$ Y compact and simply connected

$\dim Y \geq 3$

$h^{0,i}(Y) = \begin{cases} 1 & \text{if } i=0, \dim Y \\ 0 & \text{else} \end{cases}$

$\det \Omega_Y$ trivial line bundle

Z is irreducible symplectic $:\Leftrightarrow$ Z compact, simply connected

$h^{2,0}(Z) = 1$

$\exists\, \sigma$ holomorphic symplectic form

Examples

1) $\mathbb{P}^3 \supset X = \{f = 0\}$, $f \in \mathbb{C}[x_0, \dots, x_3]_4$ degree

$\mathrm{Jac}(f) \neq 0$ on X

K3 surface, simply connected by Lefschetz

$$\sigma = \mathrm{Res}_X \left(\frac{x_0\, dx_1 \wedge dx_2 \wedge dx_3 + \dots + x_3\, dx_1 \wedge \dots \wedge dx_2}{f} \right)$$ symplectic

Fact: X irreducible symplectic surface $\Leftrightarrow$ X K3 surface

2) S K3 surface, $X = \mathrm{Hilb}^n(S)$, $\dim X = 2n$

Hilbert scheme of n points on S with scheme structure

$n = 4$

4 distinct pts | 3 pts, one with tangent direct | 2-jet + pt | point with tangent plane + pt

3) $Y \subset \mathbb{P}^5$ smooth cubic hypersurface

$$F = F(Y) = \{ \ell \in Gr(2,6) \mid \ell \subseteq Y \} \qquad \text{"Fano variety of lines"}$$

$$P = \{ (y, \ell) \mid y \in \ell \subset Y \}$$

$$\begin{tikzcd} & P \arrow[dl, "q"'] \arrow[dr, "p"] & \\ F & & Y \end{tikzcd}$$

$$H^{3,1}(Y) = \mathbb{C} \cdot \omega \rightsquigarrow q_* p^* \omega =: \sigma \in H^{2,0}(F)$$

symplectic form, generator

Gadget X irreducible symplectic, $H^{2,0}(X) = \mathbb{C} \cdot \sigma$

$$\Rightarrow \quad q(\alpha) := n \int_X (\sigma\bar\sigma)^{n-1} \alpha^2 + (1-2n) \int_X \sigma^n \bar\sigma^{n-1} \alpha \int_X \sigma^{n-1} \bar\sigma^n \alpha$$

Properties

- defined over $\mathbb{Z}$ (topological invariant)
- non-degenerate of sign $(3, n)$
- $(H^2(X, \mathbb{Z}), q)$ Hodge lattice of K3 type
- $q(\alpha)^n = \text{const} \cdot \int_X \alpha^{2n}$

Singular symplectic varieties

Definition X (singular) irreducible symplectic variety

$:\Leftrightarrow$ X normal Kähler variety

$H^1(X, \mathcal{O}_X) = 0$

$H^0(X^{reg}, \Omega^2_X) = \mathbb{C} \cdot \sigma$, σ symplectic on X^{reg}

$\forall \pi : Y \to X$ resolution of sing. : $\pi^*\sigma$ extends to a hol. form on Y

Remark These singular varieties behave "very smoothly":

- $H^2(X, \mathbb{Z})$ carries a pure HS of wt 2
- $\exists\, q : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ as in the smooth case $\rightsquigarrow$ Hodge lattice of K3 type
- good local deformation theory

<u>But</u>: do not always have a crepant resolution!

§ 3 Hyperkähler manifolds and Twistor families

Definition (M,g) compact Riemannian mfd is hyperkähler

$$:\Leftrightarrow \quad \mathrm{Hol}(M,g) = Sp(n) := Sp_{2n}(\mathbb{C}) \cap U_{2n}$$

(M,g) hyperkähler $\Rightarrow \exists\ I, J, K \in \mathrm{End}(T_M)$ integrable almost complex structures s.th.

$$IJ = K \quad \text{and} \quad g \text{ is Kähler wrt } I, J, K.$$

Put $\omega_I = g(I(\cdot), \cdot)$, ω_J, ω_K analogously

Kähler forms for (M,I), (M,J), (M,K)

Theorem (Beauville) X irreducible symplectic manifold, $\alpha \in \mathcal{K}_X$ Kähler class $\Rightarrow \exists!$ HK metric g on X s.th. $\alpha = [\omega_I]$.

Conversely: (M, g, I, J, K) compact hyperkähler

$\Rightarrow$ $\sigma_I := \omega_J + i\,\omega_K$ is a holomorphic symplectic form on (M, I)

Thus:

$$\boxed{\text{Compact hyperkähler manifolds} \ \overset{1:1}{\longleftrightarrow}\ \text{Irreducible symplectic manifolds}}$$

$$\xleftarrow[\text{Kähler class}]{\text{choose}} \qquad \xrightarrow[\text{Kähler metric}]{\text{forget}}$$

Twistor Construction: X irreducible symplectic, α Kähler class on X $\rightsquigarrow$ I, J, K, $X = (M, I)$

Observation: $(\alpha I + \beta J + \gamma K)^2 = -\mathrm{id} \iff t = (\alpha, \beta, \gamma) \in S^2$

and $I_t = \alpha I + \beta J + \gamma K$ integrable

Endow $M \times S^2$ with almost complex structure

$$\mathcal{J} \text{ s.th } \mathcal{J}_{(x,t)} = \left(I_t(x), I_{\mathbb{P}^1}(t)\right)$$

where $I_{\mathbb{P}^1}(t) := \left(v \mapsto t \times v\right)$

- $\mathcal{J}$ is integrable $\rightsquigarrow$ $\mathcal{X} = (M \times S^2, \mathcal{J})$ holomorphic Twistor family

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

Twistor line in the period domain:

$$\Omega_\Lambda \xrightarrow{\cong} Gr^{po}(2, \Lambda \otimes \mathbb{R})$$

$$\omega \longmapsto \left(\mathrm{Re}\, H^{2,0}_\omega, \mathrm{Im}\, H^{2,0}_\omega\right)$$

fix $W \subseteq \Lambda \otimes \mathbb{R}$ positive 3-dim'l

$$Gr^{po}(2, \Lambda \otimes \mathbb{R}) \supset Gr^{po}(2, W) \cong \mathbb{P}^1_{\mathbb{C}}$$

$$\mathbb{P}^1_W \xrightarrow{\cong} Gr^{po}(2, W)$$

Period map Given X, ω Kähler class on X,

consider $\begin{array}{c}\mathcal{X}\\ \downarrow\\ \mathbb{P}^1\end{array}$ twistor family, $\mathbb{P}^1$ simply connected

$\Rightarrow H^2(\mathcal{X}_t, \mathbb{Z}) =: \Lambda$ independent of $t \in \mathbb{P}^1$

$$\rightsquigarrow \quad \wp : \mathbb{P}^1 \to \Omega_\Lambda \quad \text{period map}$$

$$\mathbb{P}^1 \xrightarrow{\cong} \mathbb{P}^1_W \subset \Omega_\Lambda \quad \text{for } W = \langle \operatorname{Re} H^{2,0}(X), \operatorname{Im} H^{2,0}(X), \omega \rangle$$

$\rightarrow$ **Results for sing. varieties**

- Existence of metrics (Eyssidieux, Guedj, Zeriahi)
- Decomposition Theorem (Höring, Peternell, Greb, Kebekus, Guenancia, Druel, ...)

§4 Torelli Theorem for irreducible symplectic mfds

fix lattice Λ, sign $(3, n)$

$$\mathcal{M}_\Lambda := \left\{ (X,\mu) \;\middle|\; \begin{array}{l} X \text{ irreducible symplectic} \\ \mu : H^2(X,\mathbb{Z}) \to \Lambda \text{ isometry} \end{array} \right\} \Big/ \text{iso}$$

$\mathcal{M}_\Lambda$: marked moduli space

μ: marking

Period map $\quad \wp : \mathcal{M}_\Lambda \to \Omega_\Lambda \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$

$$(X,\mu) \mapsto \mu_{\mathbb{C}}\left(H^{2,0}(X)\right)$$

Theorem (Huybrechts 1998) $\mathcal{N} \subset \mathcal{M}_\Lambda \neq \emptyset$ connected component $\Rightarrow$ $\wp : \mathcal{N} \to \Omega_\Lambda$ is surjective

proof uses twistor families

Q: $\wp$ injective? Counterexample by Debarre, Namikawa

Theorem (Verbitsky 2009) $\mathcal{N} \subset \mathcal{M}_\Lambda \neq \emptyset$ connected component

$\Rightarrow \wp : \mathcal{N} \to \Omega_\Lambda$ generically injective

idea of proof: $\mathcal{N}$ is non-Hausdorff, inseparable points $\longleftrightarrow$ birational varieties

- factorize $\wp : \mathcal{N} \to \overline{\mathcal{N}} \xrightarrow{\overline{\wp}} \Omega_\Lambda$

$\uparrow$ Hausdorff quotient

uses Twistor families

- show that $\overline{\wp}$ is a topological cover

- Ω_Λ simply connected $\Rightarrow \overline{\wp}$ homeomorphism

Claim follows as MT-general HS does not allow for non-trivial birational geometry $\blacksquare$

Huybrechts + Verbitsky = Global Torelli Theorem

Q: What about singular symplectic varieties?

Theorem (Bakker – L. 2016 + 2019) Global Torelli in the formulation of Huybrechts – Verbitsky holds for singular symplectic varieties.

idea of proof: Do not have twistor families.

Consider $G = O(\Lambda) \circlearrowright \mathcal{M}_\Lambda \qquad g.(X,\mu) := (X, g\circ\mu)$

$$\begin{array}{c} G \circlearrowright \mathcal{M}_\Lambda \\ \downarrow p \text{ equivariant} \\ G \circlearrowright \Omega_\Lambda \end{array} \qquad \text{ergodic action}$$

$$\Omega_\Lambda \ni \omega \quad \leftrightarrow \quad \Lambda \otimes \mathbb{C} = H^{2,0}_\omega \oplus H^{1,1}_\omega \oplus H^{0,2}_\omega$$

rational rank $\quad \operatorname{rrk}(\omega) = \operatorname{rk}\left(\left(H^{2,0}_\omega \oplus H^{0,2}_\omega\right) \cap \Lambda\right)$

$$\operatorname{rrk}(\omega) = \begin{cases} 0 & \Leftrightarrow \quad G.\omega \text{ dense } \subseteq \Omega_\Lambda \\ 1 & \Rightarrow \quad \text{countably many totally real subvar.} \\ 2 & \Rightarrow \quad H^{1,1}_\omega = \operatorname{Pic}_\omega \otimes \mathbb{C}, \text{ countably many pts} \end{cases}$$

Image ($\wp$) $\supseteq$ $\{$rrk $= 0$ points$\}$ dense

+ open $\rightsquigarrow$ "almost surjective"

surjectivity: degeneration of HS $\rightsquigarrow$ deg. of variety

injectivity: note that $\{$rrk 0 points$\} \subseteq \Omega_\Delta$

still simply connected.

$\blacksquare$

§5 Open Problems

Twistor spaces for singular varieties?

→ How to define almost complex structures for singular varieties?
Integrability Criterion?

→ How to define hyperkähler metrics on singular varieties?

→ X singular hyperkähler variety $\Rightarrow$ $\exists$ HK metric on X^{reg}

$\Rightarrow$ twistor family
$$\begin{array}{c} \mathfrak{X}^0 \underset{C^\infty}{\cong} X^{reg} \times \mathbb{P}^1 \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

Can $\mathfrak{X}^0$ be compactified to some family?

More moderate question:

→ Given surjectivity of the period map, can we deduce existence of twistor(-like) families?

$$\begin{tikzcd} & \mathcal{M}_\Lambda \arrow[d] \\ \mathbb{P}^1 \arrow[ur, dashed, "\exists\, \Phi"] \arrow[r, "\varphi"] & \Omega_\Lambda \end{tikzcd}$$

φ can always be lifted, but we cannot show existence of a family

Unrelated to twistor:

→ Describe moduli of polarized varieties.

Compactifications? Kodaira dimension.

L. Giovenzana: Singularities of perfect cone canonical

General questions about symplectic varieties:

→ Construction of examples

→ Degenerations

→ MMP for Kähler manifolds

→ Non-Kähler situations