Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

New results and open problems about Bergman kernel asymptotics

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Motivation
Definitions
Examples
Main results
Perspectives

Motivation

- Tian-Yau-Donaldson program

- Berezin-Toeplitz quantization

- Arithmetic geometry (asymptotics of the analytic torsion)

- Quantization of Chern-Simons theory

- Random Kähler geometry, quantum chaos

- Quantum Hall effect

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Bergman Projection

- (X, Θ) Hermitian manifold, $\dim X = n$

- volume form $dv_X = \Theta^n/n!$

- $(L, h) \to X$ holomorphic Hermitian line bundle

- $L^2(X, L) =$ space of L^2 sections

- $(s, s') = \int_X \langle s(x), s'(x) \rangle_h \, dv_X(x), \; s, s' \in L^2(X, L).$

- $H^0_{(2)}(X, L) =$ space of L^2 holomorphic sections

- $P : L^2(X, L) \to H^0_{(2)}(X, L)$ Bergman projection

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Bergman Kernel

- $\{s_j : j = 1, \ldots, d_p\}$ ONB of $H^0_{(2)}(X, L)$.

- $P(\cdot, \cdot) : X \times X \to L \boxtimes L^*$
  $$P(x, y) = \sum_{j=1}^{d_p} s_j(x) \otimes s_j(y)^* \text{ Bergman kernel}$$

- $$P(x, x) = \sum_{j=1}^{d_p} |s_j(x)|^2_h \text{ Bergman density function}$$

- $(Ps)(x) = \int_X P(x, y)s(y) dv_X(y), \; s \in L^2(X, L)$

- Bergman kernel does not depend on the choice of ONB

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Example 1

- $X = \mathbb{D} \subset \mathbb{C}$, (L, h) trivial

- $H_{(2)}^0(X, L) = L^2(\mathbb{D}, d\lambda) \cap \mathcal{O}(\mathbb{D})$

- ONB: $\sqrt{\frac{j+1}{\pi}} z^j$, $j = 0, 1, \ldots \rightsquigarrow$ Bergman kernel

$$P(z, w) = \frac{1}{\pi} \sum_{j=0}^{\infty} (j+1) z^j \overline{w}^j = \frac{1}{\pi} \frac{1}{(1 - z\overline{w})^2}$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Goal

- We are actually interested in a semiclassical limit!

- $L^p = L^{\otimes p}$

- P_p the Bergman projection on $H^0_{(2)}(X, L^p)$

- Asymptotics of $P_p(x, y)$ and $P_p(x, x)$ as $p \to \infty$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Example 2

- $X = \mathbb{P}^n$, Fubini-Study metric:

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(|z_0|^2 + \ldots + |z_n|^2)$$

- $L = \mathcal{O}(1)$, $h_{FS} = (|z_0|^2 + \ldots + |z_n|^2)^{-1}$

- $H^0_{(2)}(X, L^p) = H^0(X, L^p) =$ space of homogeneous polynomials in $n+1$ variables of degree p

- Basis $s_\alpha \sim z^\alpha$, $\alpha \in \mathbb{N}_0^{n+1}$, $|\alpha| = p$, $\|s_\alpha\|_{L^2}^2 = \dfrac{\alpha!}{(n+p)!}$

- $P_p(x, y) = \displaystyle\sum_{|\alpha|=p} \frac{(n+p)!}{\alpha!} s_\alpha(x) \otimes s_\alpha(y)^*$

- $P_p(x, x) = \dfrac{(n+p)!}{p!} = p^n + b_{n-1}p^{n-1} + \ldots + b_n$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1**- invariant sR structures & Bergman kernels**

Motivation
Definitions
Examples
Main results
Perspectives

Curvature

- Hermitian holomorphic line bundle $(L, h) \to X$

- Curvature form $c_1(L, h) = \frac{\sqrt{-1}}{2\pi} (\nabla^L)^2$, ($\nabla^L$ Chern connection)

- local holomorphic frame s of L on $U \subset X \rightsquigarrow$

$$|s(x)|_h^2 = e^{-2\varphi(x)}, \quad x \in U$$

where $\varphi : U \to \mathbb{R}$ is smooth, called local weight

- $c_1(L, h)|_U = dd^c \varphi = \frac{\sqrt{-1}}{\pi} \partial \overline{\partial} \varphi$

- (L, h) positive $:\Leftrightarrow c_1(L, h)$ positive $\Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)$ positive definite

- (L, h) semipositive $:\Leftrightarrow c_1(L, h)$ semipositive $\Leftrightarrow \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)$ positive semidefinite

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Asymptotic expansion

Theorem (Catlin 1998, Zelditch 1999)

(X, ω) compact Kähler, $(L, h) \to X$, with $c_1(L, h) = \omega$. Then

$$P_p(x, x) = b_0(x)p^n + b_1(x)p^{n-1} + \ldots = \sum_{j=0}^{\infty} b_j(x)p^{n-j}, p \to \infty$$

where $b_0 = 1$.

- Tian (1990): $P_p(x, x) = b_0(x)p^n + O(p^{n-1})$

- $b_1 = \frac{1}{8\pi}r(\omega)$, $r(\omega) =$ scalar curvature of ω (Z. Lu)

- $\pi^2 b_2 = -\frac{\Delta r(\omega)}{48} + \frac{1}{96}|R^{TX}|^2 - \frac{1}{24}|\mathrm{ric}_\omega|^2 + \frac{1}{128}r(\omega)^2$

  (Z. Lu, X. Wang)

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Compatibility with Riemann-Roch-Hirzebruch

$$\int_X P_p(x,x)dv_X = \int_X \sum_j |S_j|_h^2 \, dv_X = \dim H^0(X,L^p)$$

$$\int_X P_p(x,x)dv_X = \int_X (b_0(x)p^n + b_1(x)p^{n-1} + b_2(x)p^{n-2} + O(p^{n-3})) \, dv_X$$

$$= p^n \int_X \frac{\omega^n}{n!} + p^{n-1} \int_X \frac{r(\omega)}{8\pi} \frac{\omega^n}{n!} + p^{n-2} \int_X b_2(x) \frac{\omega^n}{n!} + O(p^{n-3})$$

$$= p^n \int_X \frac{c_1(L)^n}{n!} + p^{n-1} \int_X \frac{c_1(X)}{2} \frac{c_1(L)^{n-1}}{(n-1)!} +$$

$$+ p^{n-2} \int_X \{\mathrm{Td}(T^{(1,0)}X)\}^{(4)} \frac{c_1(L)^{n-2}}{(n-2)!}$$

$$+ O(p^{n-3})$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Asymptotic expansion variation

Theorem

(X, Θ) *compact Hermitian,* $(L, h) \to X$ *positive,* $\omega = c_1(L, h)$. *Then*

$$P_p(x, x) = b_0(x)p^n + b_1(x)p^{n-1} + \ldots = \sum_{j=0}^{\infty} b_j(x)p^{n-j}, p \to \infty$$

where $b_0 = c_1(L, h)^n / \Theta^n$ *and*

$$b_1 = \frac{1}{8\pi} b_0 \Big[r(\omega) - 2\Delta_\omega \log(\det b_0) \Big]$$

$\alpha_1(x), \ldots, \alpha_n(x)$ eigenvalues of $c_1(L, h)$ w.r.t. $\Theta \rightsquigarrow$

$$b_0(x) = \alpha_1(x) \ldots \alpha_n(x)$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Proof (local index theorem: Dai/Liu/Ma, Ma/−)

- Localization uses spectral gap of the Kodaira Laplacian
  $\square_p = \overline{\partial}^* \overline{\partial} : \Omega^{0,0}(X, L^p) \to \Omega^{0,0}(X, L^p)$

- $\mathrm{Spec}\,(\square_p) \subset \{0\} \cup [C_0 p - C_1, \infty), \ C_0 = \inf_{x \in X} \alpha_1(x)$

- $f \in C_0^\infty(\mathbb{R}), \ F = \widehat{f} \rightsquigarrow$

$$\left|F(\square_p)(x,y) - P_p(x,y)\right|_{C^l} = O(p^{-\infty})$$

- $F(\square_p)(x,y)$ depends only on geometric data on $B(x,\varepsilon)$

- $\rightsquigarrow P_p(x,y)$ depends only on local data

- Work on $B^{TX}(0,\varepsilon) \equiv B(x,\varepsilon)$ with a local model Laplacian

- Rescale coordinates and develop the rescaled operator in Taylor series

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Interpretation of the first term b_0

- $P_p(x,x) = b_0(x)p^n + O(p^{n-1}) \rightsquigarrow$ holomorphic sections are spread everywhere over X ($b_0(x) \neq 0$).

- They concentrate where the curvature is strong

- $P_p(x,y)$ localizes near each fixed point x and equals approximatively a "peak section" which is close to a Gaussian $p^n \exp(-|x-y|^2/\sqrt{p})$.

- To prove this kind of localization is a key point.

- Can put peak sections near every point and they decay quickly enough that they nearly don't overlap each other.

- Heuristically then, in the limit we can find an L^2-orthonormal basis of sections parametrized by the points of X, each point x corresponding to a section localized entirely at x.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- **invariant sR structures & Bergman kernels**

Motivation
Definitions
Examples
Main results
Perspectives

Interpretation of the second term b_1

- $P_p(x,x) = b_0(x)p^n + b_1(x)p^{n-1} + O(p^{n-2})$

- $b_1(x)$ is essentially the scalar curvature of the metric $\omega = c_1(L, h)$

- Scalar curvature measures the difference in volume of a small geodesic ball compared with the volume of a Euclidean ball of the same radius.

- So the scalar curvature tells us how closely we can push together the peaked sections making up our L^2-orthonormal basis from above.

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Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Tian's approximation

Theorem

$(L, h) \to X$ positive, Kodaira map:
$\Phi_p : X \to \mathbb{P}(H^0(X, L^p)^*)$, $x \mapsto \{S \in H^0(X, L^p) : S(x) = 0\}$ Then

$$\Big| \frac{1}{p} \Phi_p^*(\omega_{FS}) - c_1(L, h) \Big|_{C^\ell} \leq \frac{C_\ell}{p}$$

Proof.
.

$$\frac{1}{p} \Phi_p^*(\omega_{FS}) = c_1(L, h) + \frac{\sqrt{-1}}{2\pi p} \partial \overline{\partial} \log P_p(x, x)$$

$$c_1(L, h) + \frac{\sqrt{-1}}{2\pi p} \partial \overline{\partial} \log p^n \big(b_0(x) + O\big(\frac{1}{p}\big) \big)$$

□

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Generalizations

- Semipositive bundles (asymptotics on the positive part)

- Noncompact manifolds (asymptotics on compact sets)

- Orbifolds

- Symplectic manifolds

- Singular metrics

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Motivation
Definitions
Examples
Main results
Perspectives

Open questions

- Semipositive bundles (asymptotics near degenerate points)

- Noncompact manifolds (uniform asymptotics near infinity)

- Complex spaces (asymptotics near singularities)

- Manifolds with boundary (asymptotics near the boundary)

- Partial Bergman kernels

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Sub-Riemannian (sR) geometry

Sub-Riemannian (sR) geometry is the study of metric distributions $(Y, E \subset TY, g^E)$ inside the tangent space.
Subbundle E is assumed to be *bracket-generating*.
Peculiar phenomena (Hausdorff dimension & abnormal geodesics..) arise.

References:

- R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*. 2002.

- M. Gromov, *Carnot-Carathéodory spaces seen from within*, 1996,
  (in Bellaïche & Risler, *Sub-Riemannian geometry*)

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Bracket-generating distributions

$E \subset TY$ bracket generating: $C^\infty(E)$ generates $C^\infty(TY)$ under Lie bracket $[,]$.

Examples:
1. Contact case: $E^{2m} = \ker\alpha \subset TY^{2m+1}$; rank $d\alpha|_E = 2m$.
Normal form (Darboux): $\alpha = dy_3 - y_2 dy_1$; $E = \mathbb{R}[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3}]$
Generation (1 step):$[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3}] = \partial_{y_3}$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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2. Martinet case: $E^2 = \ker\alpha \subset TY^3$, $\underbrace{Z^2}_{\text{union of hypersurfaces}} = \{\alpha \wedge d\alpha = 0\} \subset Y$

with $TY \pitchfork E$.
Normal form: $\alpha = dy_3 - y_2^2 dy_1$; $E = \mathbb{R}[\partial_{y_2}, \partial_{y_1} + y_2^2 \partial_{y_3}]$
Generation (2 step):$[\partial_{y_2}, \partial_{y_1} + y_2^2 \partial_{y_3}] = 2y_2 \partial_{y_3}$, $[\partial_{y_2}, [\partial_{y_2}, \partial_{y_1} + y_2^2 \partial_{y_3}]] = 2\partial_{y_3}$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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3. Goursat case: $E^2 \subset TY^n$ (... some general definition ...)
Normal form (eg.): $E = \mathbb{R}[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3} + \ldots + y_{n-1}\partial_{y_n}]$
Generation (n-2 step): $[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3} + \ldots + y_{n-1}\partial_{y_n}] = \partial_{y_3}$,
$[[\partial_{y_2}, \partial_{y_1} + y_2\partial_{y_3} + \ldots + y_{n-1}\partial_{y_n}], \partial_{y_1} + y_2\partial_{y_3} + \ldots + y_{n-1}\partial_{y_n}] = \partial_{y_4} \ldots$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Flag, metric & dimension

In general defines canonical flag:

$$\{0\} = E_0 \subset \underbrace{E_1}_{=E} \subset \ldots \subset E_{r(y)} = TY$$

by $E_{j+1} = [E, E_j]$, $j \geq 1$.
Step$= r(y)$,   Growth vector $= k^E(y) = \left(\dim E_0, \dim E_1, \ldots, \dim E_{r(y)}\right)$.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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E called equiregular if $k^E(y)$ is constant.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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(Chow-Rashevsky '37) E bracket-generating $\implies$ any two $y_1, y_2 \in Y$ connected by
horizontal path $\gamma \in C^{0,1}([0,1]_t; Y), \dot\gamma(t) \in E_{\gamma(t)}$ a.e.

(Y, d^E) is a metric space with $d^E = \inf_{\gamma \text{ horizontal}} \int_0^1 dt \, |\dot\gamma(t)|$.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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(Y, d^E) is a metric space with $d^E = \inf_{\gamma \text{ horizontal}} \int_0^1 dt \, |\dot{\gamma}(t)|$.

(Ball-Box Thm)

$$\underbrace{Q(y)}_{\text{Hausdorff dimension}} := \lim_{\varepsilon \to 0} \frac{\ln \text{vol} B_\varepsilon(y)}{\ln \varepsilon} = \sum_{j=1}^{r(x)} j \left[k_j(y) - k_{j-1}(y) \right] > n$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Geodesics

A geodesic connecting two points y_1, y_2 is a distance minimizer

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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(Bismut '84) (Y, d^E) metric complete $\implies$ any two $y_1, y_2 \in X$ connected by geodesic $\gamma \in C^{0,1}\left([0,1]_t ; Y\right)$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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(Bismut '84) $\left(Y, d^E\right)$ metric complete $\implies$ any two $y_1, y_2 \in X$ connected by geodesic $\gamma \in C^{0,1}\left([0, 1]_t ; Y\right)$

Hamiltonian trajectories $H\left(y, \xi\right) := \left\| \xi|_{E_y} \right\|^2$ project to minimizers (always horizontal).

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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Hamiltonian trajectories $H(y, \xi) := \left\| \xi|_{E_y} \right\|^2$ project to minimizers (always horizontal).

Not all minimizers obtained this way!

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Abnormal geodesics

R. Montgomery '94 found an *abonormal* minimizer (not a Hamiltonian projection).

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Abnormal geodesics

R. Montgomery '94 found an *abonormal* minimizer (not a Hamiltonian projection).

Example. $Y = \mathbb{R}^3$, $E = \ker\left[dy_3 - y_2^2 dy_1\right]$, has vanishing hypersurface
$Z = \{\alpha \wedge d\alpha = 0\} = \{y_2 = 0\}$.
Consider $\gamma(t) = (t, 0, 0)$ along y_1-axis.
Minimizes regardless of metric (i.e. topological minimizer)!
C^1-isolated among horizontal curves.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

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Consider $\gamma(t) = (t, 0, 0)$ along y_1-axis.
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C^1-isolated among horizontal curves.

Lack understanding of abnormals in general
Open question: Are abnormal minimizers smooth?

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Sub-Riemannian (sR) geometry
Examples
Hausdorff dimension
Abnormal geodesics

Abnormal geodesics

R. Montgomery '94 found an *abonormal* minimizer (not a Hamiltonian projection).

Example. $Y = \mathbb{R}^3$, $E = \ker \left[dy_3 - y_2^2 dy_1 \right]$, has vanishing hypersurface
$Z = \{ \alpha \wedge d\alpha = 0 \} = \{ y_2 = 0 \}$.
Consider $\gamma(t) = (t, 0, 0)$ along y_1-axis.
Minimizes regardless of metric (i.e. topological minimizer)!
C^1-isolated among horizontal curves.

Lack understanding of abnormals in general
Open question: Are abnormal minimizers smooth?

Well understood in cases:

- Contact case: none.
- Martinet case: Integral curves of $\ker \alpha|_Z =: L^E \to Z$ (topological)

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

sR Laplacian

Let $\left(Y, E \subset TY, g^E\right)$ sR manifold.
Choose an auxiliary density μ to define

$$\text{sR Laplacian}: \quad \Delta_{g^E, \mu} := \left(\nabla^{g^E}\right)^*_\mu \circ \nabla^{g^E}$$

where $g^E\left(\nabla^{g^E} f, U\right) = U(f)$, $\forall U \in C^\infty(E)$ is sR-gradient.

Changing the density: $\Delta_{g^E, h\mu} = h^{-1/2} \Delta_{g^E, \mu} h^{1/2} + h^{-1/2}\left(\Delta_{g^E, \mu} h^{1/2}\right)$

Characteristic variety: $\Sigma = \left\{\sigma\left(\Delta_{g^E, \mu}\right) = H^E = 0\right\} = E^\perp$.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

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(Hormander '67) E bracket generating $\implies \Delta_{g^E,\mu}$ is hypoelliptic

(Rothschild & Stein '76) $\|f\|^2_{H^{1/r}} \lesssim \left\langle \Delta_{g^E,\mu} f, f\right\rangle + \|f\|^2_{L^2}$ where $r = \max_{x \in X} r(x)$.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

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Discrete spectrum (φ_j, λ_j); $\Delta_{g^E,\mu}\varphi_j = \lambda_j\varphi_j$, on a compact manifold.
Spectral asymptotics questions: Weyl law, trace formula, propagation, ergodicity ...
(mostly open)

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

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Do Hausdorff dimension, abnormal geodesics play a role?

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

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Does the spectrum see the Hausdorff dimension?

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
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Does the spectrum see the Hausdorff dimension?

> ### Theorem (Ben Arous 1989, Léandre 1992... Barilari 2011, Colin de Verdiere-Hillairet-Trélat)
>
> *There exist* $a_j(y) \in C^\infty(Y)$, $j = 0, 1, \ldots$,
>
> $$e^{-t\Delta_{g^E,\mu}}(y,y) \sim t^{-Q(y)/2} \left[\sum_{j=0}^{\infty} a_j(y) t^j \right].$$

The expansion is in general <u>not</u> uniform in $x \in X$. Does not yield trace asymptotics.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

sR heat trace

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Theorem (Métivier 1976, Colin De Verdiere-Hillairet-Trélat)

If E is equiregular

$$N(\lambda) \sim \underbrace{\frac{\lambda^{Q/2}}{\Gamma(Q/2+1)}\int_X a_0}_{=vol\{H^E \le \lambda\}}.$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

Spectrum and dynamics

Does the spectrum see the abnormals?

Theorem (Melrose 1984)

(X^3, E^2) *3D contact. Then*

$$sing \; spt \left(tr \; e^{it\sqrt{\Delta_{g^E, \mu}}} \right) \subset \{0\} \cup \{lengths \; of \; (normal) \; geodesics\}$$

$$N(\lambda) \sim c\lambda^2 + O\left(\lambda^{3/2}\right).$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
sR wave trace

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Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

sR Laplacian
sR heat trace
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Theorem (S.)

(X^4, E^3) *4D quasi-contact. Suppose any invariant subset of characteristic foliation* $L^E \subset TX$ *is of zero or full measure. Then one has quantum ergodicity (QE).*

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

Circle bundles

Natural place for sR-structures: $\left(\underbrace{Y^n}_{S^1 L}, \underbrace{E^{n-1}}_{HX \text{ horizontal}}, \underbrace{g^E}_{\pi^* g^{TX}} \right)$ with

$\left(L, h^L, \nabla^L \right) \to \left(X^{n-1}, g^{TX} \right)$ is a Hermitian line bundle with connection.
Equivalently consider sR structure invariant by a free and transversal S^1 action.

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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E bracket generating $\iff R^L$ vanishes to finite order (i.e. $r = \max_{x \in X} r_x < \infty$)

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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Decomposition: $X = \cup_{j=2}^r X_j$; $X_j = \{x | r_x = j\}$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

Bochner Laplacian

Fourier: $C^\infty(Y) = \oplus_{p=-\infty}^\infty C^\infty(X, L^p)$; $\underbrace{\Delta_{gE, \mu_Y}}_{\text{sR Laplacian}} = \oplus_{p=-\infty}^\infty \underbrace{\Delta_p}_{=(\nabla^{L^p})^* \nabla^{L^p} \text{ Bochner}}$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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Theorem (Marinescu-S.)

The first eigenfunction/eigenvalue (ψ_0^p, λ_0^p) of the Bochner Laplacian Δ_p satisfy

$$\lambda_0^p \sim c_0 p^{2/r}$$

$$\left|\psi_0^p(x)\right| = O\left(p^{-\infty}\right), \quad x \notin X_r.$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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Theorem (Marinescu-S.)

Assume X_r submanifold with R^L non-degenerate

$$\lambda_0^p \sim p^{2/r}\left[c_0 + c_1 p^{-1/r} + c_2 p^{-2/r} + \dots\right]$$
$$N\left(ap^{2/r}, bp^{2/r}\right) \sim p^{\dim(X_r)} C_{a,b}$$

R. Montgomery '95 ($\dim Y = 2$, $r = 3$), Helffer-Mohamed '96, Helffer-Kordyukov '09 (Y_r hypersurface of transverse vanishing).

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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If X cpx. and L holomorphic one has Kodaira Laplacian
$\Box_p : \Omega^{0,*}(X; L^p) \to \Omega^{0,*}(X; L^p)$.
2D Weitzenbock formula:

$$2\Box_p = \Delta_p + k\left[R^L(w, \overline{w})\right], \quad \text{on } \Omega^{0,1}.$$

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

Bergman kernel

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Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

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Local index theory technique of Bismut-Lebeau '91, Dai-Liu-Ma '06 gives

Theorem (Marinescu-S.)

For $dim X = 2$ & R^L semi-positive of finite order

$$P_p(x,x) \sim p^{2/r_x}\left[\sum_{j=0}^{N} b_j(x) p^{-j/r_x}\right]$$

where $r_x - 2 = ord(R_x^L)$.

R. Berman 2009 (on positive part away from base locus), Hsiao-Marinescu 2014 (on positive part when twisted by canonical).

Bergman Kernels
Sub-Riemannian (sR) geometry
sR spectral geometry
S^1- invariant sR structures & Bergman kernels

Circle bundles
Bochner Laplacian
Bergman kernel

Thank you.