TRACE DEFECT FORMULAE FOR GEOMETRIC OPERATORS
New trends and open problems in Global Analysis and Geometry
based on joint work with S. AZZALI and joint work with G. HABIB

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Non local operators
PART I

Non local operators
Pseudodifferential operators

Notations

- $M$ an $n$-dimensional smooth closed manifold;
- $\pi : E \to M$ a finite rank vector bundle;
- $C^\infty(M, E)$ the space of smooth sections of $E$;
- $\Psi_{cl}(M, E)$ the algebra of polyhomogeneous (or classical) pseudodifferential operators acting on $C^\infty(M, E)$; we write $\Psi_{cl}(M)$ if $E = M \times \mathbb{C}$.

Example

- $(M, g)$ a Riemannian manifold, $E = M \times \mathbb{C}$, $\Delta_g = - \sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j$ the Laplace-Beltrami operator: $(\Delta_g + \pi_g)^{-1} \in \Psi_{cl}^{-2}(M)$;
- $M$ a spin manifold and $E = S$ the spinor bundle, $D^2$ the square of the Dirac operator $D = \sum_{i=1}^n \gamma_i \partial_i$: $\log(D^2 + \pi_D) \notin \Psi_{cl}(M, E)$.

Classes of pseudodifferential operators determined by their order

For $\Gamma \subset \mathbb{C}$, let $\Sigma^\Gamma(M, E) := \{ A \in \Psi_{cl}(M, E), \text{ord}(A) \in \Gamma \}$. **Examples:** The class $\Psi_{cl}^\mathbb{Z}(M, E)$ (resp. $\Psi_{cl}^{\mathbb{Q}/\mathbb{Z}}(M, E)$) of integer order (resp. noninteger order) classical pseudodifferential operators.
# Pseudodifferential operators

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## Classes of pseudodifferential operators determined by their order

For $\Gamma \subset \mathbb{C}$, let $\Sigma^\Gamma(M, E) := \{A \in \Psi_{\text{cl}}(M, E), \text{ord}(A) \in \Gamma\}$. **Examples:** The class $\Psi_{\text{cl}}^\mathbb{Z}(M, E)$ (resp. $\Psi_{\text{cl}}^{\mathbb{Q}/\mathbb{Z}}(M, E)$) of integer order (resp. noninteger order) classical pseudodifferential operators.
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For \( \Gamma \subset \mathbb{C} \), let \( \Sigma^\Gamma(M, E) := \{ A \in \Psi_{\text{cl}}(M, E), \text{ord}(A) \in \Gamma \} \). **Examples:** The class \( \Psi_{\text{cl}}^\mathbb{Z}(M, E) \) (resp. \( \Psi_{\text{cl}}^{\mathbb{Q}/\mathbb{Z}}(M, E) \)) of integer order (resp. noninteger order) classical pseudodifferential operators.
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- **M** an **n-dimensional smooth closed manifold**;
- **π**: **E** → **M** a finite rank vector bundle;
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- **Ψ_{cl}(M, E)** the algebra of **polyhomogeneous (or classical) pseudodifferential operators** acting on **C^∞(M, E)**; we write **Ψ_{cl}(M)** if **E** = **M** × **C**.

Example

- **(M, g)** a **Riemannian manifold**, **E** = **M** × **C**, **Δ_g** = −∑_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} √g \partial_j the **Laplace-Beltrami operator**: (**Δ_g** + **π_g**)^{−1} ∈ **Ψ_{cl}^{−2}(M)**;
- **M** a **spin manifold** and **E** = **S** the spinor bundle, \( D^2 \) the square of the **Dirac operator** \( D = \sum_{i=1}^n \gamma_i \partial_i : \log(D^2 + π_D) \notin **Ψ_{cl}(M, E)**.

Classes of pseudodifferential operators determined by their order

For \( Γ ⊂ \mathbb{C} \), let \( \Sigma^Γ(M, E) := \{ A ∈ **Ψ_{cl}(M, E) , ord(A) ∈ Γ \} \). **Examples**: The class **Ψ_{cl}^{\mathbb{Z}}(M, E)** (resp. **Ψ_{cl}^{\mathbb{Q}/\mathbb{Z}}(M, E)**) of integer order (resp. noninteger order) classical pseudodifferential operators.
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Locality versus non-locality

Definition

A ∈ Ψ_cl(M, E) is local if it satisfies the two equivalent conditions:

- it preserves the support \( \text{Supp}(A\phi) \subset \text{Supp}(\phi) \) for \( \phi \in C^\infty(M) \);
- (locality relation) \( \text{Supp}(\phi) \cap \text{Supp}(\psi) = \emptyset \implies A\phi \psi = 0 \) for \( \phi, \psi \in C^\infty(M) \).

Local pseudodifferential operators

A ∈ Ψ_cl(M, E)

- is in general only micro-local, it preserves the support of singularities \( \text{WF}(Au) \subset \text{WF}(u) \), so in particular \( \text{Supp}_{\text{sing}}(Au) \subset \text{Supp}_{\text{sing}}(u) \) \( \forall u \in \mathcal{D}'(M) \).
- it is local if and only if it is a differential operator.

\( \epsilon \)-locality, \( \epsilon \geq 0 \)

A properly supported operator \( A \in \Psi_{\text{cl}}(M, E) \) is \( \epsilon \)-local (finite propagation) i.e., it satisfies the two equivalent conditions:

- it preserves the support modulo an \( \epsilon \)-perturbation \( \text{Supp}(A\phi) \subset \text{Neigh}_\epsilon(\text{Supp}(\phi)) \) for all \( \phi \in C^\infty(M) \);
- \( \phi \perp^\epsilon \psi : \iff d(\text{Supp}(\phi), \text{Supp}(\psi)) > \epsilon \iff A\phi \psi = 0 \) for all \( \phi \in C^\infty(M) \).

A 0-local operator \( A \) is local: \( \phi \perp^0 \psi \implies A\phi \psi = 0 \).
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Locality versus non-locality

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\( A \in \Psi_{cl}(M, E) \) is *local* if it satisfies the two equivalent conditions:

- *it preserves the support* \( \text{Supp}(A\phi) \subset \text{Supp}(\phi) \) for \( \phi \in C^\infty(M) \);
- *(locality relation)* \( \text{Supp}(\phi) \cap \text{Supp}(\psi) = \emptyset \implies \phi A \psi = 0 \) for \( \phi, \psi \in C^\infty(M) \).

**Local pseudodifferential operators**

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A properly supported operator \( A \in \Psi_{cl}(M, E) \) is *\( \epsilon \)-local* (finite propagation) i.e., it satisfies the two equivalent conditions:

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- \( \phi \top_\epsilon \psi :\iff d(\text{Supp}(\phi), \text{Supp}(\psi)) > \epsilon \implies \phi A \psi = 0 \) for all \( \phi \in C^\infty(M) \).

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Local pseudodifferential operators

\( A \in \Psi_{cl}(M, E) \)

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- it is local if and only it is a differential operator.

$\epsilon$-locality, $\epsilon \geq 0$

A properly supported operator $A \in \Psi_{\text{cl}}(M, E)$ is $\epsilon$-local (finite propagation) i.e., it satisfies the two equivalent conditions:

- it preserves the support modulo an $\epsilon$-perturbation $\text{Supp}(A\phi) \subset \text{Neigh}_\epsilon(\text{Supp}(\phi))$ for all $\phi \in C^\infty(M)$;
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A 0-local operator $A$ is local: $\phi \mathbf{T}^0 \psi \implies \phi A \psi = 0$. 

Sylvie Paycha, University of Potsdam, on leave from Université Clermont-Auvergne
Locality versus non-locality

Definition

\( A \in \Psi_{cl}(M, E) \) is local if it satisfies the two equivalent conditions:

- it preserves the support \( \text{Supp}(A\phi) \subset \text{Supp}(\phi) \) for \( \phi \in C^\infty(M) \);
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Local pseudodifferential operators

\( A \in \Psi_{cl}(M, E) \)

- is in general only micro-local, it preserves the support of singularities \( \text{WF}(Au) \subset \text{WF}(u) \), so in particular \( \text{Supp}_{\text{sing}}(Au) \subset \text{Supp}_{\text{sing}}(u) \) \( \forall u \in \mathcal{D}'(M) \).
- it is local if and only if it is a differential operator.

\( \epsilon \)-locality, \( \epsilon \geq 0 \)

A properly supported operator \( A \in \Psi_{cl}(M, E) \) is \( \epsilon \)-local (finite propagation) i.e., it satisfies the two equivalent conditions:

- it preserves the support modulo an \( \epsilon \)-perturbation \( \text{Supp}(A\phi) \subset \text{Neigh}_\epsilon(\text{Supp}(\phi)) \) for all \( \phi \in C^\infty(M) \);
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Locality versus non-locality

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Locality versus non-locality

Definition

$A \in \Psi_cl(M, E)$ is **local** if it satisfies the two equivalent conditions:

- it preserves the support $\text{Supp}(A\phi) \subset \text{Supp}(\phi)$ for $\phi \in C^\infty(M)$;
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Local pseudodifferential operators

$A \in \Psi_cl(M, E)$

- is in general only micro-local, it preserves the support of singularities $\text{WF}(Au) \subset \text{WF}(u)$, so in particular $\text{Supp}_{\text{sing}}(Au) \subset \text{Supp}_{\text{sing}}(u)$ $\forall u \in D'(M)$.
- it is local if and only it is a differential operator.

$\epsilon$-locality, $\epsilon \geq 0$

A properly supported operator $A \in \Psi_cl(M, E)$ is **$\epsilon$-local** (finite propagation) i.e., it satisfies the two equivalent conditions:

- it preserves the support modulo an $\epsilon$-perturbation $\text{Supp}(A \phi) \subset \text{Neigh}_\epsilon(\text{Supp}(\phi))$ for all $\phi \in C^\infty(M)$;
- $\phi \triangledown^\epsilon \psi \iff d(\text{Supp}(\phi), \text{Supp}(\psi)) > \epsilon \implies \phi A \psi = 0$ for all $\phi \in C^\infty(M)$.

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Locality versus non-locality

Definition

\( A \in \Psi_{\mathrm{cl}}(M, E) \) is \textit{local} if it satisfies the two equivalent conditions:

- \textit{it preserves the support} \( \text{Supp}(A\phi) \subset \text{Supp}(\phi) \) for \( \phi \in C^\infty(M) \);
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Local pseudodifferential operators

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Definition

A ∈ Ψ\text{cl}(M, E) is local if it satisfies the two equivalent conditions:

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Locality versus non-locality

Definition

A ∈ Ψcl(M, E) is local if it satisfies the two equivalent conditions:

- it preserves the support Supp(Aφ) ⊂ Supp(φ) for φ ∈ C∞(M);
- (locality relation) Supp(φ) ∩ Supp(ψ) = ∅ =⇒ φ A ψ = 0 for φ, ψ ∈ C∞(M).

Local pseudodifferential operators

A ∈ Ψcl(M, E)

- is in general only micro-local, it preserves the support of singularities WF(Au) ⊂ WF(u), so in particular SuppSing(Au) ⊂ SuppSing(u) ∀u ∈ D'(M).
- it is local if and only it is a differential operator.

ε-locality, ε ≥ 0

A properly supported operator A ∈ Ψcl(M, E) is ε-local (finite propagation) i.e., it satisfies the two equivalent conditions:

- it preserves the support modulo an ε-perturbation Supp(Aφ) ⊂ Neighε(Supp(φ)) for all φ ∈ C∞(M);
- φ ⊩ ψ :⇐⇒ d(Supp(φ), Supp(ψ)) > ε =⇒ φ A ψ = 0 for all φ ∈ C∞(M).

A 0-local operator A is local: φ ⊩ 0 ψ =⇒ φ A ψ = 0.
Pseudodifferential operators on manifolds are "tamely" non-local

"Tame" non-locality for pseudodifferential operators

For any $A \in \Psi_{\text{cl}}(M, E)$ and any $\epsilon > 0$, there exists $A_0 \in \Psi_{\text{cl}}(M, E)$ $\epsilon$-local such that

$$A - A_0 =: S_A \in \Psi^{-\infty}(M, E)$$

has smooth kernel supported outside the diagonal.

Notations

- $U = (U_i)_{i \in I}$ is a finite open cover of $M$;
- $(\chi_i)_{i \in I}$ is a partition of unity subordinated to $U$;
- $A \in \Psi^{-\infty}(M, E) := \cap_{r \in \mathbb{R}} \Psi_{r, \text{cl}}(M, E)$ has smooth Schwartz kernel.

"Tame" non-locality (following Shubin)

For $A \in \Psi_{\text{cl}}(M, E)$

$$A = \sum_{i,j} \chi_i A \chi_j = \sum_{\text{Supp}(\chi_i) \cap \text{Supp}(\chi_j) \neq \emptyset} A_{ij} + \sum_{\text{Supp}(\chi_i) \cap \text{Supp}(\chi_j) = \emptyset} A_{ij}, \quad (1)$$

$$A_0 =: \text{Op}(\sigma(A))$$

is $\epsilon$-local

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For any \( A \in \Psi_{\text{cl}}(M, E) \) and any \( \epsilon > 0 \), there exists \( A_0 \in \Psi_{\text{cl}}(M, E) \) \( \epsilon \)-local such that

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where $A_0 = \text{Op}(\sigma(A))$ is $\epsilon$-local and $S_A \in \Psi^{-\infty}(M, E)$.
Pseudodifferential operators on manifolds are "tamely" non-local

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Local linear forms
Local linear forms
$\mathcal{T}^0$- locality

$\mathcal{T}^0$- locality on $C^\infty(M)$

$\phi \mathcal{T}^0 \psi :\iff \text{Supp}(\phi) \cap \text{Supp}(\psi) = \emptyset.$

$\mathcal{T}^0$- locality on linear forms

A "linear" form $\Lambda$ on $\Sigma^r(M, E)$ is $\mathcal{T}^0$- local if for any $\phi, \psi \in C^\infty(M)$

$\phi \mathcal{T}^0 \psi \implies \Lambda(\phi A \psi) = 0$ (2)
$T^0$- locality on $C^\infty(M)$

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$T^0$- locality on linear forms

A "linear" form $\Lambda$ on $\Sigma T(M, E)$ is $T^0$- local if for any $\phi, \psi \in C^\infty(M)$

$\phi T^0 \psi \implies \Lambda(\phi A \psi) = 0 \quad (2)$
\(T^0\)-locality

**T\(^0\)-locality on** \(C^\infty(M)\)

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**T\(^0\)-locality on linear forms**

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\( T^0 \)- locality on linear forms

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Local linear forms: the canonical trace and the residue

A $T^0$-local form $\Lambda$ is local (proved for $E = M \times \mathbb{C}$)

- $\Lambda(A) = \Lambda(\text{Op}(\sigma(A)))$ only depends on the symbol $\sigma(A)$
- in fact, $\Lambda$ is local, i.e. of the form

$$\Lambda(A) = \int_M \omega_A^\Lambda(x) \, dx,$$

for some linear form $\lambda$ on the symbol class of $\Sigma^\Gamma(M, E)$ and under additional continuity assumptions.

Characterisation of local "linear" forms (with S. AZZALI 2016)

Let $\Lambda : \Sigma^\Gamma(M, E) \to \mathbb{C}$ be a local linear form:

- if $\Gamma = \mathbb{Z}$, then $\Lambda$ is proportional to the Wodzicki residue:

$$\text{Res}(A) = \int_M \text{Res}_x(A) \, dx; \quad \text{Res}_x(A) = \int_{|\xi| = 1} \text{tr}_x \sigma_n(A)(x, \cdot).$$

- if $\Gamma = \mathbb{C} \setminus \mathbb{Z}$, then $\Lambda$ proportional to the canonical trace:

$$\text{TR}(A) = \int_M \text{TR}_x(A) \, dx; \quad \text{TR}_x(A) = \int_{\mathbb{R}^n} \text{tr}_x \sigma(A)(x, \cdot).$$
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\]

- if $\Gamma = \mathbb{C} \setminus \mathbb{Z}$, then $\Lambda$ proportional to the canonical trace:

\[
\text{TR}(A) = \int_M \text{TR}_x(A) \, dx; \quad \text{TR}_x(A) = \int_{\mathbb{R}^n} \text{tr}_x (\sigma(A)(x, \cdot)).
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Let $\Lambda : \Sigma^\Gamma(M, E) \to \mathbb{C}$ be a local linear form:

- if $\Gamma = \mathbb{Z}$, then $\Lambda$ is proportional to the Wodzicki residue:

$$\text{Res}(A) = \int_M \text{Res}_x(A) \, dx; \quad \text{Res}_x(A) = \int_{|\xi_x| = 1} \text{tr}_x (\sigma_{-\eta}(A)(x, \cdot)) \, .$$

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Defect formulae measure defects of regularised traces (built from the canonical trace) in terms of the Wodzicki residue (which is local).

**Defect formulae (with S. SCOTT 2007)**

Let $A(z) \in \Psi_{cl}(M, E)$ be a holomorphic family of order $-qz + a$.

\[
A(0) \text{ is differential } \implies \lim_{z \to 0} (\text{TR}(A(z))) = \frac{1}{q} (\text{Res}(A'(0))) \text{ is local.}
\]

**ζ-regularised trace of differential (so local) operators**

Take $A(z) = A Q^{-z}$ for $A(0) = A$ differential and $Q$ elliptic pseudodifferential operator of order $q > 0$ (e.g. a Laplacian) with spectral cut: the ζ-regularised trace of $A$ with weight/regulator $Q$ reads

\[
\zeta_{A,Q}(0) := \lim_{z \to 0} (\text{TR}(A Q^{-z})) = -\frac{1}{q} \text{Res} \left( A \left\{ \log(Q) \right\} \right).
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Consequence: the index as a residue (on closed manifolds)

Notations

- \((M, g)\) Riemannian closed manifold;
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How defect formulae come in \((A = \text{Id}, Q = \Delta, q = 2)\)

\[
\text{ind}(D_+) = \dim(\ker(D_+)) - \dim(\ker(D_-)) = \text{Tr}(\pi_{D_+}) - \text{Tr}(\pi_{D_-})
\]

\[
= \text{Tr}((D_-D_+ + \pi D_+)^{-z}) - \text{Tr}((D_+D_- + \pi D_-)^{-z}) \quad \text{Re}(z) >> 0
\]

since non zero eigenvalues of \(D_\pm\) cancel pairwise

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How defect formulae come in ($A = Id$, $Q = \Delta$, $q = 2$)

\[
\text{ind}(D_+) = \dim(\ker(D_+)) - \dim(\ker(D_-)) = \text{Tr}(\pi_{D_+}) - \text{Tr}(\pi_{D_-}) \\
= \text{Tr}((D_-D_+ + \pi_{D_+})^{-z}) - \text{Tr}((D_+D_- + \pi_{D_-})^{-z}) \quad \text{Re}(z) \gg 0
\]

since non zero eigenvalues of $D_{\pm}$ cancel pairwise

\[
= s\text{TR}((\Delta + \pi_{\Delta})^{-z}) \quad \text{(meromorphic extension)}
\]

\[
= \lim_{z \rightarrow 0} s\text{TR} \left( \frac{Id}{\Delta} \left( \Delta + \pi_{\Delta} \right)^{-z} \right) = -\frac{1}{2} \text{sRes} \left( \log \Delta \right) \quad \text{(defect formula),}
\]

Sylvie Paycha, University of Potsdam, on leave from Université Clermont-Auvergne
Consequence: the index as a residue (on closed manifolds)

Notations

- \((M, g)\) Riemannian closed manifold;
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\]
The index is local as an integral of a differential form $\omega$

$$\text{ind}(D_+) = \int_M \omega(x),$$

with $\omega$ expressed in terms of the curvature $R$.

- If $\dim M = 2k$, the Chern-Gauss-Bonnet index theorem (1850, 1945) on $\Omega(M)$ with the natural $\mathbb{Z}_2$-grading.

$$\text{ind} ((d + d^*)_+) = \chi(M) = \int_M \text{Pfaffian}(R)(x).$$

- If $\dim M = 4k$, the Hirzebruch signature theorem (1966) on $\Omega(M)$ with the Hodge-star operator $\mathbb{Z}_2$-grading.

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For us to keep in mind: Locality of the index (Atiyah and Singer (1963))

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Rescaling at a point
Rescaling at a point
Geometric operators

Deformation to the normal cone

\[ M \longmapsto M^\#: = (M \times \mathbb{R}^*) \cup (T_{x_0}M \times \{0\}) . \]

For \( \lambda \in \mathbb{R}^* \) define

\[ f_{x_0,\lambda} : U_{x_0}^\lambda = \exp_{x_0} B_r / |\lambda| \longmapsto U_{x_0} = \exp_{x_0} B_r \text{ by} \]

\[ f_{x_0,\lambda}(\exp_{x_0} u) = \exp_{x_0}(\lambda u) . \]

Rescaled operators (with G. HABIB (2008))

A differential operator \( A \) is geometric of degree \( \deg(A) \) if \( \deg(A) \) is the largest real number \( d \) (so such a number should exist!) such that for any \( x_0 \in M \), \( \lambda^{-d} f_{x_0,\lambda}^\# A \) converges as \( \lambda \to 0 \) and we denote the rescaled limit operator by

\[ A_{x_0}^{\text{resc}} := \lim_{\lambda \to 0} \left( \lambda^{-\deg(A)} f_{x_0,\lambda}^\# A \right) . \] (3)

Relation to Gilkey’s invariant polynomials

A differential operator \( A(g) = \sum_{|\alpha| \leq a} A_\alpha(X, g) \partial^\alpha_x \) whose coefficients are invariant polynomials \( A_\alpha(X, g) \) in the metric \( g \), is geometric with degree

\[ \deg(A(g)) = \min_\alpha d_\alpha ; \quad d_\alpha = \deg^{G_i}(A_\alpha) - |\alpha| . \]

At a point \( x_0 \in M \), the limit rescaled differential operator reads

\[ (4) \]
Geometric operators

**Deformation to the normal cone**

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At a point \( x_0 \in M \), the limit rescaled differential operator reads

\[ \left. \right|_{x_0}. \]
Geometric operators

Deformation to the normal cone \( M \) \( \mapsto \overline{M} := (M \times \mathbb{R}^*) \cup (T_{x_0}M \times \{0\}) \).

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\[
\left( \lambda^{-\deg(A)} f_{x_0,\lambda}^\# A \right)_{x_0}.
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Geometric operators

Deformation to the normal cone $M \mapsto \mathcal{M} := (M \times \mathbb{R}^*) \cup (T_{x_0} M \times \{0\})$.

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\[
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Geometric operators

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At a point \( x_0 \in M \), the limit rescaled differential operator reads

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At a point $x_0 \in M$, the limit rescaled differential operator reads

$$A_{x_0}^{\text{resc}} := \lim_{\lambda \to 0} \left( \lambda^{-\deg(A(g))} f_{x_0,\lambda}^\# A(g) \right).$$
Geometric operators

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\[
(\ldots).
\]
Examples

The Laplace-Beltrami operator

Let \((M, g)\) be a Riemannian manifold. The Laplace-Beltrami operator 
\[
\Delta_g = - \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j 
\]
on \(M\) is geometric of degree \(-4\). In normal coordinates around a point \(x_0 \in M\), we have

\[
\lim_{\lambda \to 0} \left( \lambda^4 f^\#_{x_0, \lambda} \Delta_g \right) = - \sum_{i=1}^{n} \partial_i^2 |_{x_0}.
\] (5)

The Dirac operator

Let \((M, g)\) be a spin manifold. The Dirac operator \(D = \sum_{i=1}^{n} c(e_i) \nabla_{e_i}\) and its square \(D^2\) are geometric of degree \(-2\):

\[
\left( D^2 \right)_{x_0}^\text{res} = - \left( \sum_{j=1}^{n} \left( \partial_j - \frac{1}{4} R_{jl}(x_0) x^j \right) \right)^2,
\] (6)

where \(R_{jl}(x) = R_{jl\alpha\beta}(x) c(e_\alpha) c(e_\beta)\).

Remark

The degree of a geometric operator is not additive on compositions!
Examples

The Laplace-Beltrami operator

Let \((M, g)\) be a Riemannian manifold. The Laplace-Beltrami operator \(\Delta_g = -\sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j\) on \(M\) is geometric of degree \(-4\). In normal coordinates around a point \(x_0 \in M\), we have

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The Laplace-Beltrami operator

Let \((M, g)\) be a Riemannian manifold. The Laplace-Beltrami operator \(\Delta_g = -\sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j\) on \(M\) is geometric of degree \(-4\). In normal coordinates around a point \(x_0 \in M\), we have

\[
\lim_{\lambda \rightarrow 0} \left( \lambda^4 f^\#_{x_0}, \lambda \Delta_g \right) = -\sum_{i=1}^{n} \partial_i^2 |_{x_0}.
\] (5)

The Dirac operator

Let \((M, g)\) be a spin manifold. The Dirac operator \(D = \sum_{i=1}^{n} c(e_i) \nabla_{e_i}\) and its square \(D^2\) are geometric of degree \(-2\):

\[
\left( D^2 \right)^{\text{resc}}_{x_0} = -\left( \sum_{j=1}^{n} \left( \partial_j - \frac{1}{4} R_{jl}(x_0) x^l \right) \right)^2,
\] (6)

where \(R_{jl}(x) = R_{jl\alpha\beta}(x) c(e_\alpha) c(e_\beta)\).

Remark

The degree of a geometric operator is not additive on compositions!
Examples

The Laplace-Beltrami operator

Let \((M, g)\) be a Riemannian manifold. The Laplace-Beltrami operator 
\[ \Delta_g = -\sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j \] on \(M\) is geometric of degree \(-4\). In normal coordinates around a point \(x_0 \in M\), we have

\[ \lim_{\lambda \to 0} \left( \lambda^4 f^\#_{x_0, \lambda} \Delta_g \right) = -\sum_{i=1}^n \partial_i^2 |_{x_0}. \tag{5} \]

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Remark

*The degree of a geometric operator is not additive on compositions!*

Sylvie Paycha, University of Potsdam, on leave from Université Clermont-Auvergne
Rescaled defect formula (with G. HABIB 2018)

Let \( A(z) \in \Psi_{cl}(M, E) \) be a holomorphic family of order \(-qz + a\).

Rescaled holomorphic families

If there is some \( d(z) \) such that \( \lim_{\lambda \to 0} \left( \lambda^{-d(z)} f_{x_0, \lambda}^\# A(z) \right) = A(z)_{x_0}^{\text{resc}} \), then

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\lim_{\lambda \to 0} \left( \lambda^{-d(0)} f_{p, z=0}^\# \left( \text{TR} \left( f_{x_0, \lambda}^\# A(z) \right) \right) \right) = \frac{1}{q} \left. \text{Res} \left( \partial_z \left( A(z)_{x_0}^{\text{resc}} \right) \right) \right|_{z=0}.
\]

Rescaled index formula (S. SCOTT 2012)

\[
\text{ind}(D_+) = -\frac{1}{2} \text{sRes} \left( \log \Delta_{x_0}^{\text{resc}} \right).
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$$
Open questions

- How to compute the residue of a logarithm:

\[
\text{Res}(\log A) = \int_M dx \left( \int_{|\xi_x|=1} \text{tr}_x (\sigma_n(\log A)(x, \cdot)) \, ds\xi \right);
\]

- Why go to non local objects in order to build local expressions from a local operator \( D \):

\[
\begin{align*}
D & \quad \rightarrow \quad \log D^2 \\
\text{local} & \quad \rightarrow \quad \text{NON local} \\
\rightarrow \quad \text{Res}(\log D^2) & \quad \rightarrow \quad \text{local}
\end{align*}
\]

Analogy with:

- the heat-kernel approach:

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\begin{align*}
D & \quad \rightarrow \quad e^{-\epsilon D^2} \\
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\rightarrow \quad \text{fp Tr} \left( e^{-\epsilon D^2} \right) & \quad \rightarrow \quad \text{local}
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\]

- quantisation procedures, here functional quantisation:

\[
\begin{align*}
A(\phi) = \langle \phi, \Delta \phi \rangle & \quad \rightarrow \quad Z := \int \phi e^{A(\phi)} D\phi \\
\text{local classical action} & \quad \rightarrow \quad \text{local amplitudes}
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- How to compute the residue of a logarithm:

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\text{Res}(\log A) = \int_M \, d\mathbf{x} \left( \int_{|\xi\mathbf{x}|=1} \text{tr}_x (\sigma_n(\log A)(x, \cdot)) \, dS\xi \right);
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  \[ \text{Res}(\log A) = \int_M dx \left( \int_{|\xi_x|=1} \text{tr}_x (\sigma_n(\log A)(x, \cdot)) \, ds\xi \right); \]

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Analogy with:
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  \[ D \rightarrow e^{-\epsilon D^2} \rightarrow \text{fp Tr} \left( e^{-\epsilon D^2} \right) \]
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Open questions

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\[
\text{Res}(\log A) = \int_M dx \left( \int_{|\xi_x|=1} \text{tr}_x (\sigma_{-n}(\log A)(x, \cdot)) \, ds \xi \right);
\]

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