TRACE DEFECT FORMULAE FOR GEOMETRIC OPERATORS

New trends and open problems in Global Analysis and Geometry

based on joint work with S. AZZALI and joint work with G. HABIB

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Non local operators

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Pseudodifferential operators

Notations

- *M an n-dimensional smooth closed manifold;*
- *$\pi : E \to M$ a finite rank vector bundle;*
- *$C^\infty(M, E)$ the space of smooth sections of E;*
- *$\Psi_{\rm cl}(M, E)$ the algebra of polyhomogeneous (or classical) pseudodifferential operators acting on $C^\infty(M, E)$; we write $\Psi_{\rm cl}(M)$ if $E = M \times \mathbb{C}$.*

Example

- *(M, g) a Riemannian manifold, $E = M \times \mathbb{C}$, $\Delta_g = -\sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j$ the Laplace-Beltrami operator: $(\Delta_g + \pi_g)^{-1} \in {\Psi_{\rm cl}}^{-2}(M)$;*
- *M a spin manifold and $E = S$ the spinor bundle, D^2 the square of the Dirac operator $D = \sum_{i=1}^n \gamma_i \partial_i$: $\log(D^2 + \pi_D) \notin \Psi_{\rm cl}(M, E)$.*

Classes of pseudodifferential operators determined by their order

For $\Gamma \subset \mathbb{C}$, let $\Sigma^\Gamma(M, E) := \{A \in \Psi_{\rm cl}(M, E), {\rm ord}(A) \in \Gamma\}$. **Examples:** The class $\Psi_{\rm cl}^{\mathbb{Z}}(M, E)$ (resp. $\Psi_{\rm cl}^{\notin \mathbb{Z}}(M, E)$) of integer order (resp. noninteger order) classical pseudodifferential operators.

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- it is local if and only it is a differential operator.

ϵ-locality, $\epsilon \geq 0$

A properly supported operator $A \in \Psi_{\mathrm{cl}}(M, E)$ is ϵ- local (finite propagation) i.e., it satisfies the two equivalent conditions:

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Locality versus non-locality

Definition

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Local pseudodifferential operators

$A \in \Psi_{\mathrm{cl}}(M, E)$

- is in general only micro-local, it preserves the support of singularities $\mathrm{WF}(Au) \subset \mathrm{WF}(u)$, so in particular $\mathrm{Supp}_{\mathrm{sing}}(Au) \subset \mathrm{Supp}_{\mathrm{sing}}(u) \quad \forall u \in \mathcal{D}'(M)$.
- it is local if and only it is a differential operator.

ϵ-locality, $\epsilon \geq 0$

A properly supported operator $A \in \Psi_{\mathrm{cl}}(M, E)$ is ϵ- local (finite propagation) i.e., it satisfies the two equivalent conditions:

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Local linear forms

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$\top^0$- locality on $C^\infty(M)$

$$\phi \top^0 \psi :\Longleftrightarrow \mathrm{Supp}(\phi) \cap \mathrm{Supp}(\psi) = \emptyset.$$

$\top^0$- locality on linear forms

A "linear" form Λ on $\Sigma^\Gamma(M, E)$ is $\top^0$- local if for any $\phi, \psi \in C^\infty(M)$

$$\phi \top^0 \psi \Longrightarrow \Lambda(\phi\, A\, \psi) = 0 \tag{2}$$

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Local linear forms: the canonical trace and the residue

A $\top^0$- local form Λ is local (proved for $E = M \times \mathbb{C}$)

- (??) $\wedge$ (??) $\Longrightarrow \Lambda(A) = \Lambda(\mathrm{Op}(\sigma(A)))$ only depends on the symbol $\sigma(A)$
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for some linear form λ on the symbol class of $\Sigma^\Gamma(M,E)$ and under additional continuity assumptions.

Characterisation of local "linear" forms (with S. AZZALI 2016)

Let $\Lambda : \Sigma^\Gamma(M,E) \to \mathbb{C}$ be a local linear form:

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$$\mathrm{Res}(A) = \int_M \mathrm{Res}_x(A)\, dx; \quad \mathrm{Res}_x(A) = \int_{|\xi_x|=1} \mathrm{tr}_x\left(\sigma_{-n}(A)(x,\cdot)\right).$$

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Local linear forms: the canonical trace and the residue

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Defect formulae: local defects/ discrepancies/ anomalies

Defect formulae measure defects of regularised traces (built from the canonical trace) in terms of the Wodzicki residue (which is local.

Defect formulae (with S. SCOTT 2007)

Let $A(z) \in \Psi_{\rm cl}(M, E)$ be a holomorphic family of order $-q\,z + a$.

$$A(0) \quad \text{is differential} \implies \lim_{z \to 0} \left(\mathrm{TR}\left(A(z)\right)\right) = \frac{1}{q}\left(\mathrm{Res}\left(A'(0)\right)\right) \text{ is local.}$$

ζ-regularised trace of differential (so local) operators

Take $A(z) = A\,Q^{-z}$ for $A(0) = A$ differential and Q elliptic pseudodifferential operator of order $q > 0$ (e.g. a Laplacian) with spectral cut: the ζ-regularised trace of A with weight/regulator Q reads

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&= \mathrm{Tr}((D_- D_+ + \pi_{D_+})^{-z}) - \mathrm{Tr}((D_+ D_- + \pi_{D_-})^{-z}) \quad \mathrm{Re}(z) >> 0 \\
&\qquad \text{since non zero eigenvalues of } D_\pm \text{cancel pairwise} \\
&= \mathrm{sTR}\left((\Delta + \pi_\Delta)^{-z}\right) \quad \text{(meromorphic extension)} \\
&= \lim_{z \to 0} \mathrm{sTR}\left(\underbrace{Id}_{A} \underbrace{(\Delta + \pi_\Delta)^{-z}}_{Q} \right) = -\frac{1}{2} \underbrace{\mathrm{sRes}(\log \Delta)}_{\text{local}} \quad \text{(defect formula)}.
\end{aligned}$$

Consequence: the index as a residue (on closed manifolds)

Notations

- (M, g) *Riemannian closed manifold;*
- $\pi : E = E_+ \oplus E_- \longrightarrow M$ *a finite rank $\mathbb{Z}_2$-graded Clifford hermitian bundle;*
- $D = D_+ \oplus D_-$ *with* $D_\pm : C^\infty(M, E_\pm) \longrightarrow C^\infty(M, E_\mp)$ *an odd elliptic differential operator of order* 1*;*
- D_+ *is formally adjoint to* D_-*, so* $\Delta := \Delta_+ \oplus \Delta_-$ *is an even elliptic essentially self-adjoint differential operator of order* 2*. Here* $\Delta_+ = D_- D_+$ *and* $\Delta_- = D_+ D_-$.

How defect formulae come in ($A = Id$, $Q = \Delta$, $q = 2$)

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For us to keep in mind: Locality of the index (Atiyah and Singer (1963))

The index is local as an integral of a differential form ω

$\mathrm{ind}(D_+) = \int_M \omega(x)$, with ω expressed in terms of the curvature R.

- If $\dim M = 2k$, the Chern-Gauss-Bonnet index theorem(1850, 1945) on $\Omega(M)$ with the natural $\mathbb{Z}_2$-grading.

$$\mathrm{ind}\left((d+d^*)_+\right) = \chi(M) = \int_M \mathrm{Pfaffian}(R)(x).$$

- If $\dim M = 4k$, the Hirzebruch signature theorem (1966) on $\Omega(M)$ with the Hodge-star operator $\mathbb{Z}_2$-grading.

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Rescaling at a point

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Deformation to the normal cone $M \longmapsto \mathbb{M} := (M \times \mathbb{R}^*) \cup (T_{x_0} M \times \{0\})$.

For $\lambda \in \mathbb{R}^*$ define $f_{x_0,\lambda} : U_{x_0}^\lambda = \exp_{x_0} B_{r/|\lambda|} \longrightarrow U_{x_0} = \exp_{x_0} B_r$ by

$$f_{x_0,\lambda}(\exp_{x_0} u) = \exp_{x_0}(\lambda\, u).$$

Rescaled operators (with G. HABIB (2008))

A differential operator A is geometric of degree $\deg(A)$ if $\deg(A)$ is the largest real number d (so such a number should exist!) such that for any $x_0 \in M$, $\lambda^{-d}\, f^\sharp_{x_0,\lambda} A$ converges as $\lambda \to 0$ and we denote the rescaled limit operator by

$$A_{x_0}^{\mathrm{resc}} := \lim_{\lambda \to 0} \left(\lambda^{-\deg(A)}\, f^\sharp_{x_0,\lambda} A \right). \tag{3}$$

Relation to Gilkey's invariant polynomials

A differential operator $A(g) = \sum_{|\alpha| \leq a} A_\alpha(X, g)\, \partial_x^\alpha$ whose coefficients are invariant polynomials $A_\alpha(X, g)$ in the metric g, is geometric with degree

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Examples

The Laplace-Beltrami operator

Let (M, g) be a Riemannian manifold. The Laplace-Beltrami operator $\Delta_g = -\sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j$ on M is geometric of degree -4. In normal coordinates around a point $x_0 \in M$, we have

$$\lim_{\lambda \to 0} \left(\lambda^4 \mathfrak{f}^\sharp_{x_0,\lambda} \Delta_g \right) = -\sum_{i=1}^n \partial_i^2 |_{x_0}. \tag{5}$$

The Dirac operator

Let (M, g) be a spin manifold.The Dirac operator $D = \sum_{i=1}^n c(e_i) \nabla_{e_i}$ and its square D^2 are geometric of degree -2:

$$\left(D^2\right)^{\mathrm{resc}}_{x_0} = -\left(\sum_{j=1}^n \left(\partial_j - \frac{1}{4} R_{jl}(x_0)\, x^l \right) \right)^2, \tag{6}$$

where $R_{jl}(x) = R_{jl\alpha\beta}(x)\, c(e_\alpha) c(e_\beta)$.

Remark

The degree of a geometric operator is not additive on compositions!

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Rescaled defect formula (with G. HABIB 2018)

Let $A(z) \in \Psi_{\mathrm{cl}}(M, E)$ be a holomorphic family of order $-q\,z + a$.

Rescaled holomorphic families

If there is some $d(z)$ such that $\lim_{\lambda \to 0} \left(\lambda^{-d(z)}\, f^{\sharp}_{x_0,\lambda} A(z) \right) = A(z)^{\mathrm{resc}}_{x_0}$, then

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Rescaled index formula (S. SCOTT 2012)

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$$\mathrm{ind}(D_+) = -\frac{1}{2}\,\underbrace{\mathrm{sRes}\left(\underbrace{\log \Delta^{\mathrm{resc}}_{x_0}}_{\text{NON local!}}\right)}_{\text{local!}}.$$

Let $A(z) \in \Psi_{\mathrm{cl}}(M, E)$ be a holomorphic family of order $-q\,z + a$.

Rescaled holomorphic families

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Open questions

- How to compute the residue of a logarithm:

$$\mathrm{Res}(\log A) = \int_M dx \left(\int_{|\xi_x|=1} \mathrm{tr}_x \left(\sigma_{-n}(\log A)(x,\cdot) \right) d_S\xi \right);$$

- Why go to non local objects in order to build local expressions from a local operator D:

$$\underbrace{D}_{\text{local}} \longrightarrow \underbrace{\log D^2}_{\text{NON local !}} \longrightarrow \underbrace{\mathrm{Res}(\log D^2)}_{\text{local}}.$$

Analogy with:

- the heat-kernel approach:

$$\underbrace{D}_{\text{local}} \longrightarrow \underbrace{e^{-\epsilon D^2}}_{\text{NON local !}} \longrightarrow \underbrace{\mathrm{fp}_{\epsilon=0}\,\mathrm{Tr}\left(e^{-\epsilon D^2}\right)}_{\text{local}}$$

- quantisation procedures, here functional quantisation:

$$\underbrace{A(\phi) = \langle \phi, \Delta\phi \rangle}_{\text{local classical action}} \longrightarrow \underbrace{Z := \int_\phi e^{A(\phi)}\,\mathcal{D}\phi}_{\text{NON local effective action}} \longrightarrow \underbrace{\langle \phi(x_1)\cdots\phi(x_k)\rangle}_{\text{local amplitudes}}.$$

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