TRACE DEFECT FORMULAE FOR GEOMETRIC OPERATORS New trends and open problems in Global Analysis and Geometry based on joint work with S. AZZALI and joint work with G. HABIB

Sylvie Paycha, University of Potsdam, on leave from Université Clermont-Auvergne

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Non local operators

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Notations

- M an n-dimensional smooth closed manifold;
- $\pi: E \to M$ a finite rank vector bundle;
- $C^{\infty}(M, E)$ the space of smooth sections of E;
- $\Psi_{cl}(M, E)$ the algebra of polyhomogeneous (or classical) pseudodifferential operators acting on $C^{\infty}(M, E)$; we write $\Psi_{cl}(M)$ if $E = M \times \mathbb{C}$.

Example

- (M, g) a Riemannian manifold, E = M × C, Δ_g = Σⁿ_{i,j=1} 1/√g ∂_ig^{ij} √g∂_j the Laplace-Beltrami operator: (Δ_g + π_g)⁻¹ ∈ Ψ_{cl}⁻²(M);
- M a spin manifold and E = S the spinor bundle, D² the square of the Dirac operator D = Σⁿ_{i=1} γ_i∂_i: log(D² + π_D) ∉ Ψ_{cl}(M, E).

Classes of pseudodifferential operators determined by their order

For $\Gamma \subset \mathbb{C}$, let $\Sigma^{\Gamma}(M, E) := \{A \in \Psi_{cl}(M, E), \operatorname{ord}(A) \in \Gamma\}$. Examples: The class $\Psi_{cl}^{\mathbb{Z}}(M, E)$ (resp. $\Psi_{cl}^{\neq\mathbb{Z}}(M, E)$) of integer order (resp. noninteger order) classical pseudodifferential operators.

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Definition

 $A\in \Psi_{
m cl}(M,E)$ is local if it satisfies the two equivalent conditions:

- *it preserves the support* $\text{Supp}(A\phi) \subset \text{Supp}(\phi)$ *for* $\phi \in C^{\infty}(M)$ *;*
- (locality relation) $\operatorname{Supp}(\phi) \cap \operatorname{Supp}(\psi) = \emptyset \Longrightarrow \phi \, A \, \psi = 0$ for $\phi, \psi \in C^{\infty}(M)$.

_ocal pseudodifferential operators

$A \in \Psi_{\rm cl}(M, E)$

- is in general only micro-local, it preserves the support of singularities $WF(Au) \subset WF(u)$, so in particular $Supp_{sing}(Au) \subset Supp_{sing}(u) \quad \forall u \in D'(M)$.
- it is local if and only it is a differential operator.

ϵ -locality, $\epsilon \geq 0$

A properly supported operator $A \in \Psi_{cl}(M, E)$ is \leftarrow local (finite propagation) i.e., it satisfies the two equivalent conditions:

- it preserves the support modulo an *c*-perturbation Supp(A φ) ⊂ Neigh_ε (Supp(φ)) for all φ ∈ C[∞](M)
- $\phi \top^{\epsilon} \psi : \iff d(\operatorname{Supp}(\phi), \operatorname{Supp}(\psi)) > \epsilon \Longrightarrow \phi A \psi = 0$ for all $\phi \in C^{\infty}(M)$.

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- is in general only micro-local, it preserves the support of singularities $WF(Au) \subset WF(u)$, so in particular $Supp_{sing}(Au) \subset Supp_{sing}(u) \quad \forall u \in D'(M)$.
- it is local if and only it is a differential operator.

ϵ -locality, $\epsilon \geq 0$

A properly supported operator $A \in \Psi_{cl}(M, E)$ is \leftarrow local (finite propagation) i.e., it satisfies the two equivalent conditions:

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Definition

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"Tame" non-locality for pseudodifferential operators

For any $A \in \Psi_{cl}(M, E)$ and any $\epsilon > 0$, there exists $A_0 \in \Psi_{cl}(M, E)$ ϵ -local such that

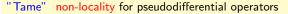
 $A - \underbrace{A_0}_{\epsilon - local} =: S_A \in \Psi^{-\infty}(M, E)$ has smooth kernel supported outside the diagonal.

Notations

- $\mathcal{U} = (U_i)_{i \in I}$ is a finite open cover of M;
- $(\chi_i)_{i \in I}$ is a partition of unity subordinated to \mathcal{U} ;
- $A \in \Psi^{-\infty}(M, E) := \cap_{r \in \mathbb{R}} \Psi^r_{cl}(M, E)$ has smooth Schwartz kernel.



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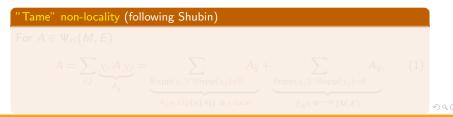
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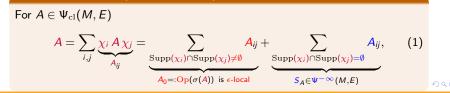
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Tame" non-locality (following Shubin)



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Local linear forms

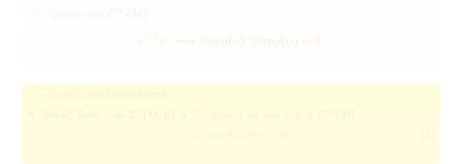
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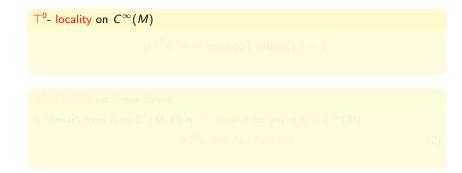
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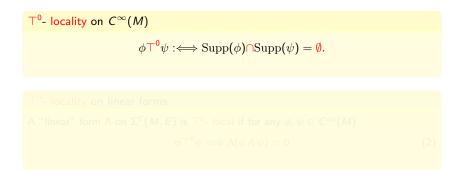
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A \top^{0} - local form Λ is local (proved for $E = M \times \mathbb{C}$)

- (??) \land (??) \Longrightarrow \land (A) = \land ($Op(\sigma(A))$) only depends on the symbol $\sigma(A)$
- in fact, Λ is local, i.e. of the form

$$\Lambda(A) = \int_M \omega_A^{\Lambda}(x) \text{ with } \quad \omega_A^{\Lambda}(x) = \Lambda_x(A) \, dx, \quad \Lambda_x(A) = \lambda(\sigma(A)(x, \cdot)),$$

for some linear form λ on the symbol class of $\Sigma^{1}(M, E)$ and under additional continuity assumptions.

Characterisation of local "linear" forms (with S. AZZALI 2016)

Let $\Lambda : \Sigma^{\Gamma}(M, E) \to \mathbb{C}$ be a local linear form:

• if $\Gamma = \mathbb{Z}$, then Λ is proportional to the Wodzicki residue:

$$\operatorname{Res}(A) = \int_{M} \operatorname{Res}_{x}(A) \, dx; \quad \operatorname{Res}_{x}(A) = \int_{|\xi_{x}|=1} \operatorname{tr}_{x} \left(\sigma_{-n}(A)(x, \cdot) \right).$$

• if $\Gamma = \mathbb{C} \setminus \mathbb{Z}$, then Λ proportional to the canonical trace:

$$\operatorname{TR}(A) = \int_{M} \operatorname{TR}_{\mathsf{x}}(A) \, d\mathsf{x}; \quad \operatorname{TR}_{\mathsf{x}}(A) = \int_{\mathbb{R}^{d}} \operatorname{tr}_{\mathsf{x}}\left(\sigma(A)(\mathsf{x},\cdot)\right).$$

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Sylvie Paycha, University of Potsdam, on leave from Université Clermont-Auvergne

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Defect formulae measure defects of regularised traces (built from the canonical trace) in terms of the Wodzicki residue (which is local.

Defect formulae (with S. SCOTT 2007)

Let $A(z) \in \Psi_{cl}(M, E)$ be a holomorphic family of order -q z + a.

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Take $A(z) = A Q^{-z}$ for A(0) = A differential and Q elliptic pseudodifferential operator of order q > 0 (e.g. a Laplacian) with spectral cut: the ζ -regularised trace of A with weight/regulator Q reads

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Consequence: the index as a residue (on closed manifolds)

Notations

- (M, g) Riemannian closed manifold;
- $\pi: E = E_+ \oplus E_- \longrightarrow M$ a finite rank \mathbb{Z}_2 -graded Clifford hermitian bundle;
- D = D₊ ⊕ D_− with D_± : C[∞](M, E_±) → C[∞](M, E_∓) an odd elliptic differential operator of order 1;
- D₊ is formally adjoint to D₋, so Δ := Δ₊ ⊕ Δ₋ is an even elliptic essentially self-adjoint differential operator of order 2. Here Δ₊ = D₋D₊ and Δ₋ = D₊D₋.

How defect formulae come in $(A = Id, Q = \Delta, q = 2)$

- $\operatorname{ind}(D_+) = \operatorname{dim}(\operatorname{Ker}(D_+)) \operatorname{dim}(\operatorname{Ker}(D_-)) = \operatorname{Tr}(\pi_{D_+}) \operatorname{Tr}(\pi_{D_-})$
 - $= \operatorname{Tr}((D_{-}D_{+} + \pi_{D_{+}})^{-z}) \operatorname{Tr}((D_{+}D_{-} + \pi_{D_{-}})^{-z}) \operatorname{Re}(z) >> 0$
 - since non zero eigenvalues of D_{\pm} cancel pairwise
 - $= \mathrm{sTR}\left((\Delta+\pi_\Delta)^{-z}
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How defect formulae come in $(A = Id, Q = \Delta, q = 2)$

$$\operatorname{ind}(D_+) = \operatorname{dim}(\operatorname{Ker}(D_+)) - \operatorname{dim}(\operatorname{Ker}(D_-)) = \operatorname{Tr}(\pi_{D_+}) - \operatorname{Tr}(\pi_{D_-})$$

 $= \operatorname{Tr}((D_{-}D_{+} + \pi_{D_{+}})^{-z}) - \operatorname{Tr}((D_{+}D_{-} + \pi_{D_{-}})^{-z}) \operatorname{Re}(z) >> 0$

since non zero eigenvalues of D_{\pm} cancel pairwise

 $= \operatorname{sTR} \left((\Delta + \pi_{\Delta})^{-z} \right) \quad (\text{meromorphic extension})$

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- (M,g) Riemannian closed manifold;
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$$= \operatorname{Tr}((D_{-}D_{+} + \pi_{D_{+}})^{-z}) - \operatorname{Tr}((D_{+}D_{-} + \pi_{D_{-}})^{-z}) \quad \operatorname{Re}(z) >> 0$$
since non zero eigenvalues of D_{\pm} cancel pairwise
$$= \operatorname{sTR}\left((\Delta + \pi_{\Delta})^{-z}\right) \quad (\text{meromorphic extension})$$

$$= \lim_{z \to 0} \operatorname{sTR}\left(\underbrace{Id}_{A} \underbrace{(\Delta + \pi_{\Delta})^{-z}}_{Q}\right) = -\frac{1}{2} \underbrace{\operatorname{sRes}(\log\Delta)}_{|\text{local}} \quad (\text{defect formula}).$$

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The index is local as an integral of a differential form ω ind $(D_+) = \int_M \omega(x)$, with ω expressed in terms of the curvature R

• If dimM = 2k, the Chern-Gauss-Bonnet index theorem(1850, 1945) on $\Omega(M)$ with the natural \mathbb{Z}_2 -grading.

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$$((d + d^*)_+) = \chi(M) = \int_M \mathsf{Pfaffian}(R)(x).$$

• If dim M = 4k, the Hirzebruch signature theorem (1966) on $\Omega(M)$ with the Hodge-star operator \mathbb{Z}_2 -grading.

$$\operatorname{ind}\left(\left(d+d^*\right)_+\right) = \operatorname{sign}(M) = \int_M \operatorname{L-form}(R)(x).$$

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Rescaling at a point

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Rescaling at a point

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Deformation to the normal cone $M \mapsto \mathbb{M} := (M \times \mathbb{R}^*) \cup (T_{x_0}M \times \{0\})$. For $\lambda \in \mathbb{R}^*$ define $f_{x_0,\lambda} : U_{x_0}^{\lambda} = \exp_{x_0} B_{r/|\lambda|} \longrightarrow U_{x_0} = \exp_{x_0} B_r$ by $f_{x_0,\lambda}(\exp_{x_0} u) = \exp_{x_0}(\lambda u).$

Rescaled operators (with G. HABIB (2008))

A differential operator A is geometric of degree deg(A) if deg(A) is the largest real number d (so such a number should exist!) such that for any $x_0 \in M$, $\lambda^{-d} f_{x_0,\lambda}^{\sharp} A$ converges as $\lambda \to 0$ and we denote the rescaled limit operator by

$$A_{x_0}^{\operatorname{resc}} := \lim_{\lambda \to 0} \left(\lambda^{-\operatorname{deg}(\mathcal{A})} f_{x_0,\lambda}^{\sharp} \mathcal{A} \right).$$
(3)

Relation to Gilkey's invariant polynomials

A differential operator $A(g) = \sum_{|\alpha| \leq \vartheta} A_{\alpha}(X, g) \partial_{x}^{\alpha}$ whose coefficients are invariant polynomials $A_{\alpha}(X, g)$ in the metric g, is geometric with degree

$$\deg(A(g)) = \min_{\alpha} d_{\alpha}; \quad d_{\alpha} = \deg^{\operatorname{Gi}}(A_{\alpha}) - |\alpha|.$$

At a point $x_0 \in M$, the limit rescaled differential operator reads

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A differential operator A is geometric of degree deg(A) if deg(A) is the largest real number d (so such a number should exist!) such that for any $x_0 \in M$, $\lambda^{-d} f_{x_0,\lambda}^{\sharp} A$ converges as $\lambda \to 0$ and we denote the rescaled limit operator by

$$A_{\mathbf{x}_{0}}^{\mathrm{resc}} := \lim_{\lambda \to 0} \left(\lambda^{-\mathrm{deg}(A)} f_{\mathbf{x}_{0},\lambda}^{\mu} A \right).$$
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Relation to Gilkey's invariant polynomials

A differential operator $A(g) = \sum_{|\alpha| \leq \vartheta} A_{\alpha}(X, g) \partial_{x}^{\alpha}$ whose coefficients are invariant polynomials $A_{\alpha}(X, g)$ in the metric g, is geometric with degree

$$\deg(A(g)) = \min_{\alpha} d_{\alpha}; \quad d_{\alpha} = \deg^{\operatorname{Gi}}(A_{\alpha}) - |\alpha|.$$

At a point $x_0 \in M$, the limit rescaled differential operator reads

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The Laplace-Beltrami operator

Let (M, g) be a Riemannian manifold. The Laplace-Beltrami operator $\Delta_g = -\sum_{i,j=1}^n \frac{1}{\sqrt{g}} \partial_i g^{ij} \sqrt{g} \partial_j$ on M is geometric of degree -4. In normal coordinates around a point $x_0 \in M$, we have

$$\lim_{\lambda \to 0} \left(\lambda^4 f^{\sharp}_{\mathbf{x}_0, \lambda} \Delta_g \right) = -\sum_{i=1}^n \partial_i^2 |_{\mathbf{x}_0}.$$
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The Dirac operator

Let (M, g) be a spin manifold. The Dirac operator $D = \sum_{i=1}^{n} c(\mathbf{e}_i) \nabla_{\mathbf{e}_i}$ and its square D^2 are geometric of degree -2:

$$\left(D^2\right)_{x_0}^{\text{resc}} = -\left(\sum_{j=1}^n \left(\partial_j - \frac{1}{4}R_{jl}(x_0)x^l\right)\right)^2,\tag{6}$$

where $R_{jl}(x) = R_{jl\alpha\beta}(x) c(e_{\alpha})c(e_{\beta})$.

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The degree of a geometric operator is not additive on compositions!

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Let $A(z) \in \Psi_{cl}(M, E)$ be a holomorphic family of order -q z + a.

Rescaled holomorphic families

If there is some d(z) such that $\lim_{\lambda \to 0} \left(\lambda^{-d(z)} f_{x_0,\lambda}^{\sharp} A(z) \right) = A(z)_{x_0}^{\text{resc}}$, then

$$\lim_{\lambda \to 0} \left(\lambda^{-d(0)} \inf_{z=0} \left(\operatorname{TR} \left(f_{x_0, \lambda}^{\sharp} \mathcal{A}(z) \right) \right) \right) = \frac{1}{q} \underbrace{\operatorname{Res} \left(\underbrace{\partial_z \left(\mathcal{A}(z)_{x_0}^{\operatorname{resc}} \right)_{|_{z=0}}}_{\operatorname{NON |ocal|}} \right)}_{\operatorname{local!}}.$$

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$$\operatorname{ind}(D_+) = -\frac{1}{2} \underbrace{\operatorname{sRes}\left(\underbrace{\log\Delta_{x_0}^{\operatorname{resc}}}_{\operatorname{NON \ local!}}\right)}_{\operatorname{local!}}.$$

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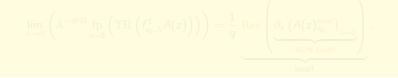
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$$\operatorname{Res}(\log A) = \int_{M} dx \left(\int_{|\xi_{x}|=1} \operatorname{tr}_{x} \left(\sigma_{-n}(\log A)(x, \cdot) \right) \, d_{S} \xi \right)$$

• Why go to non local objects in order to build local expressions from a local operator *D*:



Analogy with:

• the heat-kernel approach:



local

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NON local effective action

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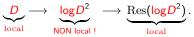
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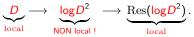
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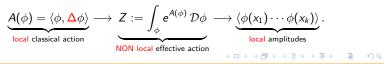


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References

- S. Azzali, S. Paycha, Spectral ζ -invariants lifted to coverings, arXiv:1603.02263
- G. Habib, S. Paycha, Regularised traces of geometric operators: Getzler's rescaling revisited (running title), in preparation
- J. Mickelsson, S. Paycha, The logarithmic residue density of a generalized Laplacian, Journal of the Australian Mathematical Society Volume **90**, N. 01, p. 53 80 (2011)
- S. Paycha, S. Scott, A Laurent expansion for regularised integrals of holomorphic symbols, *Geom. Funct. Anal.* **17** (2007), 491–536

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