Open problems in infinite approximate group theory

References

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Basic definitions

G locally compact second countable (lcsc) group, possibly discrete
d left-invariant, proper, continuous metric on G

$X \subset G$ uniformly discrete if $\inf\{d(x,y) \mid x,y \in G, x \neq y\} > 0$
$X \subset G$ relatively dense if $\sup\{d(g,X) \mid g \in G\} < \infty$
$X \subset G$ Delone if uniformly discrete and relatively dense
$X \subset G$ finite local complexity (FLC) if $X^{-1}X$ closed and discrete

$\Lambda \subset G$ approximate subgroup if $\Lambda \cdot \Lambda \subset \Lambda \cdot F$, $|F| < \infty$ and $e \in \Lambda = \Lambda^{-1}$.
$\Lambda \subset G$ uniform approximate lattice if approximate subgroup & Delone set

$\Lambda \subset G$ FLC set $\rightsquigarrow G \curvearrowright \Omega_\Lambda$ hull dynamical system of Λ (replaces "G/Γ")

$$\begin{aligned} \Omega_\Lambda &= \overline{\{g.\Lambda \mid g \in G\}} \text{ w.r.t. Chabauty–Fell topology} \\ &= \{\Lambda' \subset G \mid \forall R > 0\ \exists g \in G :\ \Lambda' \cap B_R(e) = g\Lambda \cap B_R(e)\} \end{aligned}$$

[**Properties:** Compact Hausdorff space, jointly continuous G-action, uniformly Delone iff Λ Delone, $\emptyset \in \Omega_\Lambda$ iff Λ not relatively dense]

Model sets vs. uniform approximate lattices

G, H lcsc groups, $\Gamma < G \times H$ uniform irreducible lattice, $W \subset H$ "window" (Jordan measurable compact symmetric identity neighbourhood)

Model set

$$\Lambda := \Lambda(G, H, \Gamma, W) = \pi_G((G \times W) \cap \Gamma) \subset G.$$

1. Every relatively dense subset of a model set is a uniform approximate lattice.
2. If G is compactly-generated abelian, then every uniform approximate lattice is of this form.

Holy grail

For which lcsc groups G is every uniform approximate lattice relatively dense in a model set? What about nilpotent, solvable, semisimple Lie groups?

Which groups admit interesting model sets?

Let G be a group, $\Lambda \subset G$ symmetric subset.
$F \subset G$ quasi-commensurates Λ if for every $a \in F$ there is $F_a \subset G$ finite such that

$$a\Lambda \subset \Lambda F_a \quad \text{and} \quad \Lambda a \subset F_a \Lambda.$$

If $\Lambda \subset G$ is an approximate subgroup and $F \subset G$ quasi-commensurates Λ, then $F\Lambda \cup \Lambda F^{-1}$ is an approximate subgroup, the enlargement of Λ by F.

Theorem [Björklund–Hartnick]

If Λ_o is a model set in a lcsc group G, then there exists a model set $\Lambda = \Lambda(G, H, \Gamma, W)$ with H a connected Lie group such that Λ_o is a relatively dense subset of a finite enlargement of Λ.

Open problem

For which lcsc groups G does there exist a connected Lie group $H \neq \{e\}$ and a uniform irreducible lattice in $G \times H$? (If no such lattice exists, then every model set in G is contained in a finite enlargement of a lattice.)

Approximate lattices in abelian extensions

$A \to G \xrightarrow{\pi} Q$ abelian extension, $\Lambda \subset G$ uniform approximate lattice

Λ is called π-**adapted** if the following equivalent conditions hold:

1. $\pi(\Lambda)$ is a uniform approximate lattice in Q.
2. Over a relatively dense subset of $\pi(\Lambda)$ the symmetrized fibers are uniform approximate lattices.

Assume G is a 1-connected Lie group.
If G is 2-step nilpotent and $A := Z(G)$, then every Λ is π-adapted.
If G is nilpotent, then there exists π such that every approximate lattice is π-adapted. (May choose $A := C(C_G([G, G]))$.)

Open problems

1. In the nilpotent case, can we always choose A to be central (depending on Λ or universal)?
2. In the solvable case, can we always find A (depending on Λ or universal)?

Approximate lattices in nilpotent Lie groups

Let G be a 1-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$.
G contains a model set (respectively a uniform lattice) iff $\mathfrak{g}$ admits a basis with structure constants in $\overline{\mathbb{Q}}$ (respectively $\mathbb{Q}$).

Open problem

Can G contain a uniform approximate lattice, but no model sets?

Test case

Consider the 8-dimensional 2-step nilpotent Lie group G_λ with associated Lie algebra $\mathfrak{g}_\lambda$ given by generators $\{X_1, \ldots, X_6, Y_1, Y_2\}$ and non-zero bracket relations

$$[X_1, X_2] = [X_3, X_4] = Y_1, \quad [X_3, X_5] = [X_6, X_4] = Y_2, \quad [X_5, X_6] = \lambda Y_1.$$

Does this contain a uniform approximate lattice if $\lambda \notin \overline{\mathbb{Q}}$? Can you construct such a uniform approximate lattice from a pair of model sets $\Lambda_1 \subset Z(G_\lambda)$ and $\Lambda_2 \subset G_\lambda / Z(G_\lambda)$ such that $\omega(\Lambda_2, \Lambda_2) \subset \Lambda_1$, where ω is a cocycle defining G_λ?

Continuous functions on the hull

Let G be a lcsc group and let $\Lambda \subset G$ be an FLC Delone subset.
We can construct continuous functions on the hull Ω_Λ by periodization:

$$\mathcal{P} : C_c(G) \to C(\Omega_\Lambda), \quad \mathcal{P}f(\Lambda') := \sum_{x \in \Lambda'} f(x).$$

Then with respect to the topology of uniform convergence the algebra

$$\bigcup_{n=0}^{\infty} \{F_1 \cdots F_n \mid F_i \in \mathcal{P}(C_c(G))\}$$

is dense in $C(\Omega_\Lambda)$.

Open problem

Does there exist a finite $N \in \mathbb{N}$ such that

$$\bigcup_{n=0}^{N} \{F_1 \cdots F_n \mid F_i \in \mathcal{P}(C_c(G))\} \quad ?$$

is dense in $C(\Omega_\Lambda)$?

Invariant and stationary measures on the hull

$\Lambda \subset G$ uniformly discrete approximate subgroup, $\Omega_\Lambda^\times := \Omega_\Lambda \setminus \{\emptyset\}$.
Λ is called

- a strong approximate lattice if $\Omega_\Lambda^\times$ admits a G-invariant probability.
- an approximate lattice if $\Omega_\Lambda^\times$ admits a p-stationary probability for every symmetric probability $p \ll \mathrm{Haar}_G$ with $\langle \mathrm{supp}(p) \rangle = G$.

Open problems

1. Is every approximate lattice strong? Is every uniform approximate lattice strong? Classification of invariant/stationary measures?
2. If there exists $\mathcal{F} \subset G$ with $\mathrm{Haar}_G(\mathcal{F}) < \infty$ and $\Lambda\mathcal{F} = G$, is Λ an approximate lattice?
3. Does the existence of an approximate lattice imply that G is unimodular? (Yes, if the approximate lattice is strong or finitely-generated and uniform.)
4. Is every approximate lattice in a solvable group uniform? (Yes, if G is nilpotent or $\Lambda = \Lambda^\infty$ group.)

Approximate groups as geometric objects

$(\Lambda, \Lambda^\infty)$ countable approx. group, d proper left-invariant metric on Λ^∞

$[\Lambda]_c := [\Lambda, d|_{\Lambda\times\Lambda}]_c$ coarse equivalence class of Λ (independent of d).

In the group case, if $\Lambda = \Lambda^\infty$ is finitely generated, then one can define a canonical QI type. In our case, we have two possible generalizations:

First generalization

$(\Lambda, \Lambda^\infty)$ is geometrically finitely-generated if there exists a large-scale geodesic metric d_o on Λ representing $[\Lambda]_c$. In this case, its canonical QI type is defined as $[\Lambda] := [\Lambda, d_o]$ (independent of d_o).

Second generalization

$(\Lambda, \Lambda^\infty)$ is algebraically finitely-generated if Λ^∞ is a f.g. group. In this case its external QI type is $[\Lambda]_{\mathrm{ext}} := [\Lambda, d_S|_{\Lambda\times\Lambda}]$ for a word metric d_S on Λ^∞ (independent of S).

Distortion in approximate groups

Let $(\Lambda, \Lambda^\infty)$ be a countable group which is both geometrically and algebraically finitely-generated.

$(\Lambda, \Lambda^\infty)$ is called undistorted if $[\Lambda] = [\Lambda]_{\mathrm{ext}}$, otherwise distorted.

In the group case $\Lambda = \Lambda^\infty$ is geometrically f.g. iff it is algebraically f.g. iff it is f.g.; in this case it is always undistorted. Thus distortion is a new phenomenon in approximate groups. Model sets are always undistorted. An example of a distorted approximate group is given by

$$\Lambda^\infty := \langle a, b \mid bab^{-1} = a^2 \rangle, \quad \Lambda := \langle a \rangle \cup \{b, b^{-1}\}.$$

This approximate group is exponentially distorted.

Open problem

Which types of distortion appear in approximate groups?

The QI rigidity problem

An approximate group $(\Lambda, \Lambda^\infty)$ acts geometrically on a metric space (X, d) if there is a homomorphism $\rho : \Lambda^\infty \to \mathrm{Is}(X, d)$ such that $\rho(\Lambda).x$ is relatively dense in X for every $x \in X$ and the map $\Lambda \times X \to X \times X$, $(\lambda, x) \mapsto (x, \rho(\lambda)x)$ is proper.

Theorem (Cordes–Hartnick–Tonić, Björklund–Hartnick)

If $(\Lambda, \Lambda^\infty)$ acts geometrically on a proper geodesic metric space (X, d), then $(\Lambda, \Lambda^\infty)$ is geometrically and algebraically f.g. and undistorted and $[\Lambda] = [\Lambda]_{\mathrm{ext}} = [(X, d)]$.

We say that X is QI-rigid with respect to approximate groups if every undistorted approximate group which is is quasi-isometric to X acts geometrically on X. This is the case for higher rank symmetric spaces.

Open problem

Find more examples of proper geodesic metric spaces which are QI-rigid with respect to approximate groups.

Hyperbolic approximate groups

A geometrically finitely-generated approximate group is hyperbolic if it is quasi-isometric to a Gromov-hyperbolic geodesic metric space.
All known examples of geometrically f.g. hyperbolic approximate groups so far are of one of the following types:

1. Model sets in Gromov-hyperbolic lcsc groups (e.g. real and p-adic rank one Lie groups); these are quasi-isometric to compactly-generated lcsc groups, although not necessarily to f.g. discrete groups, and undistorted.
2. Quasi-kernels of quasimorphisms constructed by small cancellation theory; these are always distorted (N. Heuer–D. Kielak)

Open problem

Construct examples of geometrically and algebraically f.g. hyperbolic approximate groups which are undistorted and not quasi-isometric to any compactly-generated lcsc group.

Diffraction, I: Pure point diffraction

Let (G, K) be a Gelfand pair with commutative space $X = G/K$, let $\Lambda = \Lambda(G, H, \Gamma, W)$ be a regular model set in G and let Λ' be the associated weighted model set in X. Then the spherical diffraction measure of Λ' is pure point.

The proof uses the fact that for the unique invariant measure ν on Ω_Λ the space $L^2(\Omega_\Lambda, \nu)^K$ decomposes discretely over the Hecke algebra $L^1(K\backslash G/K)$. This is based on an isomorphism

$$L^2(\Omega_\Lambda, \nu)^K \cong L^2((K \times \{e\})\backslash(G \times H)/\Gamma).$$

The pure point property of the diffraction is easily destroyed, e.g. by removing a single point from Λ.

Open problem

For a general approximate lattice in G, what can you say about its pure point part? Is it always non-empty? Is it large in some sense?

Diffraction, II: Diffraction coefficients

The spherical diffraction of a regular model set in a commutative space is a pure point measure, i.e. given by a countable sum of weighted Dirac measures. The weights are given by a certain integral transform called the shadow transform associated with the Gelfand pair (G, K) and the lattice $\Gamma < G \times H$.

We have recently computed the shadow transform explicitly for certain Gelfand pairs associated with compact extensions of the Heisenberg group. The virtually nilpotent Gelfand pairs have been classified.

Open problem

For every virtually nilpotent Gelfand pair, compute the associated shadow transforms in terms of special functions, and use this to obtain more explicit diffraction formulas in this case. Compare the resulting formulas to formulas obtained by physicists studying quasi-crystals under magnetic fields.

Relative Property (FH) for approximate lattices

A lcsc group G has Property (FH) relative to a subset $A \subset G$ if every affine isometric action of G on a separable Hilbert space has bounded A-orbits.

Open problem

Let G be a lcsc group and let $\Lambda \subset G$ be an approximate subgroup with enveloping group Λ^∞. Is it true that

$$G \text{ has (FH)} \quad \Leftrightarrow \quad \Lambda^\infty \text{ has (FH) relative } \Lambda \quad ?$$

If Λ is a model set, this is true. It is also true if Λ is contained in a model set $\Lambda(G, H, \Gamma, W)$ with H almost connected. Even in the group case $\Lambda = \Lambda^\infty$ one needs additional integrability assumptions on the cocycle if Λ is non-uniform.

Delone sets in non-locally compact Polish groups

The notion of a Delone set can also be defined for certain non-locally compact Polish groups. For example a subset of a separable Banach space $(E, \|\cdot\|)$ is Delone if it is uniformly discrete and relatively dense with respect to the metric $d(x, y) := \|x - y\|$. This definition also makes sense for general Banach spaces.
In this more general setting, a Delone (approximate) subgroup is still called a uniform (approximate) lattice.

Open problems

1. Show that a separable Hilbert space does not admit a uniform lattice. (Some Banach spaces like ℓ^∞ do, and there are even separable examples like c_o.)
2. Can a product of a Hilbert space with a Banach space contain a uniform lattice?
3. Does a separable Hilbert space admit a uniform approximate lattice?
4. What about more general Hilbert/Banach-Lie groups?