G_2 structures on nilmanifolds and their moduli spaces

Prospects in Geometry and Geometric Analysis
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Goal of this talk

We review G_2 structures on 7-dimensional manifolds. We are especially interested in studying them on nilmanifolds, compact quotients of connected, simply connected nilpotent Lie groups. We dwell on *closed* and *(purely) coclosed* G_2 structures, showing how the latter can be constructed from certain $SU(3)$ structures in dimension 6. We will say something on their moduli spaces.

Content of the talk

1. G_2 structures on 7-manifolds
2. Nilpotent Lie algebras and nilmanifolds
3. G_2 structures on NLAs and nilmanifolds
4. Purely coclosed G_2 structures on NLAs and nilmanifolds
5. Moduli spaces of G_2 structures

Content of the talk

G_2 structures on 7-manifolds

Definition

A G_2 *structure* on a smooth 7-manifold M is a reduction of the structure group of the frame bundle from $GL(7, \mathbb{R})$ to $G_2 \subset SO(7)$.

A G_2 structure on M is $\varphi \in \Omega^3(M)$ pointwise equivalent to

$$\varphi_0 = e^{123} - e^{167} + e^{257} - e^{356} + e^{145} + e^{246} + e^{347}$$

A G_2 structure determines a metric g_φ and an orientation o_φ on M.
$\Omega^3_+(M)$, the set of G_2 structures, is an open subset of $\Omega^3(M)$.

The G_2 package

Let M be a manifold endowed with a G_2 structure φ. The G_2 package on M consists of $(g_\varphi, o_\varphi, *_\varphi)$

Existence of G_2 structures

Let M be a manifold with a G_2 structure.

- M is orientable
- Since G_2 is simply connected, M admits a spin structure.

☞ If a 7-manifold admits a G_2 structure, then the first and second Stiefel-Whitney classes vanish, $w_1(M) = 0 = w_2(M)$

Proposition [Gray, 1968]

A smooth 7-manifold M admits a G_2 structure if and only if $w_1(M) = 0 = w_2(M)$

☞ The existence of G_2 structures is purely topological
Every parallelizable 7-manifold, such as S^7, has a G_2 structure

G_2 structures from G_2 holonomy

If (M^n, g) is a connected Riemannian manifold, the *holonomy group of* g, $\mathrm{Hol}(g)$, is the subgroup of $O(n)$, well defined up to conjugation, determined by parallel transport along loops based at some $p \in M$.
Berger showed that G_2 is the only possible *non generic* holonomy group of a simply connected, irreducible, non-symmetric Riemannian 7-manifold.
By the holonomy principle, (M, g) with $\mathrm{Hol}(g) \subset G_2$ has a G_2 structure φ such that $g_\varphi = g$ and $\nabla\varphi = 0$. Moreover, $\mathrm{Ric}(g) = 0$.

Examples of manifolds with holonomy G_2

- Non-complete examples were constructed by Bryant (1985); complete examples are due to Bryant and Salamon (1989)
- Compact examples were constructed by Joyce (1996), Corti-Haskins-Nordström-Pacini (2015) and Joyce-Karigiannis (2021)

Classification of G_2 structures - I

Proposition [Fernández, Gray, 1982]

Let M be a manifold endowed with a G_2 structure φ and let ∇^φ be the Levi-Civita connection of g_φ. Then

$$\nabla^\varphi \varphi = 0 \Leftrightarrow d\varphi = 0 = d(*_\varphi \varphi)\,.$$

In other words, $\mathrm{Hol}(g_\varphi) \subset G_2$ if and only if φ is *closed* and *coclosed*.

The *intrinsic torsion* of a G_2 structure is governed by $d\varphi \in \Omega^4(M) \cong \Omega^3(M)$ and $d(*_\varphi\varphi) \in \Omega^5(M) \cong \Omega^2(M)$. G_2 acts naturally on these spaces, and there are orthogonal decompositions $\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M)$ and $\Omega^3(M) = \langle\varphi\rangle \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M)$;

$$\Omega^2_{14}(M) = \{\tau \in \Omega^2(M) \mid \tau \wedge *_\varphi\varphi = 0\}$$
$$\Omega^3_{27}(M) = \{\tau \in \Omega^3(M) \mid \tau \wedge \varphi = 0, \tau \wedge *_\varphi\varphi = 0\}$$

Classification of G_2 structures - II

Proposition [Bryant, 2005]

Let φ be a G_2 structure on a 7-manifold M. There exist $\tau_i \in \Omega^i(M)$, $i = 0, 1$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$, the torsion forms, such that

$$\begin{cases} d\varphi &= \tau_0 *_\varphi \varphi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3 \\ d(*_\varphi \varphi) &= 4\tau_1 \wedge *_\varphi \varphi + \tau_2 \wedge \varphi \end{cases}$$

- There are 16 classes of G_2 structures
- g_φ has holonomy (contained in) G_2 if and only if $\tau_i = 0$ for $i = 0, \ldots, 3$
- Metrics with holonomy (contained in) G_2 are also called *torsion-free* G_2 structures

Pure G_2 structures

Definition

A G_2 structure is called *pure* if all the torsion forms vanish, but one

The four pure classes

$\tau_0 \neq 0$	nearly parallel (NP)	$d\varphi = \tau_0 *_\varphi \varphi$
$\tau_1 \neq 0$	locally conformally parallel (LCP)	$d\varphi = 3\tau_1 \wedge \varphi$ $d(*_\varphi \varphi) = 4\tau_1 \wedge *_\varphi \varphi$
$\tau_2 \neq 0$	closed	$d(*_\varphi \varphi) = \tau_2 \wedge \varphi$
$\tau_3 \neq 0$	purely coclosed	$d\varphi = *_\varphi \tau_3$ $\Leftrightarrow \varphi \wedge d\varphi = 0$

Nearly parallel and locally conformally parallel G_2 structures

NP G_2 structures [Friedrich, Kath, Moronianu, Semmelmann, 1997]

- can be equivalently defined in terms of a Killing spinor
- have Einstein underlying Riemannian metric
- exist on the sphere S^7 and on many homogeneous spaces

Locally conformally parallel G_2 structures

- on simply connected manifolds, they can be rescaled to give torsion-free G_2 structures
- on compact manifolds, they are equivalent to mapping tori of compact simply connected nearly Kähler 6-manifolds [Ivanov, Parton, Piccinni, 2006]

Closed G_2 structures

- Closed G_2 structures are important for the constructions of Joyce and Joyce-Karigiannis of holonomy G_2 manifolds, which start with a closed G_2 structure on a manifold/orbifold
- They define *calibrations* on M: for every $V \subset T_pM$ 3-dimensional subspace, $\varphi|_V \leq \mathrm{vol}\,(g_\varphi|_V)$. They are sometimes thought of as the symplectic analogue of G_2 structures
- A basic open question concerns the existence of *exact* G_2 structures, i.e. $\varphi = d\beta$, on *compact* manifolds

Open question

Does the sphere S^7 admit a closed G_2 structure?

Coclosed G_2 structures

There is a class of G_2 structures which contains both purely coclosed and nearly parallel G_2 structures, namely that of *coclosed* G_2 structures, defined by $d(*_\varphi \varphi) = 0$. They exist naturally on hypersurfaces of Riemannian 8-manifolds with holonomy Spin(7). They are not very hard to obtain:

Theorem [Crowley, Nordström, 2012]

Let M be a 7-manifold admitting a G_2 structure. Then M also admits a coclosed G_2 structure.

Purely coclosed G_2 structures

In this talk we focus on the construction of examples of manifolds with purely coclosed G_2 structures

Content of the talk

Nilpotent Lie algebras

The *ascending central series* of a Lie algebra $\mathfrak{g}$ is defined by

$$\mathfrak{g}_0 := \{0\}, \quad \mathfrak{g}_k := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subset \mathfrak{g}_{k-1}\}, k \geq 1 .$$

- $\mathfrak{g}_k \subset \mathfrak{g}_{k+1}$ and every $\mathfrak{g}_k$ is an ideal
- $\mathfrak{g}_1 = \mathfrak{z}(\mathfrak{g})$

Definition

A Lie algebra $\mathfrak{g}$ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathfrak{g}_n = \mathfrak{g}$. A nilpotent Lie algebra $\mathfrak{g}$ is *m-step nilpotent* if $\mathfrak{g}_{m-1} \neq \mathfrak{g}$ but $\mathfrak{g}_m = \mathfrak{g}$. m is the *nilpotency step* of $\mathfrak{g}$

- We shorten nilpotent Lie algebra to NLA
- NLAs have non-trivial center

Nilmanifolds

Definition

A connected, simply connected Lie group is *nilpotent* if its Lie algebra is.

- A nilpotent Lie group is diffeomorphic to $\mathbb{R}^n$, for some n

Definition

A *nilmanifold* $\Gamma\backslash G$ is the compact quotient of a connected, simply connected nilpotent Lie group G by a lattice $\Gamma \subset G$.

- A connected, simply connected nilpotent Lie group G admits a lattice $\Leftrightarrow$ $\mathfrak{g}$ admits a *rational structure*, that is, $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ [Mal'cev]
- Nilmanifolds are aspherical spaces and $\pi_1(\Gamma\backslash G) \cong \Gamma$.
- The natural inclusion $(\Lambda\mathfrak{g}^*, d) \hookrightarrow (\Omega(N), d)$ is a quasi-isomorphism [Nomizu]

Content of the talk

G_2 structures on Lie algebras and nilmanifolds

Every nilmanifold is parallelizable, hence a 7-dimensional nilmanifold admits a coclosed G_2 structure. We are interested in G_2 structures which come from the Lie algebra.

Definition

Let $\mathfrak{g}$ be a real Lie algebra of dimension 7. A G_2 *structure* on $\mathfrak{g}$ is $\varphi \in \Lambda^3\mathfrak{g}^*$ for which there exists a coframe $(e^1, \ldots, e^7)$ such that

$$\varphi = e^{123} - e^{167} + e^{257} - e^{356} + e^{145} + e^{246} + e^{347}$$

Definition

A G_2 structure on a 7-dimensional nilmanifold $N = \Gamma \backslash G$ is *left-invariant* if it is induced by a G_2 structure on $\mathfrak{g}$, under the inclusion $(\Lambda\mathfrak{g}^*, d) \hookrightarrow (\Omega(N), d)$

G_2 structures on nilmanifolds

Facts

- Nilmanifolds (not tori) cannot carry torsion-free G_2 structures
- Nilmanifolds cannot carry left-invariant nearly parallel G_2 structures (Milnor proved that a non abelian nilpotent Lie algebra cannot carry an Einstein scalar product)
- Nilmanifolds cannot carry locally conformally parallel G_2 structures (By the result of Ivanov, Parton, Piccinni, a compact manifold with a LCP G_2 structure has $b_1 = 1$. But a nilmanifold always has $b_1 \geq 2$)

Closed G_2 structures on NLAs and nilmanifolds

There is a classification of 7-dimensional NLAs by Gong; it consists of 140 Lie algebras and 9 one-parameter families; in addition, there are 35 decomposable nilpotent Lie algebras. Using this, Conti and Fernández proved:

Theorem [Conti, Fernández, 2011]

Up to isomorphism, there are exactly 12 nilpotent Lie algebras of dimension 7 that admit a closed G_2 structure.

Using these Lie algebras, one obtains many examples of (nil)manifolds endowed with closed G_2 structures.

Corollary

There are many examples of nilmanifolds with (left-invariant) closed G_2 structures.

Coclosed G_2 structures on NLAs

7-dimensional nilmanifolds admit coclosed G_2 structures.
☞ What about *left-invariant* coclosed G_2 structures?

Theorem [Bagaglini, Fernández, Fino, 2017]

- 24/35 of 7-dimensional decomposable NLAs admit a coclosed G_2 structure
- 7/9 of 7-dimensional indecomposable 2-step NLAs admit a coclosed G_2 structure
- All of them admit coclosed G_2 structures inducing nilsolitons

These results are obtained using Gong's classification
In an unpublished paper, Bagaglini tackled the other cases; the paper, however, does not seem to be complete

Purely coclosed G_2 structures on NLAs

☞ What about purely coclosed G_2 structures?

Theorem [del Barco, Moroianu, Raffero, 2022]

The 7-dimensional 2-step NLAs (decomposable and indecomposable) which admit a coclosed G_2 structure also admit a purely coclosed one, except for $\mathfrak{h}_3 \oplus \mathbb{R}^4$, where $\mathfrak{h}_3$ is the Heisenberg Lie algebra.

The proof does not use Gong's classification, but relies on existing results on the structure of metric 2-step NLAs

Content of the talk

(Purely) coclosed G_2 structures on NLAs

☞ What about purely coclosed G_2 structures on decomposable NLAs with nilpotency step ≥ 3?

☞ What about coclosed and purely coclosed G_2 structures on indecomposable NLAs with nilpotency step ≥ 3?

Problem

Study the existence of coclosed and purely coclosed G_2 structures on NLAs of nilpotency step 3 and 4. This breaks down to two problems:

- find a way to *construct* purely coclosed G_2 structures on a NLA
- prove that a given NLA does not admit any purely coclosed G_2 structure ($\Rightarrow$ obstructions!)

Linear SU(3) structures

Let V be a 6-dimensional vector space. For a fixed $\tau \in \Lambda^3(V^*)$ we have a map

$$k_\tau \colon V \to \Lambda^5(V^*), \quad k_\tau(x) = \imath_x \tau \wedge \tau$$

There is a natural isomorphism $V \otimes \Lambda^6(V^*) \to \Lambda^5(V^*)$, given by $(v, o) \mapsto \imath_v o$. Its inverse is $\mu \colon \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*)$. We obtain a map

$$K_\tau := \mu \circ k_\tau \colon V \to V \otimes \Lambda^6(V^*)$$

which determines a function $\lambda \colon \Lambda^3(V^*) \to \left(\Lambda^6(V^*)\right)^{\otimes 2}$ by

$$\lambda(\tau) := \frac{1}{6}\mathrm{tr}\big((K_\tau \otimes 1_{\Lambda^6(V^*)}) \circ K_\tau\big) \in \left(\Lambda^6(V^*)\right)^{\otimes 2}$$

Linear SU(3) structures

Set

$$\Lambda_{\pm}(V^*) := \{\tau \in \Lambda^3(V^*) \mid \pm\lambda(\tau) > 0\}$$

For $\psi_- \in \Lambda_-(V^*)$, $J := |\lambda(\psi_-)|^{-1/2} K_{\psi_-}$ is an almost complex structure on V; set $\psi_+ := -J^*\psi_-$. Set also

$$\Lambda_0(V^*) = \{\omega \in \Lambda^2(V^*) \mid \omega^3 \neq 0\}$$

and let $(\omega, \psi_-) \in \Lambda_0(V^*) \times \Lambda_-(V^*)$ be such that

- $\omega \wedge \psi_- = 0$
- $h(x, y) := \omega(x, Jy)$ is positive definite.

Definition

$(\omega, \psi_-) \in \Lambda_0(V^*) \times \Lambda_-(V^*)$ as above is an SU(3) *structure* on V; h is the corresponding SU(3) *metric*

G_2 structures from SU(3) structures

- $\mathfrak{g}$, a 7-dimensional Lie algebra with non-trivial center $\mathfrak{z}(\mathfrak{g})$
- $V \subset \mathfrak{g}$, be a codimension 1 subspace, cooriented by $X \in \mathfrak{z}(\mathfrak{g})$
- $\omega \in \Lambda^2\mathfrak{g}^*$ and $\psi_- \in \Lambda^3\mathfrak{g}^*$, such that $\imath_X\omega = 0$, $\imath_X\psi_- = 0$, and they define an SU(3) structure on V.

If $\bar{\omega}$ and $\bar{\psi}_-$ are the pull-back to V of ω and ψ_-, we have $\bar{\omega} \in \Lambda_0(V^*)$, $\bar{\psi}_- \in \Lambda_-(V^*)$ and $\bar{\omega} \wedge \bar{\psi}_- = 0$. This determines $\bar{\psi}_+ \in \Lambda^3(V^*)$, which we extend to $\psi_+ \in \Lambda^3\mathfrak{g}^*$ by declaring $\imath_X\psi_+ = 0$. Finally, pick $\eta \in \mathfrak{g}^*$ with $\eta(X) \neq 0$.

Then $\varphi := \omega \wedge \eta + \psi_+$ is a G_2 structure on $\mathfrak{g}$, with metric $g_\varphi = h + \eta \otimes \eta$, where h is the SU(3) metric on V. Clearly, $*_\varphi\varphi = \frac{\omega^2}{2} + \psi_- \wedge \eta$.

Purely coclosed G_2 structures from SU(3) structures

Theorem [−, Garvín, Muñoz, 2023]

In the above setting, the G_2 structure $\varphi = \omega \wedge \eta + \psi_+$ is coclosed if

- $d\psi_- = 0$;
- $\omega \wedge d\omega = \psi_- \wedge d\eta$.

Furthermore, φ is purely coclosed if

- $\omega^2 \wedge d\eta = -2\psi_+ \wedge d\omega$.

Using this theorem and Gong's list, we are able to construct purely coclosed G_2 on many 3-step and 4-step NLAs of dimension 7. For the computations, we used the free software `SageMath`

Obstructions to coclosed G_2 structures

In order to make sure that a certain NLA does not admit coclosed G_2 structures we need obtructions to their existence. Two obstructions were found by Bagaglini, Fino and Fernández. With Garvín and Muñoz we found a third one. Together, they are sufficient to rule out all the cases in could not apply our constructions successfully.

Results

Using Gong's list, our construction theorem, and the obstructions, we obtain the following results.

Theorem [–, Garvín, Muñoz, 2023]

Every 7-dimensional decomposable NLA admitting a coclosed G_2 structure also admits a purely coclosed one, except for $\mathfrak{h}_3 \oplus \mathbb{R}^4$.

Theorem [–, Garvín, Muñoz, 2023]

Every 7-dimensional indecomposable NLA of nilpotency step ≤ 4 admitting a coclosed G_2 structure also admits a purely coclosed one.

Remarks and conjectures

- Differently from the case of closed G_2 structures, the large majority of 7-dimensional NLAs admits purely coclosed G_2 structures
- We expect that *all* indecomposable 7-dimensional nilpotent Lie algebras with a coclosed G_2 structure also have a purely coclosed one.
- Nothing is known about *exact* purely coclosed G_2 structures, those for which $*_\varphi\varphi = d\beta$. None of the ones we constructed are. The same question is open for closed G_2 structures. Fino, Martín-Merchán and Raffero recently proved that no compact quotient of a Lie group by a discrete subgroup admits an exact G_2 structure which is left-invariant.

Conjecture

No compact quotient of a Lie group by a discrete subgroup admits an exact purely coclosed G_2 structure.

Content of the talk

Moduli space of torsion-free G_2 structures

Definition

Let M be a compact 7-manifold endowed with torsion-free G_2 structure. The *(Teichmuller) moduli space of torsion-free* G_2 *structures* is

$$\mathcal{M}(M) = \{\varphi \in \Omega^3_+(M) \mid d\varphi = 0, d *_\varphi \varphi = 0\}/\mathrm{Diff}_0(M)\,.$$

Theorem [Joyce, 1996]

$\mathcal{M}(M)$ is a smooth manifold of dimension $b_3(M)$.

Moduli space of closed G_2 structures

Definition

Let M be a compact 7-manifold endowed with closed G_2 structures. The *(Teichmuller) moduli space of closed* G_2 *structures* is

$$\mathcal{M}^c(M) = \{\varphi \in \Omega^3_+(M) \mid d\varphi = 0\}/\mathrm{Diff}_0(M)\,.$$

Proposition [folklore]

$\mathcal{M}^c(M)$ is infinite dimensional.

Remark

If M is a compact symplectic manifold, the moduli space of symplectic structures on M, $\mathcal{M}^s(M) = \{\omega \in \Omega^2_+(M) \mid d\omega = 0\}/\mathrm{Diff}_0(M)$, is a smooth manifold of dimension $b_2(M)$.

Moduli space of left-invariant closed G_2 structures on NLAs

Definition

Let $\mathfrak{g}$ be a Lie algebra endowed with closed G_2 structures. The *moduli space of closed* G_2 *structures on* $\mathfrak{g}$ is

$$\mathcal{M}^c(\mathfrak{g}) = \{\varphi \in \Lambda^3_+\mathfrak{g} \mid d\varphi = 0\}/\mathrm{Aut}_0(\mathfrak{g})\,.$$

Proposition [–, Gil-García, 2023]

Let $N = \Gamma\backslash G$ be a nilmanifold. There is a natural inclusion $\mathcal{M}^c(\mathfrak{g}) \hookrightarrow \mathcal{M}^c(N)$.

Question

For a nilmanifold $N = \Gamma\backslash G$, $\mathcal{M}^c(N)$ contains a full copy of $\mathcal{M}^c(\mathfrak{g})$. Is the dimension of this "finite-dimensional approximation" of $\mathcal{M}^c(N)$ related to $b_3(N) = b_3(\mathfrak{g})$?

A final result

Theorem [−, Gil-García, 2023]

On 7-dimensional non-abelian NLAs with a closed G_2 structure there is no relation between $\dim \mathcal{M}^c(\mathfrak{g})$ and $b_3(\mathfrak{g}) = b_3(N)$. More precisely:

- $\dim \mathcal{M}^c(\mathfrak{g}) < b_3(\mathfrak{g})$ if $\mathfrak{g}$ is decomposable;
- both $\dim \mathcal{M}^c(\mathfrak{g}) < b_3(\mathfrak{g})$ and $\dim \mathcal{M}^c(\mathfrak{g}) > b_3(\mathfrak{g})$ $\mathfrak{g}$ are possible if $\mathfrak{g}$ is indecomposable.

☞ Similar results hold for coclosed G_2 structures

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