ON THE GEOMETRY OF ANTI-QUASI-SASAKIAN MANIFOLDS

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Prospects in Geometry and Global Analysis

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Plan of the talk

1. Almost contact metric structures
2. Anti-quasi-Sasakian manifolds
   - Definition and characterization
   - Transverse geometry
   - Examples
3. Curvature properties

Almost contact structures

Definition

An almost contact manifold is a smooth manifold M^{2n+1} endowed with a (1,1)-tensor field φ, a vector field ξ (the *Reeb vector field*), and a 1-form η, satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

It follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and

$$TM = \mathcal{D} \oplus \langle \xi \rangle, \qquad J_{\mathcal{D}}^2 = -I,$$

where $\mathcal{D} := \operatorname{Ker} \eta = \operatorname{Im} \varphi$ and $J_{\mathcal{D}} := \varphi|_{\mathcal{D}}$.

Almost contact metric manifolds

A Riemannian metric g is compatible with the structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad \forall X, Y \in \mathfrak{X}(M)$$

so that

$$\eta = g(\xi, \cdot), \qquad \|\xi\| = 1, \qquad \mathcal{D} = \langle \xi \rangle^{\perp},$$
$$g(\varphi X, \varphi Y) = g(X, Y) \qquad \forall X, Y \in \Gamma(\mathcal{D})$$

$(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold.

The fundamental 2-form is defined by

$$\Phi(X, Y) = g(X, \varphi Y).$$

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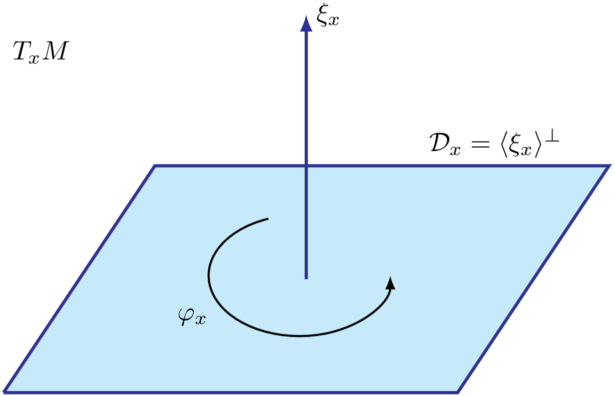
Remark: Almost contact (metric) structures correspond $U(n) \times \{1\}$-reductions of the structural group of the frame bundle.


ξ_x
T_xM
$\mathcal{D}_x = \langle \xi_x \rangle^\perp$
φ_x

Let (M, φ, ξ, η) be an almost contact manifold. On $M \times \mathbb{R}$ one can define an almost complex structure

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

$$\begin{aligned} M \text{ is called normal} &\iff J \text{ is integrable} \\ &\iff N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi \equiv 0 \end{aligned}$$

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Considering the decomposition

$$TM^{\mathbb{C}} = \mathcal{D}^{\mathbb{C}} \oplus \mathbb{C}\xi = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1} \oplus \mathbb{C}\xi,$$

where $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$ are the eigendistributions associated to eigenvalues i and $-i$ of $J_{\mathcal{D}}^{\mathbb{C}}$, normality is also equivalent to

$$[\xi, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}, \quad [\mathcal{D}^{1,0}, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}.$$

Some remarkable classes of almost contact metric manifolds:

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- Sasakian: $d\eta = 2\Phi$ and $N_\varphi = 0$, or equivalently

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- cokähler: $d\eta = 0$, $d\Phi = 0$ and $N_\varphi = 0$, or equivalently

$$\nabla\varphi = 0.$$

  $\rightsquigarrow$ they are locally isometric to $\mathbb{R} \times N^{2n}$, N^{2n} Kähler.

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The quasi-Sasakian structure has constant rank $2r+1$ if

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$d\eta(\xi, \cdot) = 0 \Rightarrow$ the manifold cannot have even rank.

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Theorem (Kanemaki)

$(M, \varphi, \xi, \eta, g)$ is quasi-Sasakian if and only if there exists $A \in \mathfrak{T}^1_1(M)$ such that

$$g(AX, Y) = g(X, AY), \qquad A\varphi = \varphi A,$$
$$(\nabla_X \varphi)Y = \eta(X)AY - g(X, AY)\xi.$$

In this case A is given by $A = -\varphi \circ \nabla\xi + k\eta \otimes \xi$ for some smooth function k.

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Objective: define a new class of almost contact metric manifolds such that the transverse geometry with respect to ξ is given by a Kähler structure endowed with a closed 2-form ω of type $(2, 0)$.

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The almost complex structure J on $M \times \mathbb{R}$ is not integrable, since

$$[J, J]\Big(\big(X, a\frac{d}{dt}\big), \big(Y, b\frac{d}{dt}\big)\Big) = \Big(2d\eta(X, Y)\xi, 2d\eta(X, \varphi Y)\frac{d}{dt}\Big).$$

Considering the decomposition $TM^{\mathbb{C}} = \mathcal{D}^{\mathbb{C}} \oplus \mathbb{C}\xi = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1} \oplus \mathbb{C}\xi$, anti-normality is equivalent to

$$[\xi, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}, \quad [\mathcal{D}^{1,0}, \mathcal{D}^{1,0}]_{\mathcal{D}^{\mathbb{C}}} \subset \mathcal{D}^{1,0}, \quad [\mathcal{D}^{1,0}, \mathcal{D}^{0,1}] \subset \mathcal{D}^{\mathbb{C}}.$$

Remark: M is normal $\Leftrightarrow d\eta = 0$.

For any anti-normal almost contact manifold (M, φ, ξ, η) it turns out that:

$$\mathcal{L}_\xi \varphi = 0, \quad \mathcal{L}_\xi \eta = 0 \quad \mathcal{L}_\xi d\eta = 0,$$

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- $N_\varphi(X, Y, Z) := g(N_\varphi(X, Y), Z) = 0 \ \forall X, Y, Z \in \Gamma(\mathcal{D}) \implies J$ is integrable;
- $d\eta$ projects onto a closed 2-form ω of type $(2, 0)$;
- if the φ-invariant distribution $\mathcal{E} = \mathcal{D} \cap \ker(d\eta)$ has constant rank $2q$, then $\dim M = 2q + 4p + 1$, where

$$\eta \wedge (d\eta)^{2p} \neq 0, \quad d\eta^{2p+1} = 0.$$

We say that the structure has rank $4p + 1$.

Anti-quasi-Sasakian manifolds

Definition

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called anti-quasi-Sasakian (aqS) if

$$d\Phi = 0, \qquad N_\varphi = d\eta \otimes \xi.$$

Remark: $\mathrm{qS} \cap \mathrm{aqS} = \{\text{cokähler}\}$.

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Remark: qS $\cap$ aqS $=$ {cokähler}.

Theorem

An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is anti-quasi-Sasakian if and only if there exists $A \in \mathfrak{T}^1_1(M)$ such that

$$g(AX, Y) + g(X, AY) = 0, \qquad A\varphi = -\varphi A,$$
$$(\nabla_X \varphi)Y = 2\eta(X)AY + \eta(Y)AX + g(X, AY)\xi.$$

In this case A is uniquely determined by $A = -\varphi \circ \nabla\xi$.

It follows that for any aqS manifold $\varphi A = \nabla\xi$ is skew-symmetric w.r.t. g. Hence:

- ξ is Killing ($\mathcal{L}_\xi g = 0$);
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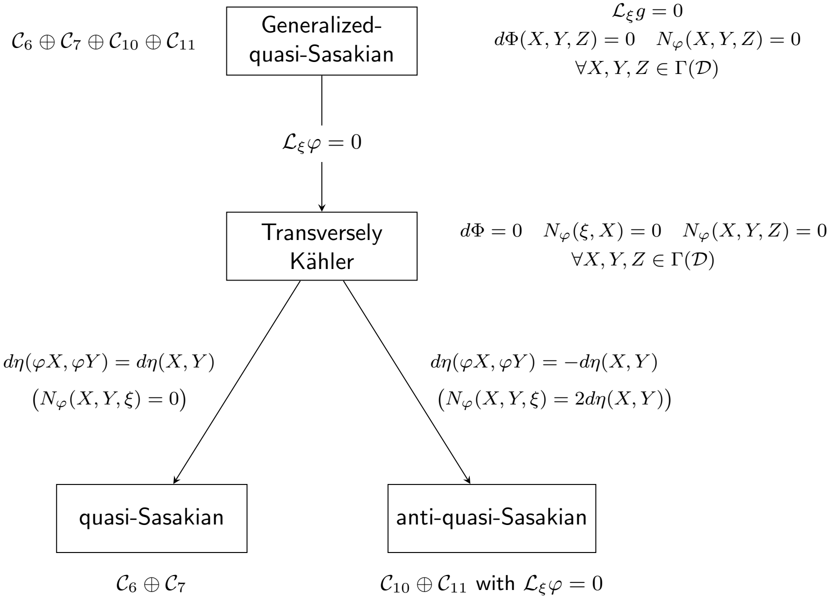
Theorem (Boothby-Wang type theorem)

Every anti-quasi-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ locally fibers onto a Kähler manifold $(M/\xi, J, g)$ endowed with a closed J-anti-invariant 2-form ω.
In particular, if ξ is regular with compact leaves, then M is a principal $\mathbb{S}^1$-bundle over M/ξ and η is a connection form on M, whose curvature form is $d\eta = \pi^*\omega$.


$\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11}$
Generalized-quasi-Sasakian
$\mathcal{L}_\xi g = 0$
$d\Phi(X,Y,Z) = 0 \quad N_\varphi(X,Y,Z) = 0$
$\forall X,Y,Z \in \Gamma(\mathcal{D})$
$\mathcal{L}_\xi \varphi = 0$
Transversely Kähler
$d\Phi = 0 \quad N_\varphi(\xi, X) = 0 \quad N_\varphi(X,Y,Z) = 0$
$\forall X,Y,Z \in \Gamma(\mathcal{D})$
$d\eta(\varphi X, \varphi Y) = d\eta(X,Y)$
$\big(N_\varphi(X,Y,\xi) = 0\big)$
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$\big(N_\varphi(X,Y,\xi) = 2d\eta(X,Y)\big)$
quasi-Sasakian
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Theorem (Converse of the B.W. type thm.)

Let (B, J, k) be a Kähler manifold endowed with a closed (2,0)-form ω. If $[\omega] \in H^2(B, \mathbb{Z})$, then there exists a principal $\mathbb{S}^1$-bundle M over B endowed with an anti-quasi-Sasakian structure (φ, ξ, η, g) such that η is a connection form on M whose curvature form is $d\eta = \pi^*\omega$.

Examples

- Let (B^{4n}, J, k) be a Kähler manifold, U a coordinate neighborhood such that $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$, $i = 1, \dots, 2n$. Consider

$$\omega = \sum_{i=1}^{p}(dx_i \wedge dx_{n+i} - dy_i \wedge dy_{n+i}), \qquad 1 \leq p \leq n,$$

  $\omega = d\beta$ is an exact 2-form of type $(2,0)$ and rank $4p$.

  The trivial bundle $U \times \mathbb{S}^1$ is endowed with an aqS structure (φ, ξ, η, g), where

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- Complex unit disc $D^{2n} \subset \mathbb{C}^{2n}$ endowed with the Kähler structure of constant holomorphic sectional curvature $c < 0$.
- Hermitian symmetric spaces of non-compact type of complex dimension $2n$.

- Hyperkähler manifolds $(B^{4n}, J_1, J_2, J_3, g)$:

$$J_i^2 = -I, \qquad g(J_iX, J_iY) = g(X, Y), \qquad \nabla J_i = 0,$$

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V. Cortés, *A note on quaternionic Kähler manifolds with ends of finite volume* (2022), arXiv:2205.13806.

A remarkable class of examples (I)

Example (The weighted Heisenberg Lie group)

Consider a (4n+1)-dimensional Lie group G with Lie algebra $\mathfrak{g} = \mathrm{span}\{\tau_1, \dots, \tau_{4n}, \xi\}$ such that $[\tau_r, \tau_{3n+r}] = [\tau_{n+r}, \tau_{2n+r}] = 2\lambda_r \xi$ ($\lambda_r \in \mathbb{R}$, $r = 1, \dots, n$). Define:

$$\varphi_i \xi = 0,\ \varphi_i(\tau_r) = \tau_{in+r},\ \varphi_i(\tau_{in+r}) = -\tau_r,\ \varphi_i(\tau_{jn+r}) = \tau_{kn+r},\ \varphi_i(\tau_{kn+r}) = -\tau_{jn+r},$$

for $i = 1, 2, 3$ and $r = 1, \dots, n$.

Then $\varphi_1\varphi_2 = \varphi_3 = -\varphi_2\varphi_1$ and the left invariant structures $(\varphi_1, \xi, \eta, g)$, $(\varphi_2, \xi, \eta, g)$ are *anti-quasi-Sasakian*, while $(\varphi_3, \xi, \eta, g)$ is *quasi-Sasakian*.

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Consider a (4n+1)-dimensional Lie group G with Lie algebra $\mathfrak{g} = \mathrm{span}\{\tau_1, \dots, \tau_{4n}, \xi\}$ such that $[\tau_r, \tau_{3n+r}] = [\tau_{n+r}, \tau_{2n+r}] = 2\lambda_r \xi$ ($\lambda_r \in \mathbb{R}$, $r = 1, \dots, n$). Define:

$$\varphi_i \xi = 0, \;\; \varphi_i(\tau_r) = \tau_{in+r}, \;\; \varphi_i(\tau_{in+r}) = -\tau_r, \;\; \varphi_i(\tau_{jn+r}) = \tau_{kn+r}, \;\; \varphi_i(\tau_{kn+r}) = -\tau_{jn+r},$$

for $i = 1, 2, 3$ and $r = 1, \dots, n$.

Then $\varphi_1\varphi_2 = \varphi_3 = -\varphi_2\varphi_1$ and the left invariant structures $(\varphi_1, \xi, \eta, g)$, $(\varphi_2, \xi, \eta, g)$ are *anti-quasi-Sasakian*, while $(\varphi_3, \xi, \eta, g)$ is *quasi-Sasakian*.

Moreover,

- if $\lambda_1 = \dots = \lambda_n = 1$, then $(\varphi_3, \xi, \eta, g)$ is Sasakian.
- G is 2-step nilpotent. If $\lambda_r \in \mathbb{Q}$ for every $r = 1, \dots, n$, then G admits a cocompact descrete subgroup Γ, so that an aqS strucure is induced on the compact nilmanifold G/Γ (Malčev)
- G is transversely flat.

A remarkable class of examples (II)

We call $Sp(n)$-almost contact metric manifold any smooth manifold M^{4n+1} such that the structural group of the frame bundle is reducible to $Sp(n) \times \{1\}$.

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If $(M, \varphi_i, \xi, \eta, g)_{i=1,2,3}$ is an $Sp(n)$-almost contact metric manifold such that

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Double aqS-Sasakian and $SU(2)$-structures

If $\mathbf{dim}\, \boldsymbol{M} = \mathbf{5}$ (i.e. $n = 1$), then $Sp(1) = SU(2)$.

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Proposition (D. Conti, S. Salamon)

$SU(2)$-structures are in one-to-one correspondence with quadruplets $(\eta, \omega_1, \omega_2, \omega_3)$, where $\eta \in \Lambda^1(M)$, $\omega_i \in \Lambda^2(M)$ are such that

- $\omega_i \wedge \omega_j = \delta_{ij} v$, *for some* $v \in \Lambda^4(M)$ *s.t.* $\eta \wedge v \neq 0$;
- $\omega_1(X, \cdot) = \omega_2(Y, \cdot) \Rightarrow \omega_3(X, Y) \geq 0$.

The 2-forms ω_i are related to the underlying almost contact metric structures $(\varphi_i, \xi, \eta, g)$ by $\omega_i = -\Phi_i$.

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Special subclasses:

- contact hypo: $d(\eta \wedge \omega_1) = d(\eta \wedge \omega_2) = 0$, $d\eta = -2\omega_3$;
- K-contact hypo: contact hypo + ξ Killing;
- double aqS-Sasakian: $d\omega_1 = d\omega_2 = 0$, $d\eta = -2\omega_3$.

It is easily seen that:

$$\{\text{double aqS-Sasakian}\} \subset \{K\text{-contact hypo } SU(2)\}.$$

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Conversely:

Proposition

Let $(M^5, \eta, \omega_1, \omega_2, \omega_3)$ be a manifold with a K-contact hypo $SU(2)$-structure and let $(\varphi_i, \xi, \eta, g)$ $(i = 1, 2, 3)$ the underlying almost contact metric structures. Then, $(\varphi_3, \xi, \eta, g)$ is always Sasakian. Moreover, $(\varphi_2, \xi, \eta, g)$ and $(\varphi_3, \xi, \eta, g)$ are in $\mathcal{C}_{10} \oplus \mathcal{C}_{11}$, so they are aqS if and only if $\mathcal{L}_\xi \varphi_2 = 0$ or $\mathcal{L}_\xi \varphi_3 = 0$.

Curvature properties

Remark: For a double aqS-Sasakian manifold $K(\xi, X) = 1$, for every $X \in \Gamma(\mathcal{D})$.

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Then $A\xi = \psi\xi = 0$, $\eta \circ A = \eta \circ \psi = 0$, A, ψ are skew-symmetric w.r.t. g and

$$A\varphi = \psi = -\varphi A,$$
$$\varphi\psi = A = -\psi\varphi,$$
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Consider also the 2-forms $\mathcal{A} := g(\cdot, A\cdot)$ and $\Psi := g(\cdot, \psi\cdot)$.

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$$d\mathcal{A} = 0, \quad d\Phi = 0, \quad d\eta = 2\Psi.$$

Remark: In general A, ψ are *not* almost contact structures, i.e. $A|_{\mathcal{D}}^2 = \psi|_{\mathcal{D}}^2 \neq -I$.

Proposition

Let $(M, \varphi, \xi, \eta, g)$ *be an anti-quasi-Sasakian manifold. Then*

1. $R(\xi, X)Y = (\nabla_X \psi)Y$;
2. $R(\xi, X)\xi = \psi^2 X = A^2 X$.

In particular M *has non-negative* ξ*-sectional curvatures, and* $K(\xi, X) = \lambda^2$, *for every unit* $X \in \Gamma(\mathcal{D})$ *such that* $\psi^2 X = -\lambda^2 X$.

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Theorem

Let $(M, \varphi, \xi, \eta, g)$ be anti-quasi-Sasakian manifold. Then the following are equivalent:

(a) $K(\xi, X) = 1$ *for every* $X \in \Gamma(\mathcal{D})$;

(b) $\psi^2 = A^2 = -I + \eta \otimes \xi$;

(c) $(A, \varphi, \psi, \xi, \eta, g)$ *is a double aqS-Sasakian structure.*

Remark: Up to homothetic deformations of the structure, the theorem holds true if M has constant ξ-sectional curvature $K(\xi, X) = \lambda^2 > 0$.

Proposition

Let $(M, \varphi, \xi, \eta, g)$ *be an anti-quasi-Sasakian manifold. Then:*

1. $\mathrm{Ric}(\xi, \xi) = |\psi|^2$;
2. $\mathrm{Ric}(\xi, X) = 0$;
3. $\mathrm{Ric}(X, Y) = \mathrm{Ric}^T(X', Y') - 2g(\psi X, \psi Y)$,

for every $X, Y \in \Gamma(\mathcal{D})$ *basic vector field projecting on* X', Y' *with respect to* $\pi : M \to M/\xi$.

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for every $X, Y \in \Gamma(\mathcal{D})$ *basic vector field projecting on* X', Y' *with respect to* $\pi : M \to M/\xi$.

Theorem

Let $(M, \varphi, \xi, \eta, g)$ *be a transversely Einstein, non coKähler, aqS manifold. Then:*

$$\psi^2|_{\mathcal{D}} = -\lambda^2 I, \; \lambda \in \mathbb{R}^* \;\Leftrightarrow\; M \text{ is } \eta\text{-Einstein}.$$

In this case M *turns out to be transversely Ricci-flat,* $\dim M = 4n + 1$, *and*

$$\mathrm{Ric} = -2\lambda^2 g + (4n + 2)\lambda^2 \eta \otimes \eta, \quad s = -4n\lambda^2.$$

These results give obstructions to the existence of anti-quasi-Sasakian structures.

Theorem

If $(M, \varphi, \xi, \eta, g)$ is an anti-quasi-Sasakian manifold with constant sectional curvature, then it is flat and cokähler.

These results give obstructions to the existence of anti-quasi-Sasakian structures.

Theorem

If $(M, \varphi, \xi, \eta, g)$ is an anti-quasi-Sasakian manifold with constant sectional curvature, then it is flat and cokähler.

Proof: If (M, g) has constat sectional curvature κ, then M is Einstein and

$$R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y).$$

Hence,

$$\psi^2 = R(\xi, \cdot)\xi = \kappa(-I + \eta \otimes \xi),$$

that is

$$\psi^2|_{\mathcal{D}} = -\kappa I.$$

If $\kappa \neq 0$, M is η-Einstein, non Einstein. □

Theorem

There exist no compact regular, non-cokähler, aqS manifolds with $\mathrm{Ric} > 0$*.*
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Proof: If $(M, \varphi, \xi, \eta, g)$ is a compact regular, non cokähler, aqS manifold, then M/ξ is compact Kähler with a non-vanishing closed (hence holomorphic) $(2, 0)$-form. Hence M/ξ cannot have positive definite Riemannian Ricci tensor. On the other hand,

$$\mathrm{Ric}^T(X', X') = \mathrm{Ric}(X, X) + 2\|\psi X\|^2$$

$X' \in \mathfrak{X}(M/\xi)$ and $X \in \Gamma(\mathcal{D})$ basic vector field projecting on X'. In both the cases of the statement $\mathrm{Ric}^T > 0$, which is not possible. □

Theorem

There exist no compact regular, non-cokähler, aqS manifolds with $\mathrm{Ric} > 0$*.*
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In both the cases of the statement $\mathrm{Ric}^T > 0$, which is not possible. □

Corollary

There exist no compact *homogeneous* *aqS manifolds of* *maximal rank* *with* $\mathrm{Ric} \geq 0$*.*

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Thank you!

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The canonical connection

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an anti-quasi-Sasakian manifold and (M^{2m}, J, h) a Kähler manifold. Then the product $M^{2n+1} \times M^{2m}$ is naturally endowed with an anti-quasi Sasakian with structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$:

$$\tilde{\varphi}X = (\varphi X_1, JX_2), \quad \tilde{\xi} = (\xi, 0), \quad \tilde{\eta}(X) = \eta(X_1),$$

$$\tilde{g}(X,Y) = g(X_1, Y_1) + h(X_2, Y_2),$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \mathfrak{X}(M^{2n+1} \times M^{2m})$.

If M^{2n+1} has rank $4p+1$, then $M^{2n+1} \times M^{2m}$ has the same rank.

Question: Is an anti-quasi-Sasakian manifold decomposable as Riemannian product of an anti-quasi-Sasakian manifold of maximal rank and a Kähler manifold?

Theorem

Let $(M, \varphi, \xi, \eta, g)$ be an anti-quasi-Sasakian manifold. Then, there exists a metric connection $\bar{\nabla}$ such that

- $\bar{\nabla}\varphi = 0$, $\bar{\nabla}\xi = 0$,
- *the torsion $\bar{T}$ is totally skew-symmetric on $\mathcal{D}$ and $\bar{T}(\xi, \cdot) = 0$.*

The connection $\bar{\nabla}$ is uniquely determined by $\bar{\nabla} = \nabla + H$, with

$$H(X, Y) = \eta(X)\psi Y + \eta(Y)\psi X + g(X, \psi Y)\xi,$$

and its torsion is given by

$$\bar{T}(X, Y) = d\eta(X, Y)\xi.$$

Example

For a double aqS-Sasakian manifold $(M, \varphi_i, \xi, \eta, g)$, $i = 1, 2, 3$, the canonical connections associated to the two aqS structures $(\varphi_1, \xi, \eta, g)$ and $(\varphi_2, \xi, \eta, g)$ coincide. This is the Tanaka-Webster connection of the Sasakian structure $(\varphi_3, \xi, \eta, g)$.

Theorem

Let (M,φ,ξ,η,g) *be an anti-quasi-Sasakian manifold with* $\bar{\nabla}\psi = 0$ $(\Leftrightarrow \bar{\nabla}\bar{T} = 0)$.

- *The distributions*

$$\mathcal{E}^{2q} = \mathcal{D} \cap \mathrm{Ker}(d\eta), \quad \mathcal{E}^{4p+1} = {\mathcal{E}^{2q}}^{\perp}$$

  are integrable with totally geodesic leaves and M *is locally isometric to* $N^{2q} \times M^{4p+1}$, *with* N^{2q} *Kähler manifold,* M^{4p+1} *aqS manifold of maximal rank.*

- *If* M *is connected, then* ψ^2 *has constant eigenvalues.*
  For every nonvanishing eigenvalue μ,

$$\langle \xi \rangle \oplus \mathcal{D}_\mu$$

  is integrable with totally geodesic leaves. Every leaf is endowed with a double aqS-Sasakian structure, up to a homothetic deformation.

Theorem

There exist no connected, locally symmetric, non cokähler, anti-quasi-Sasakian manifolds with $\bar{\nabla}\psi = 0$.

Proof:

- A double aqS-Sasakian manifold cannot be locally symmetric:
  - every locally symmetric Sasakian manifold has constant sectional curvature 1 (Okumura);
  - any double aqS-Sasakian manifold cannot have constant sectional curvature 1.
- Assume $(M, \varphi, \xi, \eta, g)$ locally symmetric, non cokähler, aqS manifold with $\bar{\nabla}\psi = 0$:
  - for a nonzero eigenvalue μ of ψ^2, any maximal integral submanifold N of $\langle\xi\rangle \oplus \mathcal{D}_\mu$, is totally geodesic and hence locally symmetric;
  - up to a homothetic deformation of the structure which preserves the local symmetry, N is endowed with a double aqS-Sasakian structure, which is not possible.

Chinea-Gonzalez classification

Let V be a $(2n+1)$-dimensional real vector space endowed with an almost contact metric structure $(\varphi, \xi, \eta, \langle , \rangle)$.
Let $\mathcal{C}(V)$ be the space all $(0,3)$-tensors having the same symmetries of $\nabla\Phi$, i.e.

$$\begin{aligned}\alpha(X,Y,Z) &= -\alpha(X,Z,Y) = \\ &= -\alpha(X,\varphi Y,\varphi Z) + \eta(Y)\alpha(X,\xi,Z) + \eta(Z)\alpha(X,Y,\xi)\end{aligned}$$

Then one has an orthogonal decomposition

$$\mathcal{C}(V) = \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_{12}$$

with $\mathcal{C}_i$ $(i = 1, \ldots, 12)$ irreducible under the action of $U(n) \times 1$.
This provides 2^{12} invariant subspaces.
The null subspace $\{0\}$ corresponds to the class of cokähler manifolds $(\nabla\Phi = 0)$.

Definition (Puhle)

We say that an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is generalized-quasi-Sasakian (gqS) if

$$\mathcal{L}_\xi g = 0, \quad d\Phi(X,Y,Z) = 0, \quad N_\varphi(X,Y,Z) = 0 \quad \forall X,Y,Z \in \Gamma(\mathcal{D}),$$

where $N_\varphi(X,Y,Z) = g(N_\varphi(X,Y),Z)$.

It turns out that $(M, \varphi, \xi, \eta, g)$ is

- generalized-quasi-Sasakian $\Leftrightarrow$ it belongs to $\mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11}$;
- transversely Kähler $\Leftrightarrow$ it is gqS and $\mathcal{L}_\xi \varphi = 0$;
- quasi-Sasakian $\Leftrightarrow$ it belongs to $\mathcal{C}_6 \oplus \mathcal{C}_7$;
- anti-quasi-Sasakian $\Leftrightarrow$ it belongs to $\mathcal{C}_{10} \oplus \mathcal{C}_{11}$ and $\mathcal{L}_\xi \varphi = 0$.