Geodesic completeness of pseudo-Riemannian Lie groups

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Left-invariant vector fields: $X_1 = x\frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial y}$

Left-invariant metrics: $g(X_i, X_j) = \text{const.}, \quad i, j = 1.2$

$$g = \frac{1}{x^2}(c_1 dx^2 + c_2(dxdy + dydx) + c_3 dy^2), \quad c_1 c_3 - c_2^2 \neq 0.$$

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$\longrightarrow$ Question: Which of these metrics are *complete*?

Geodesic equations

(M, g) (connected) pseudo-Riemannian manifold
$\mathcal{X}_\gamma(M)$: vector fields along a curve γ in M

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γ is called a **geodesic** if it satisfies $\nabla_{\dot\gamma}\dot\gamma = 0$.

In local coordinates $(x_1, ..., x_n)$

$$\ddot\gamma^k(t) + \dot\gamma^i(t)\dot\gamma^j(t)\Gamma^k_{ij}(\gamma(t)) = 0$$

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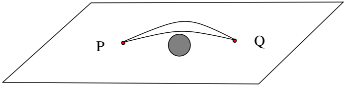
A geodesic γ is called

$\rightarrow$ complete if it is defined in $\mathbb{R}$.

$\longrightarrow$ (M, g) is said to be complete if all of its geodesics are complete.

Remark



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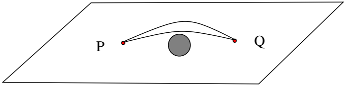

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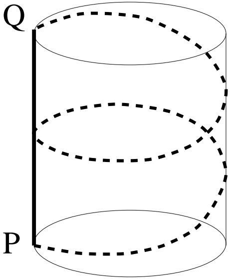

→ There might be more than one geodesic between two points.
→ A geodesic is not necessarily minimizing.

Riemannian vs. pseudo-Riemannian geometry

Hopf-Rinow theorem

For a Riemannian manifold (M, g) the following condition are equivalent:

(MC) As a metric space with Riem. distance d, (M, d) is Cauchy-complete.

(GC) (M, g) is geodesically complete.

(HB) Every closed and bounded subset of M is compact.

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Companion results:
For a complete Riemannian manifold (M, g)

- $\to$ Any two points can be joined by a minimizing geodesic segment.
- $\to$ The length of any diverging curve is infinite.

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Theorem

(G, g): G a Lie group with g a left-invariant Riemannian metric is geodesically complete.

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Example: Clifton-Pohl torus

$(M = \mathbb{R}^2 \backslash \{(0,0)\}, g)$, $g = 2\dfrac{dxdy}{x^2+y^2}$

Geodesic equations: $\ddot{x} = \frac{2x}{x^2+y^2}(\dot{x})^2$, $\ddot{y} = \frac{2y}{x^2+y^2}(\dot{y})^2$

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Nevertheless... (Marsden, 1973)

Compact PR homogeneous spaces are geodesically complete.

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→ Key idea:

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→ Key advantage:

geodesics seen as curves in $\mathbb{R}^n$ with standard topology;

easier to control behaviour at infinity.

$\gamma(t)$ curve in G and $x(t)$ the associated curve in $\mathfrak{g}$ given by $x(t) = \gamma^{-1}(t)\dot{\gamma}(t)$

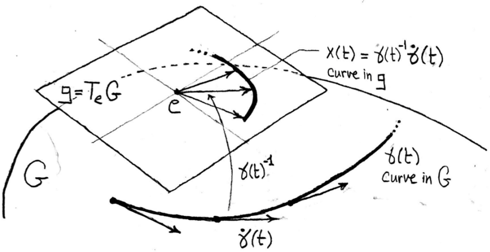

Figure: Associate curve in $\mathfrak{g}$

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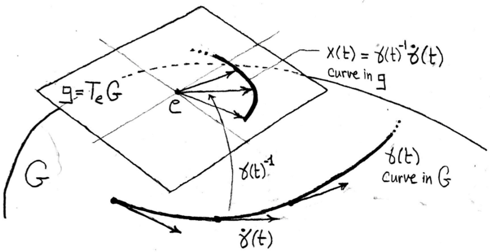

Figure: Associate curve in $\mathfrak{g}$

Theorem (Arnold, 1966)

Then $\gamma(t)$ is a geodesic iff $x(t)$ is an integral curve of

$$\dot{x}(t) = \mathrm{ad}^{\dagger}_{x(t)} x(t)$$

Clairaut first integrals

Interesting facts:

- → If γ is geodesic, X is Killing field, then $g(\dot{\gamma}(t), X_{\gamma(t)}) = \text{constant}$.
- → If X is a right-invariant vector field, then X is Killing for any LI metric.

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The construction:

Let (e_i) be a basis of $\mathfrak{g}$.
Denote by Y_i the extension of e_i as a right-invariant vector field.

Then $\omega^i = g(Y_i, \cdot)$ is a first integral of the geodesic equations.
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Definition (Elshafei, F., Sánchez, Zeghib, 2023)

The *Clairaut metric* associated to g from (e_i) is the Riemannian metric defined by

$$h = \sum_i \omega^i \otimes \omega^i$$

Transformation law:

For $u, v \in \mathfrak{g}$

$$\begin{aligned}\omega^i(p.u) &= g_p(Y_i(p), p.u) = g_p(ei.p, p.u) = g_1(\mathrm{Ad}_{p^{-1}}(e_i), u) \\ &= g_1(e_i, ((\mathrm{Ad}_p)^{-1})^\dagger)(u)) = \omega_1^i(((\mathrm{Ad}_p)^\dagger)^{-1}(u))\end{aligned}$$

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and a concrete expression for h is

$$h(p.u, p.v) = \sum_i \; g_1((\mathrm{Ad}_{p^{-1}})(e_i), u) \; g_1((\mathrm{Ad}_{p^{-1}})(e_i), v).$$

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Proposition (EFSZ, 23)

Let h and $\hat{h}$ be two Clairaut metrics associated to g from the bases (e_i) and $(\hat{e}_i)$, resp., then h and $\hat{h}$ are bi-Lipschitz bounded.

Completeness

→ $\omega^i(\dot{\gamma}(t))$ is constant for any inextensible geodesic γ of g

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Corollary

Any bi-invariant PR metric g on G is complete.

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Corollary

Any bi-invariant PR metric g on G is complete.

Corollary (Marsden, 1973)

Any left-invariant PR metric g on a compact Lie group G is complete.

Wick rotation

(G, g) PR Lie group.

→ Choose an ON basis (e_i) for g_1

→ Construct a LI Riem. metric $\tilde{g}$ by imposing that (e_i) is ON for $\tilde{g}_1$.

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More precisely, consider

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Let $g, \tilde{g}$ be Wick rotated metrics and h the Clairaut metric associated to g from (e_i). Then, h is unique (independent of the chosen (e_i)) and

$$h_p(p.u, p.v) = \tilde{g}_1(\mathrm{Ad}^*_{p^{-1}}(\psi(u)), \mathrm{Ad}^*_{p^{-1}}(\psi(v)))$$

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Remark: If g is Riemannian then $\tilde{g} = g$.

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$\longrightarrow$ $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathrm{Sym}(\mathfrak{g})$ by

$$(\varphi.m)(u,v) = m(\varphi^{-1}u, \varphi^{-1}v).$$

with $\varphi \in \mathrm{Aut}(\mathfrak{g}), m \in \mathrm{Sym}(\mathfrak{g})$ and $u, v \in \mathfrak{g}$.

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Proposition (EFSZ, 23)

G connected Lie group, g LI PR metric, $\varphi \in \mathrm{Aut}(\mathfrak{g})$, g^φ the LI metric such that $(g^\varphi)_1 = \varphi.g_1$. Then:

(1) g^φ is complete if and only if so is g.

(2) LI PR metrics in each orbit of $\mathrm{Sym}(\mathfrak{g})$ are all complete or incomplete.

(3) Clairaut metrics associated to LI metrics on the same orbit are all bi-Lipschitz bounded.

2-dimensional example revisited

Recall

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and the LI PR metrics

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$\longrightarrow$ Three special metrics

$$g^{(1)} = \frac{1}{x^2}(dx^2 + dy^2) \qquad (c_1 = 1, c_2 = 0, c_3 = 1)$$

$$g^{(-1)} = \frac{1}{x^2}(dx^2 - dy^2) \qquad (c_1 = 1, c_2 = 0, c_3 = 1)$$

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$\longrightarrow$ Can be studied to characterize completeness on G, as shall be seen.

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LI Riemannian (hyperbolic) metric, therefore complete.

Clairaut metric: $h^{(1)} = \frac{1}{x^4}\left(x^2 dx^2 + (1 + y^2)dy^2 + xy(dxdy + dydx)\right)$

After some work: $h^{(1)}$ is proved to be complete.

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$\longrightarrow \quad g^{(-1)} = \frac{1}{x^2}(dx^2 - dy^2)$

$\gamma(t) = \left(\frac{1}{1-t}, \frac{1}{1-t}\right)$ incomplete (null) geodesic.

Clairaut metric: $h^{(-1)} = \frac{1}{x^4}\left(x^2dx^2 + (1+y^2)dy^2 - xy(dxdy + dydx)\right)$

$h^{(1)}$ is automatically incomplete.

$\longrightarrow \quad g^{(1)} = \frac{1}{x^2}(dx^2 + dy^2)$

LI Riemannian (hyperbolic) metric, therefore complete.

Clairaut metric: $h^{(1)} = \frac{1}{x^4}\left(x^2dx^2 + (1+y^2)dy^2 + xy(dxdy + dydx)\right)$

After some work: $h^{(1)}$ is proved to be complete.

$\longrightarrow \quad g^{(-1)} = \frac{1}{x^2}(dx^2 - dy^2)$

$\gamma(t) = \left(\frac{1}{1-t}, \frac{1}{1-t}\right)$ incomplete (null) geodesic.

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$h^{(1)}$ is automatically incomplete.

$\longrightarrow \quad g^{(0)} = \frac{1}{x^2}(dxdy + dydx)$

$\gamma(t) = \left(\frac{1}{1-t}, 0\right)$ incomplete (null) geodesic.

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Three classes and their (in)completeness

Consider the action $\mathrm{Aut}(\mathfrak{g})$ on $\mathrm{Sym}(\mathfrak{g})$.

→ Basis (e_1, e_2) s.t. $[e_1, e_2] = e_2$.

→ $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g} \in \mathrm{Aut}(\mathfrak{g})$ satisfies $[\varphi(e_1), \varphi(e_2)] = \varphi(e_2)$

→ W.r.t basis (e_1, e_2), φ is represented by the matrix

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→ Orbit of $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$M^T B M = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} = \begin{pmatrix} 1+\alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix}$$

Up to (positive or negative) scaling, this orbit contains Euclidean (or negative definite) scalar products and, moreover, all of them.

→ Orbit of $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

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This orbit corresponds to all Lorentzian scalar products such that $\langle e_2, e_2 \rangle = 0$.

Therefore:

Prop.(2) proves that all the LI Lorentzian metrics on G are incomplete.
Prop.(3) shows the existence of only 3 bi-Lipschitz Clairaut classes.

A question

$\longrightarrow$ What classes of Lie groups have all of their PR LI metrics complete?

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Positive answers

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- $\rightarrow$ Compact (Marsden, 1973)
- $\rightarrow$ 2-step nilpotent (Guediri, 1994)
- $\rightarrow$ $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ (Bromberg-Medina, 2008)

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However...

Proposition (Elshafei, F., Reis, 2023)
G quadratic Lie group.
For every possible signature, there is a open set of LI PR complete metrics.

Preservation of completeness

Proposition

(M, R) connected complete Riemannian manifold.
Choose $x_0 \in M$ and let $M \ni x \mapsto d_R(x)$ be the distance function from x_0.
If h is a Riemannian metric on M with pointwise norm $\| \cdot \|_h$ satisfying:

$$\|v_x\|_h \geq \frac{\|v_x\|_R}{a + b\, d_R(x)}, \qquad \forall x \in M, v_x \in T_x M$$

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Remarks

→ $\varphi(r) = a + b\, r$ can be replaced with any Lipschitz function φ s.t. $\int_0^\infty \frac{1}{\varphi(r)}\, dr = \infty$.

→ similiar types of bounds have been known since the 70s
[Abraham-Marsden, Foundations of mechanics, 1987]

Groups of linear growth

Let R be a LI Riemannian metric on a Lie group G.

Consider the map

$$r : G \longrightarrow \mathbb{R} \qquad \text{where} \qquad r(p) = d_R(1, p)$$

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Definition (EFSZ, 23)

A Lie group G has (at most) linear growth if there exist

→ a LI Riemannian metric R on G

→ a Euclidean scalar product with norm $\| \cdot \|$ on $\mathfrak{g}$

$$\text{s.t.} \quad \frac{\|u\|}{a + b\, r(p)} \leq \|\mathrm{Ad}_p(u)\| \leq (a + b\, r(p))\|u\|$$

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Remark: since R is LI, $r(p) = r(p^{-1})$, the two inequalities are equivalent,
linear growth is independent of choice of LI R and $\| \cdot \|$.

Linear growth of Clairaut metrics

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All the left-invariant pseudo-Riemannian metrics of a Lie group with linear growth are geodesically complete.

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Proof

Take LI PR g, choose ON basis, construct Wick rotated $\tilde{g}$ and Clairaut h. From the definition of h, can be computed that

$$h_p(p.u, p.u) \geq \frac{\tilde{g}_1(u,u)}{\|\mathrm{Ad}_p\|^2}$$

Since $\|\mathrm{Ad}_p\| \leq a + b\, r(p)$ then

$$\|v_p\|_h \geq \frac{\|v_p\|_{\tilde{g}}}{a + b\, r(p)}$$

and the result follows. □

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Remark: $\mathrm{Aff}^+(\mathbb{R})$ is not of linear growth.

Some groups of linear growth

Theorem (EFSZ, 23)

The following classes of Lie groups have linear growth

- → abelian, compact [in fact, bounded growth]
- → 2-step nilpotent
- → the semidirect product $K \ltimes_\rho \mathbb{R}^n$ where K is *pseudo-compact* and $\rho(K)$ is pre-compact in $\mathrm{GL}(n, \mathbb{R})$
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Corollary

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Proposition (EFSZ, 23)

A k-step nilpotent Lie group, with $k \geq 3$, does not have linear growth.

An aside

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Lemma (F., Agricola, 2017)

G a compact Lie group.
There exists a Riemannian metric which is left-invariant for both $G \times \mathfrak{g}$ and $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$.

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K be a pseudo-compact Lie group, H be another Lie group s.t. there is a homomorphism $\rho : K \longrightarrow \mathrm{Aut}(H)$ with $\rho(K)$ pre-compact.

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Corollary (EFSZ, 23)

Let K and H be two Lie groups as in Lemma above.
Then any pair of LI Riemannian metrics on $K \ltimes_\rho H$ and $K \times H$ are bi-Lipschitz bounded.

Thank you very much for the attention !

Proof of Lemma

Sketch.

- → $G = K \ltimes_\rho H$
- → K is pseudo-compact $\Rightarrow$ a bi-invariant metric on K
- → Moreover, $\rho(K)$ pre-compact $\Rightarrow$ an $\mathrm{Ad}(K)$-invariant positive definite inner product on $\mathfrak{g}$
- → R Riemannian metric on G induced by left translations
- → Isometry group of (G, R) contains $K \times G$
- → The map

$$(K \times G) \times G \longrightarrow G \quad \text{such that} \quad (k, x, y) \longmapsto xyk^{-1}$$

  is an action of $K \times G$ on G which preserves the Riemannian metric R.
- → Restrict this action to $K \times H$.
- → The action of $K \times H$ on G is transitive and free
- → Yields G as the principal homog. space $(K \times H)/\{1_G\}) \cong K \times H$.