Boundary value problems on domains with cusps

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Prospects in Geometry and Global Analysis 2023

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$\Omega \subset \mathbb{R}^n$ compact domain

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In particular: If f, g is smooth, then u is smooth.

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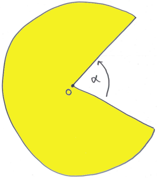

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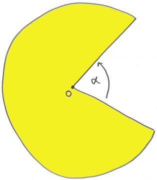

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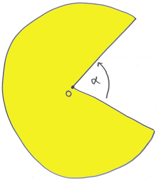

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But for $\alpha \in (0,\pi)$ there are f, g with $u \notin H^2$

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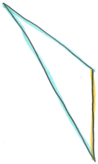

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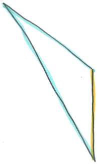
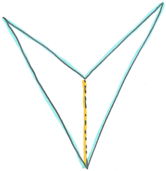


for $g_2 = 0$
reflect:



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for $f, \rho \colon \Omega \setminus \{\text{singularities}\} \to (0, \infty)$ continuous.

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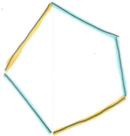

Theorem (Kondratiev/Mazya ∼67)

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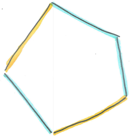

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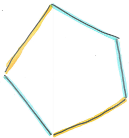

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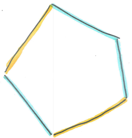

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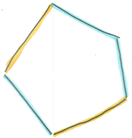

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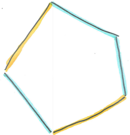

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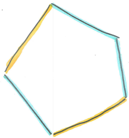

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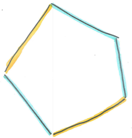

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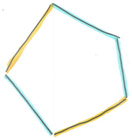

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hope to apply to other domains, boundary conditions, operators

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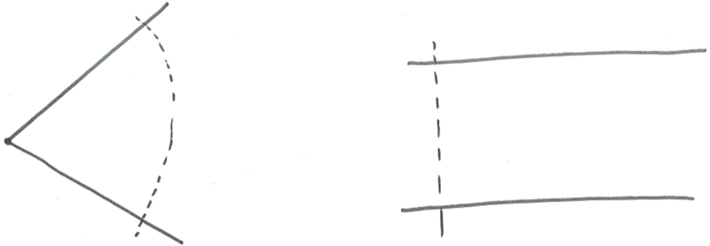

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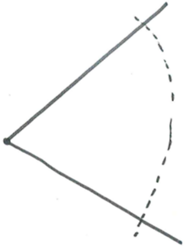

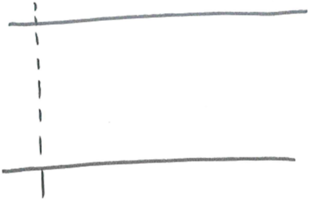

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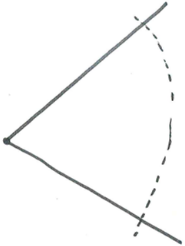

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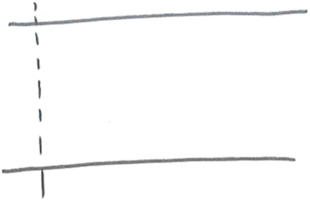

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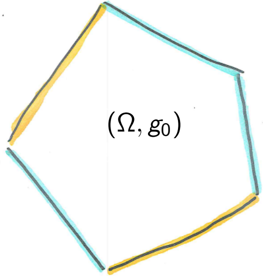

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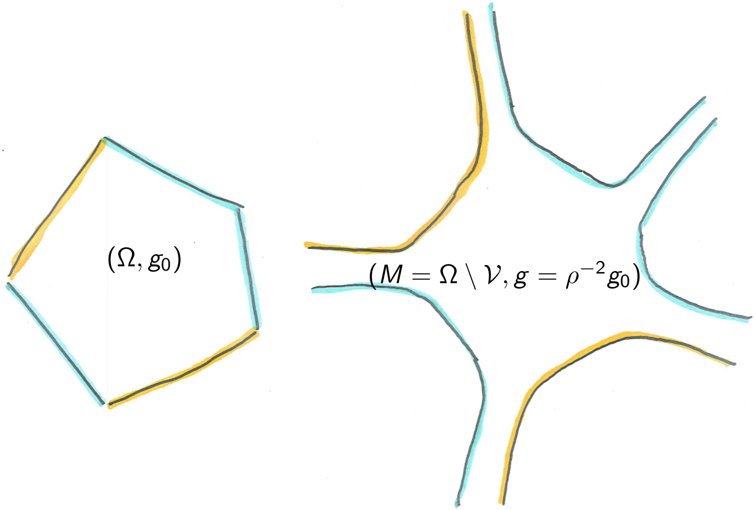

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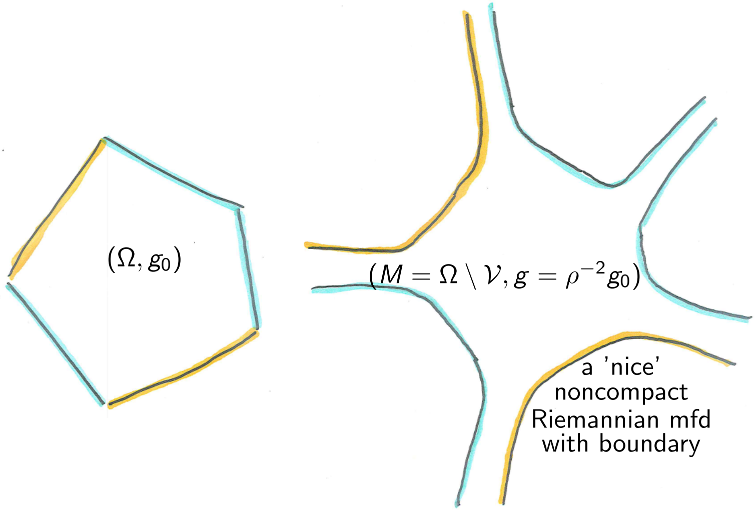

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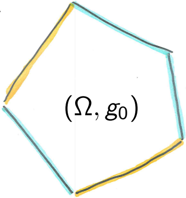

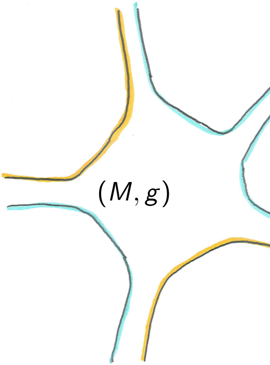

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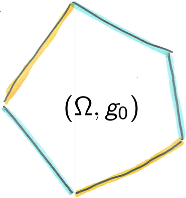

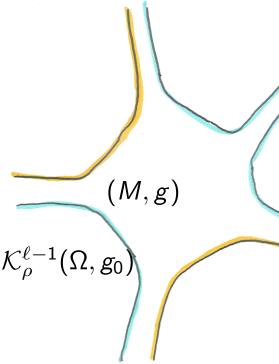

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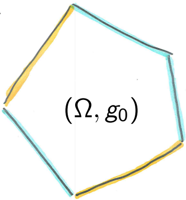

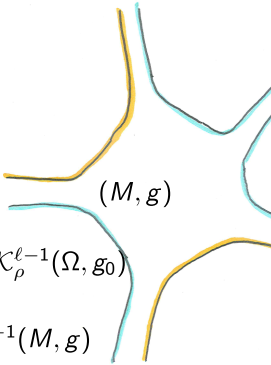

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$$\Updownarrow$$

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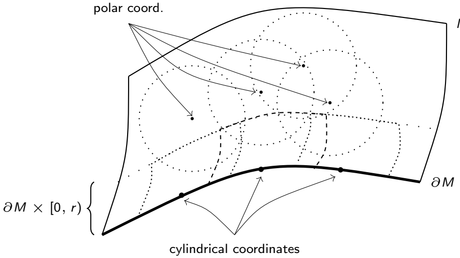

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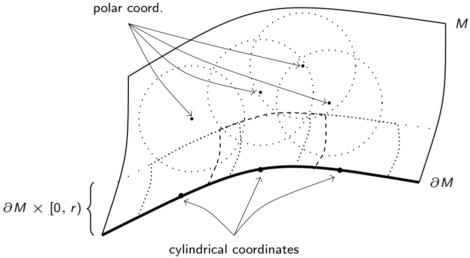

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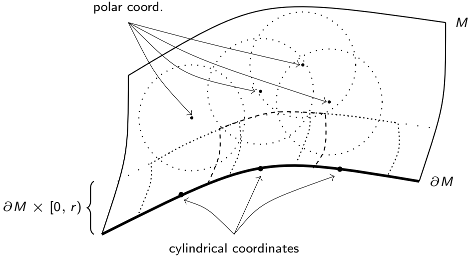

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- **implies regularity estimates, trace/extension theorems, ... from the local versions on $\mathbb{R}^n$.**

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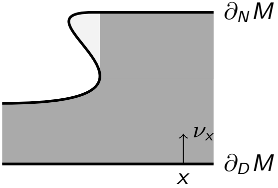

Well-posedness for Laplacian on bounded geometry manifolds

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is an isomorphism for all $k \geq 1$.

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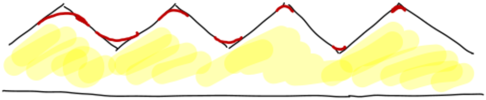
- 

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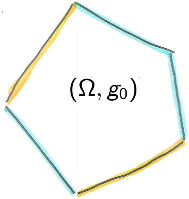

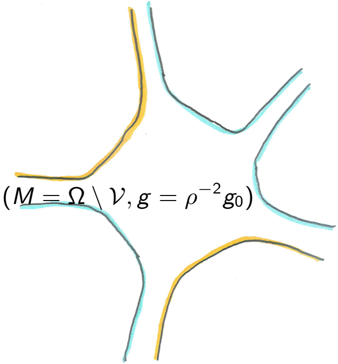

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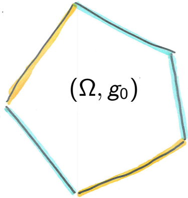

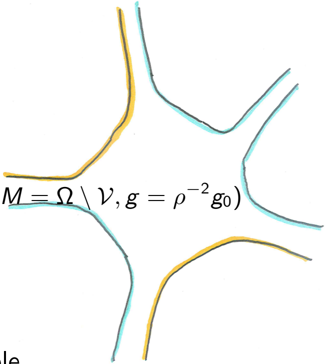

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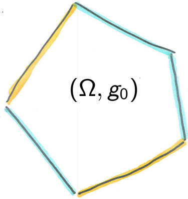

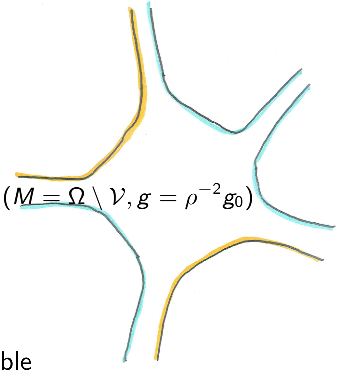

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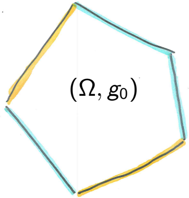

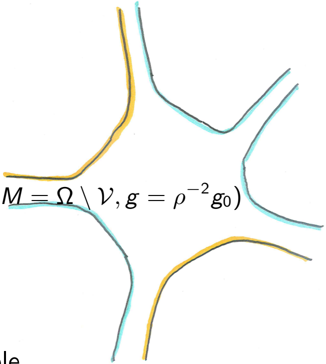

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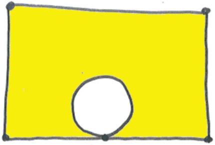
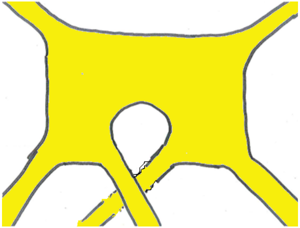
no adjacent edges in $\partial_N \Omega$ $\longleftrightarrow$ distance to $\partial_D M$ is finite

Example for a domain with cusps





$\rho \sim \text{dist}\,(., \text{cusp point})^2$ near the cusp point
$\rho \sim \text{dist}\,(., \text{corners})$ near the corners

'Strange' singularities in planar domains

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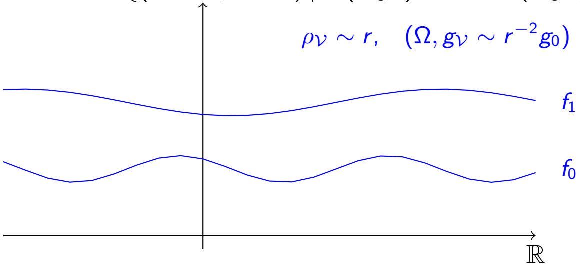

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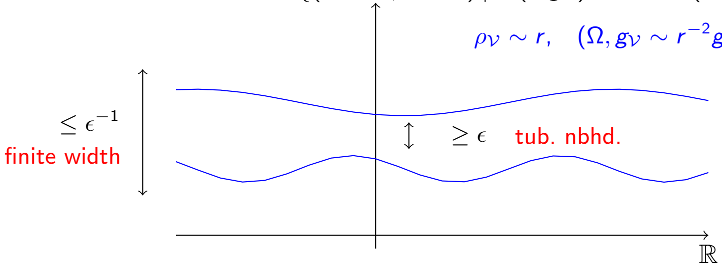
$\rho_{\mathcal{V}} \sim r, \quad (\Omega, g_{\mathcal{V}} \sim r^{-2} g_0)$



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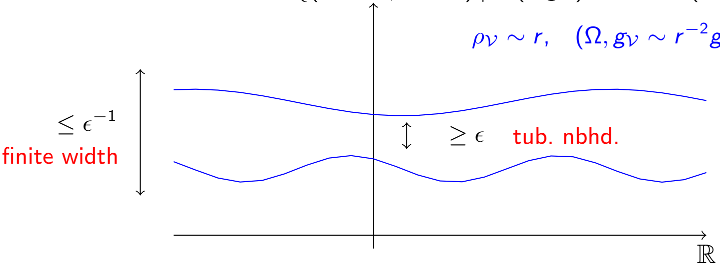

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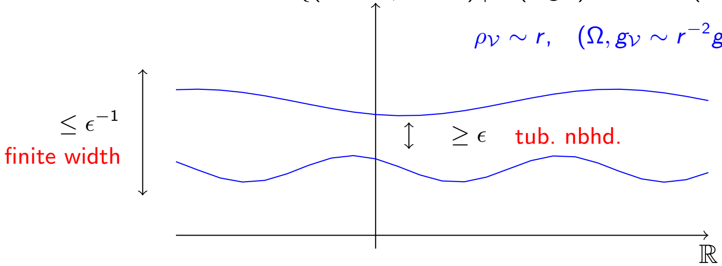
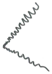
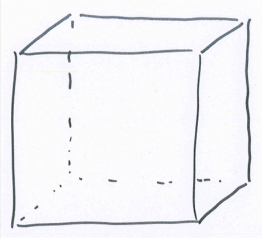


blow-down by r^2 - an oscillating singularity: 

Higher dimensions



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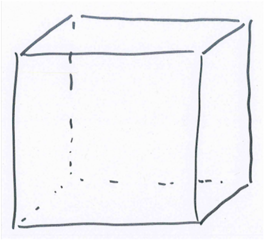
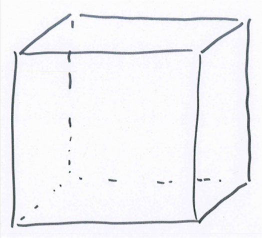


need to blow up corners and edges

Higher dimensions



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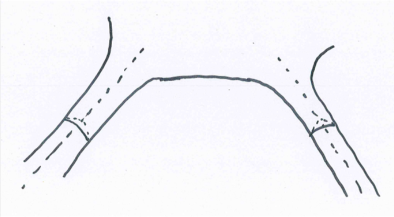

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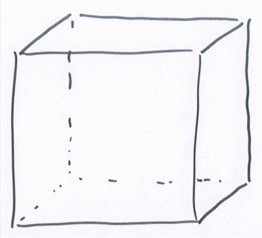

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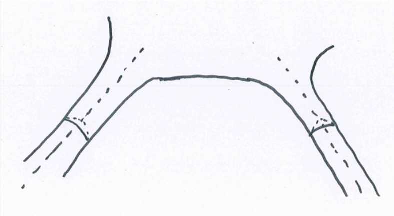

(M, g) mfd of bounded geometry and finite width to ∂M

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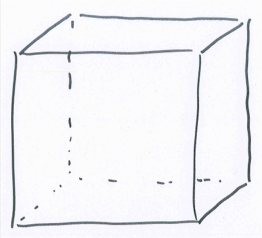

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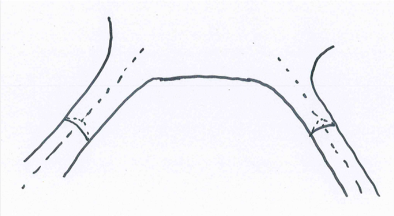

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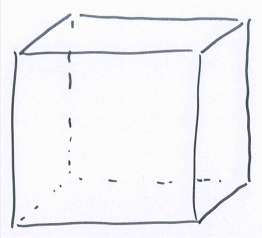

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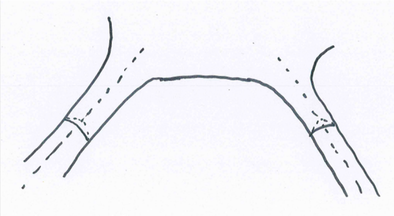

(M, g) mfd of bounded geometry and finite width to ∂M

But a new difficulty arises:(g_E.. Euclidean metric, $g = \rho^{-2} g_E$.)

$$\Delta_{g_E} u = \rho^{\frac{n+2}{2}} \left(4 \frac{n-1}{n-2} \Delta_g (\rho^{-\frac{n-2}{2}} u) + \mathrm{scal}_g \rho^{-\frac{n-2}{2}} u\right) =: \rho^{\frac{n+2}{2}} L_g (\rho^{-\frac{n-2}{2}} u)$$

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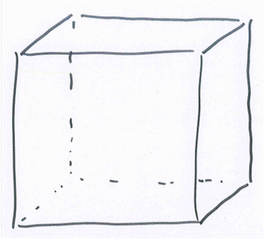

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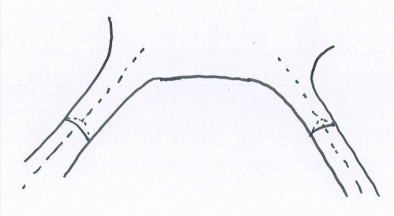

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– well-posedness of L_g on Sobolev scale gives well-pos. of Δ_{g_0} on Kondratiev scale

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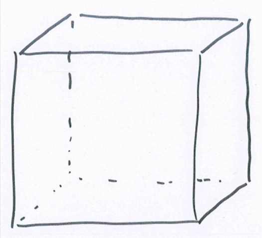

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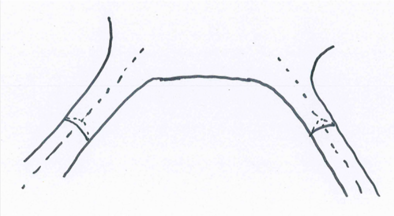

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– immediate if $\mathrm{scal}_g \geq 0$

Higher dimensions

Theorem (Ammann-G.-Nistor)

Let (M, g_0) be such that there is a $\rho\colon M \to (0,1]$ such that $(M, g := \rho^{-2} g_0)$ is a manifold with boundary $\partial M = \partial_D M \sqcup \partial_N M$ and of bounded geometry and finite width to $\partial_D M$. Assume that ρ is g-admissible and that (M, g_0) satifies a Hardy-Poincaré inequality, i.e. there is a $c > 0$ such that for all $u \in H^1_{loc}(M, g_0)$ with $u|_{\partial_D M} = 0$ we have $\int_M \rho^{-2} u^2 \leq c \int_M |du|^2$.
Then

$$\Delta_{g_0}\colon \rho \mathcal{K}^{\ell+1}_{\rho}(M, g_0) \cap \{u|_{\partial_D M} = 0, \partial_\nu u|_{\partial_N M} = 0\} \overset{\simeq}{\to} \rho^{-1} \mathcal{K}^{\ell-1}_{\rho}(M, g_0)$$

is an isomorphism.

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Example

In the above setting, for 'singular spaces' with Euclidean metric (and with a nice blow-up as before) we have a Hardy-Poincaré inequality. (e.g. ok for the cube)

Higher dimensions - this is currently written down

- Iterative definition of a stratified space such that
  - the blow-up is bounded geometry with finite width to ∂M
  - one gets the Hardy-Poincaré inequality

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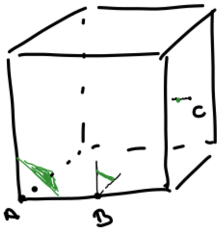
- Iterative definition of a stratified space such that
  - the blow-up is bounded geometry with finite width to ∂M
  - one gets the Hardy-Poincaré inequality
- Model near a point of the singularity set: For some $\epsilon > 0$ and an admissible stratified domain $B \subset \mathbb{R}^{\ell-1}$:

  $$K_h^\ell(B) \times \mathbb{R}^{n-\ell} := \{(t, th(t)y) \in \mathbb{R}^\ell \mid 0 < t < \epsilon, y \in B\} \times \mathbb{R}^{n-\ell},$$

  where $h\colon (0, \epsilon] \to (0, \infty)$ is a smooth function such that all derivatives $(t\partial_t)^k h$ are bounded ($k \geq 0$).

Finding the weight function



C $K^1_{h_C}(\text{point}) \times \mathbb{R}^2 = \{(t, th_C(t)) \in \mathbb{R}^2 \mid t \in (0, \epsilon)\} \times \mathbb{R}^2$
weight function near C $\sim$ (dist to face) $\cdot\, h_C$(dist to face)

B $K^2_{h_B}(\text{interval}) \times \mathbb{R} = \{(t, th_B(t)y) \in \mathbb{R}^3 \mid t \in (0, \epsilon), y \in [0, 1]\} \times \mathbb{R}$
weight function near B $\sim$ (dist to edge) $\cdot\, h_B$(dist to edge)

A $K^1_{h_A}(\Delta) = \{(t, th_A(t)y) \in \mathbb{R}^4 \mid t \in (0, \epsilon), y \in \Delta\}$
weight function near A
$\sim$ (dist to corner) $\cdot\, h_A$(dist to corner) $\cdot$ (weight fct of Δ)

Thank you for your attention!