The κ-nullity of Riemannian manifolds and their splitting tensors

Joint work with Claudio Gorodski

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August 21, 2023

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A Riemannian manifold is called *semi-symmetric* if its curvature tensor is, at each point, orthogonally equivalent to the curvature tensor of a symmetric space.

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The local classification of semi-symmetric spaces is the work of Z. I. Szabó [Sza85]: $M = S \times N$

For $\kappa \in \mathbb{R}$, the *κ-nullitty distribution* of M^n is the distribution $\mathcal{N}_\kappa$ on M defined for each $p \in M^n$ by

$$\mathcal{N}_\kappa|_p = \{Z \in T_pM : R_p(X, Y)Z = -\kappa(\langle X, Z\rangle_p Y - \langle Y, Z\rangle_p X)$$

The number $\nu_\kappa(p) := \dim \mathcal{N}_\kappa|_p$ is called the *index of κ-nullity* at p.

Outline

1. Splitting tensor
2. 0-nullity
3. 1-nullity
4. -1-nullity
5. Complete non-integrability of the conullity

Splitting tensor
Definition

Consider the orthogonal splitting $TM = \mathcal{D} \oplus \mathcal{D}^\perp$. For a vector field $X \in \Gamma(TM)$, we shall write $X = X^h + X^v$. Now we can define the *splitting tensor* of $\mathcal{D}^\perp$ as the map

$$C : \Gamma(\mathcal{D}^\perp) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$$

given by

$$C(T, X) = -(\nabla_X T)^h = C_T X$$

The splitting tensor satisfies a Ricatti-type ODE ([Fer70, Lem. 1]).

Proposition

The splitting tensor C of Δ satisfies

$$\nabla_T C_S = C_S C_T + C_{\nabla_T S} + \kappa \langle T, S \rangle I \tag{1.1}$$

for all $S,\ T \in \Gamma(\Delta)$. In particular, the operator $C_{\gamma'}$, along a unit speed geodesic γ in a leaf of Δ, satisfies

$$(C_{\gamma'})' = C_{\gamma'}^2 + \kappa I, \tag{1.2}$$

where the prime denotes covariant differentiation along γ.

Splitting tensor

Basic facts

Proposition

Let $\gamma : [0, b) \to M$ be a nontrivial unit speed geodesic with $p = \gamma(0)$ and $\gamma'(0) \in \Delta_p$ so that γ is a geodesic of the leaf of Δ through p. Then the splitting tensor $C_{\gamma'(t)} = C(t)$ of Δ at $\gamma(t)$ is given, in a parallel frame along γ, by

$$C(t) = -J_0'(t)J_0(t)^{-1}, \tag{1.3}$$

where

$$J_0(t) = \begin{cases} \cos(\sqrt{\kappa}t)I - \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}C_0 & \text{if } \kappa > 0, \\ \cosh(\sqrt{-\kappa}t)I - \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}}C_0 & \text{if } \kappa < 0, \\ I - tC_0 & \text{if } \kappa = 0, \end{cases} \tag{1.4}$$

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and $C_0 = C(0)$.In particular $J_0(t)$ is invertible for $t \in [0, b)$.

Corollary

Let $\gamma : [0, b) \to M$ be as in Proposition 1.2, where $b = \infty$.

(a) If $\kappa > 0$, then the splitting tensor $C_{\gamma'}$ has no real eigenvalues. It follows that $\rho(n - d) \geq d + 1$, where $n = \dim M$ and $d = \dim \Delta$; (cf. [Fer70, Thm. 1])

(b) If $\kappa \leq 0$, then any real eigenvalue λ of $C_{\gamma'}$ satisfies $|\lambda| \leq \sqrt{-\kappa}$.

Radon-Hurwitz number: $\rho(m = (\mathrm{odd})2^{c+4d}) = 2^c + 8d$ for $d \geq 0$ and $0 \leq c \leq 3$.

Splitting tensor

Basic facts

Lemma

Let $\gamma : [0, b) \to M$ with $\gamma' = T \in \Delta$. Denote the scalar curvature of M by scal, and put $n = \dim M$ and $d = \dim \Delta$. Then

$$\frac{1}{2}\frac{d}{dt}\mathrm{scal} = -\kappa(n-d-1)\mathrm{tr}\, C_T + \sum_{i\neq j}\langle R(C_T X_i, X_j)X_j, X_i\rangle, \tag{1.5}$$

where $\{X_i\}_{i=1}^{n-d}$ is a parallel orthonormal frame of $\Delta^\perp$ along γ.
In particular, in case $\Delta = \mathcal{N}_\kappa$, and $\nu_\kappa = n-2$ along γ, we have

$$\frac{1}{2}\frac{d}{dt}\mathrm{scal} = \mathrm{tr}\, C_T(K_{\mathcal{D}} - \kappa), \tag{1.6}$$

where $K_{\mathcal{D}}$ denotes the sectional curvature of the 2-plane distribution $\mathcal{D} = \mathcal{N}^\perp$.
Further, if in addition scal is constant, then $\mathrm{tr}\, C_T = 0$ and $\det C_T = \kappa$ along γ.

Lemma

Assume $\kappa \leq 0$, γ is a complete κ-nullity geodesic, $\nu_\kappa = n - 2$ and $K_{\mathcal{D}}$ is bounded away from κ along γ. Then $\operatorname{tr} C(t) = 0$ and $\det C(t) = \kappa$ for all $t \in \mathbb{R}$.

Splitting tensor
Basic facts

Lemma

Assume $\kappa \leq 0$, γ is a complete κ-nullity geodesic, $\nu_\kappa = n-2$ and $K_{\mathcal{D}}$ is bounded away from κ along γ. Then $\operatorname{tr} C(t) = 0$ and $\det C(t) = \kappa$ for all $t \in \mathbb{R}$.

Proof.

Note that $\frac{1}{2}\mathrm{scal} = K_{\mathcal{D}} + m\kappa$, where $m = \frac{n^2-n}{2} - 1$.

$$\frac{d}{dt}(K_{\mathcal{D}} - \kappa) = \operatorname{tr}(-J_0' J_0^{-1})(K_{\mathcal{D}} - \kappa) = -\frac{\frac{d}{dt}\det J_0}{\det J_0}(K_{\mathcal{D}} - \kappa).$$

Integration of this equation yields

$$K_{\mathcal{D}}(t) - \kappa = (K_{\mathcal{D}}(0) - \kappa)|\det J_0(t)|^{-1}.$$

□

Theorem

Let M be a simply-connected complete Riemannian n-manifold with maximal 0-conullity 2. Assume the scalar curvature function s is positive and bounded away from zero. Then M splits as the Riemannian product $\mathbb{R}^{n-2} \times \Sigma$, where Σ is diffeomorphic to the 2-sphere.

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We can substitute the hypothesis about scalar curvature by finite volume (universal cover) [SW17].

Question

Is there a simply-connected complete irreducible Riemannian n-manifold with constant 0-conullity 2 and nonnegative sectional curvature?

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Theorem

There exists an irreducible homogeneous Riemannian 5-manifold with 0-nullity 1 and finite volume.

The construction of this example follows the ideas of [SOF].

Lie algebra as a semidirect product $\mathfrak{g} = \mathbb{R} \ltimes_A V$, where V is an Abelian ideal and the action of $\mathbb{R}$ on V is determined by the adjoint action of a fixed generator $\xi \in \mathbb{R}$, which we represent by an operator $A \in \mathfrak{gl}(V)$, so that $[\xi, X] = AX$ for all $X \in V$.

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Note that $G = \mathbb{R} \ltimes_{e^A} V$, G is unimodular if and only if $\operatorname{tr} A = 0$, and $A \neq 0$ as G is non-Abelian.

Lemma (Filipkiewicz's criterion)

Suppose $G = \mathbb{R} \ltimes_A V$ is unimodular and non-nilpotent. Then there is a discrete subgroup Γ of G with G/Γ compact if and only if there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$, such that λA has a characteristic polynomial with integral coefficients.

Theorem

Let M^n be a simply-connected complete Riemannian with constant $(+1)$-conullity equal to 2, and constant scalar curvature.

1-nullity
results

Theorem

Let M^n be a simply-connected complete Riemannian with constant $(+1)$-conullity equal to 2, and constant scalar curvature. Then M is a 3-dimensional Sasakian space form, that is, isometric to one of the Lie groups $SU(2)$ (the Berger sphere), $\widetilde{SL(2,\mathbb{R})}$ (the universal covering of the unit tangent bundle of the real hyperbolic space), or Nil^3 (the Heisenberg group). In all cases, the $(+1)$-nullity distribution is orthogonal to the contact distribution.

Corollary

A complete Riemannian manifold modelled on one of the left-invariant metrics listed on Table is locally isometric to the corresponding model.

$$[e_1, e_2] = \lambda_3 e_3, \;\; [e_2, e_3] = \lambda_1 e_1, \;\; [e_3, e_1] = \lambda_2 e_2.$$

λ_1	λ_2	λ_3	M	scal	φ-sect curv	Condition
$\theta + 1/\theta$	θ	$1/\theta$	$SU(2)$	2	-1	$\theta > 0$
			$SU(2)$			$\theta > 0$
2	θ	θ	$\widetilde{SL}(2,\mathbb{R})$	$-2+4\theta$	$-3+2\theta$	$\theta < 0$
			Nil^3			$\theta = 0$

Theorem

Let M be a Riemannian n-manifold ($n \geq 3$) with (max) (-1)-conullity 2.

(a) If the scalar curvature is constant and $\mathcal{D} = \mathcal{N}_{-1}^{\perp}$ is integrable on an open subset U of (-1)-conullity 2 then U is locally isometric to the group of ridig motions of the Minkowski plane, $E(1,1) = SO_0(1,1) \ltimes \mathbb{R}^2$, with a left-invariant metric.

(b) Assume M is complete and has finite volume. Assume, in addition, that either $n = 3$ or the scalar curvature bounded away from $-n(n-1)$. Then the universal covering of M is homogeneous.

(c) If M is homogeneous and simply-connected, then M is isometric to $E(1,1)$ or $\widetilde{SL(2,\mathbb{R})}$ with a left-invariant metric.

$n = 3$:

$$C_T = \begin{pmatrix} -1 & 0 \\ 2F & 1 \end{pmatrix},$$

We can write the Levi-Cività connection as follows:

$$\nabla_T T = \nabla_T X = \nabla_T Y = 0,\ \nabla_X T = X - 2FY,\ \nabla_Y T = -Y,$$
$$\nabla_X X = -T + \alpha Y,\ \nabla_Y Y = T + \beta X,\ \nabla_X Y = 2FT - \alpha X,\ \nabla_Y X = -\beta Y,$$

The bracket relations follow:

$$[X, Y] = 2FT - \alpha X + \beta Y,\ [T, X] = -X + 2FY,\ [T, Y] = Y.$$

Using the curvature relations:

$$\begin{aligned} \alpha &= -\beta F, \\ T(\beta) &= \beta, \\ Y(F) &= -\beta(1 + F^2), \\ X(\beta) - FY(\beta) &= K_{\mathcal{D}} - 1. \end{aligned}$$

Complete non-integrability of the conullity

Results

Theorem

Let M be either:

(a) a connected complete Riemannian manifold of with nonzero constant index of κ-nullity, where $\kappa > 0$; or

(b) a connected simply-connected irreducible locally homogeneous Riemannian manifold with nonzero index of 0-nullity.

Then any two points of M can be joined by a piecewise smooth curve which is orthogonal to the distribution of κ-nullity at smooth points.

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Thank you for your attention!