Geometry of Aloff-Wallach spaces

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Outline

1. The Aloff-Wallach Space(s) $W^{k,l}$
2. Sasaki and G_2-Geometry
3. Geometric Structures on $W^{k,l}$
4. The Spectrum of a Normal Homogeneous Space
5. Normal Homogeneous Realizations of $(W^{k,l}, g)$
6. The Spectrum on $(W^{1,1}, g_{t_1,t_2})$

MOTIVATION

- Aloff-Wallach spaces were introduced by S. Aloff and N. Wallach in 1974
  - Family of manifolds of strictly positive curvature
- Family of manifolds with parameter-depending metric
  - 'Playground' for different geometric structures
  - Interesting examples for various applications
- Already lot of results:
  - Agricola, Ball, Baum, Dileo, Friedrich, Grunewald, Kath, A. Moroianu, Oliveira, Semmelmann, Stecker, . . .

Goal: Understanding geometric structures in big picture; relation to spectral properties etc.

The Aloff-Wallach Space(s) $W^{k,l}$

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- For $k, l \in \mathbb{Z}$ relatively prime we define $W^{k,l} := \mathrm{SU}(3) \big/ S^1_{k,l}$ where

$$S^1_{k,l} \hookrightarrow \mathrm{SU}(3), \quad z \mapsto \operatorname{diag}\left(z^k, z^l, z^{-(k+l)}\right)$$

- Lie-Algebra decomposition: $\mathfrak{su}(3) = \mathfrak{s}^1 \oplus \mathfrak{m}$ where

$$\begin{aligned}
\mathfrak{m} &= \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \\
\mathfrak{m}_0 &:= \operatorname{span}\left\{ \begin{pmatrix} (2l+k)\,\mathrm{i} & 0 & 0 \\ 0 & -(2k+l)\,\mathrm{i} & 0 \\ 0 & 0 & (k-l)\,\mathrm{i} \end{pmatrix} \right\}, \\
\dim(\mathfrak{m}_i) &= 2
\end{aligned}$$

- Using $B(X,Y) = -\frac{1}{2}\operatorname{Re}\left(\operatorname{tr}(XY)\right)$ we get a family of metrics g on $W^{k,l}$, parametrized by $\lambda, x, y, z \in \mathbb{R}^+$:

$$g = g_{\mathfrak{m}} = \lambda \cdot B\big|_{\mathfrak{m}_0} + \frac{1}{x} \cdot B\big|_{\mathfrak{m}_1} + \frac{1}{y} \cdot B\big|_{\mathfrak{m}_2} + \frac{1}{z} \cdot B\big|_{\mathfrak{m}_3}.$$

Isotropy Representation of $W^{k,l}$

- We choose ONB: $X_1 \in \mathfrak{m}_0$, $X_i, X_{i+1} \in \mathfrak{m}_{i+1}$
- The isotropy representation $\mathrm{Ad}_{S^1_{k,l}}$ on $W^{k,l}$ is given by

$$\mathrm{Ad}_{k,l}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R\left[(k-l)\,\theta\right] & 0 & 0 \\ 0 & 0 & R\left[(2k+l)\,\theta\right] & 0 \\ 0 & 0 & 0 & R\left[(k+2l)\,\theta\right] \end{pmatrix},$$

where $R[\alpha] = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$.

  - Obviously: X_1 is always $\mathrm{Ad}_{S^1_{k,l}}$-invariant
  - If $k = l (= 1)$, then X_2, X_3 are also Ad-invariant

Sasaki and G_2-Geometry

Almost Contact Metric Structures

- An almost contact metric structure (φ, ξ, η, g) on a Riemannian manifold (M^{2n+1}, g) is given by
  - a $(1,1)$-tensor field φ,
  - a Reeb vector field ξ,
  - and a 1-form η

  such that

$$\varphi^2 = -\operatorname{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

- An almost contact metric structure is α-Sasaki iff
  - $d\eta = 2\alpha\, \Phi$, where $\Phi(X, Y) := g(X, \varphi(Y))$,
  - ξ is Killing,
  - $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi = 0$ (normal)

- Almost 3-contact metric manifold: Mfd. M endowed with three 'compatible' almost contact metric structures $(\varphi_i, \xi_i, \eta_i, g)$.
  - Then $\dim M = 4n+3$, $n \geq 1$ and

    $$TM = \mathcal{H} \oplus \mathcal{V}, \qquad \mathcal{H} := \bigcap_{i=1,2,3} \ker \eta_i, \qquad \mathcal{V} := \operatorname{span}\{\xi_1, \xi_2, \xi_3\}.$$

  - M is hypernormal if $N_{\varphi_i} = 0$, $i \in \{1,2,3\}$.

Definition

A 3-(α,δ)-Sasaki manifold is an almost 3-contact metric manifold such that

$$d\eta_i = 2\alpha\,\Phi_i + 2(\alpha-\delta)\,\eta_j \wedge \eta_k,$$

for some $\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}$, (i,j,k) any even permutation of $(1,2,3)$.

Subcases:

- $\alpha = \delta$: 3-α-Sasaki manifold
- $\alpha = \delta = 1$: 3-Sasaki manifold

3-(α,δ)-Sasaki Structures

Facts (Agricola, Dileo, 2020):

- Every 3-(α,δ)-Sasaki manifold M is hypernormal, all ξ_i are Killing.
- Its Ricci curvature in dimension 7 is given by

$$\mathrm{Ric}^g = 2\alpha\big(6\delta - 3\alpha\big)g + 2(\alpha - \delta)\big(5\alpha - \delta\big)\sum_{i=1}^{3}\eta_i \otimes \eta_i.$$

  $\Rightarrow$ ∇^g-Einstein iff $\delta = \alpha$ or $\delta = 5\alpha$.

- $\exists$ adapted metric connection with skew torsion, the canonical connection. It satisfies for $\beta := 2(\delta - 2\alpha)$

$$\nabla_X\varphi_i = \beta\left(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k\right), \quad \nabla_X\xi_i = \beta\left(\eta_k(X)\xi_j - \eta_j(X)\xi_k\right).$$

  $\Rightarrow$ We call the 3-(α,δ)-Sasaki manifold parallel if $\beta = 0$.

3-(α,δ)-Sasaki and G_2

Facts (Agricola, Dileo, 2020):

- Every 7-dim. 3-(α,δ)-Sasaki manifold admits cocalibrated G_2-structure such that characteristic connection coincides with the canonical one
  - Cocalibrated G_2: $\omega^3 \in \Omega^3(M^7)$ such that $\delta\omega^3 = 0$ (Fernandez-Gray $W_1 \oplus W_3$)

- (proper) nearly parallel G_2 iff $\delta = 5\alpha$
  - $d\omega^3 = \lambda * \omega^3$ (Fernandez-Gray W_1)
  - nearly-parallel $G_2 \Rightarrow \left(M^7, g\right)$ is ∇^g-Einstein

Geometric Structures on $W^{k,l}$

The isotropy representation leads to two cases:

Case I: $k \neq l$ (one invariant vector field, $X_1 \in \mathfrak{m}_0$)

Theorem

The Aloff-Wallach space $(W^{k,l}, g)$ admits an α-Sasaki structure if

$$0 < 2\alpha\sqrt{\frac{k^2 + kl + l^2}{3\lambda}} = x(k+l) = ky = lz,$$

where $\xi := X_1,\ \eta := g(\cdot, X_1),\ \varphi := -\frac{1}{\alpha}\,\nabla^g X_1$.

Subcase $k = l(= 1)$ and $x = \alpha^2$:

$$x = \alpha^2 = \frac{1}{\lambda}, \quad y = z = 2\alpha^2$$

Case II: $k = l = 1$ (3 invariant vector fields, $X_1 \in \mathfrak{m}_0$, $X_2, X_3 \in \mathfrak{m}_1$)

Theorem

The Aloff-Wallach space $(W^{1,1}, g)$ admits a 3-(α,δ)-Sasaki structure if $\alpha > 0$, $\delta > 0$ as well as

$$x = \delta^2 = \frac{1}{\lambda}, \quad y = z = 2\alpha\delta,$$

with $\xi_i := X_i$ and $\varphi_i := -\frac{1}{\alpha}\nabla^g X_i - \frac{\alpha-\delta}{\alpha}\left[\eta_k \otimes \xi_j - \eta_j \otimes \xi_k\right]$.

Special subcases:

- $\delta = \alpha$: Reduction to 3-α-Sasaki structure which is ∇^g-Einstein

$$x = \alpha^2 = \frac{1}{\lambda}, \quad y = z = 2\alpha^2$$

- $\delta = 2\alpha$: Parallel case ($\beta = 0$); implies $\frac{1}{\lambda} = x = y = z = 4\alpha^2$
  $\Rightarrow$ The metric scales all $\mathfrak{m}_i$ equally!

G_2-Structure on $W^{1,1}$

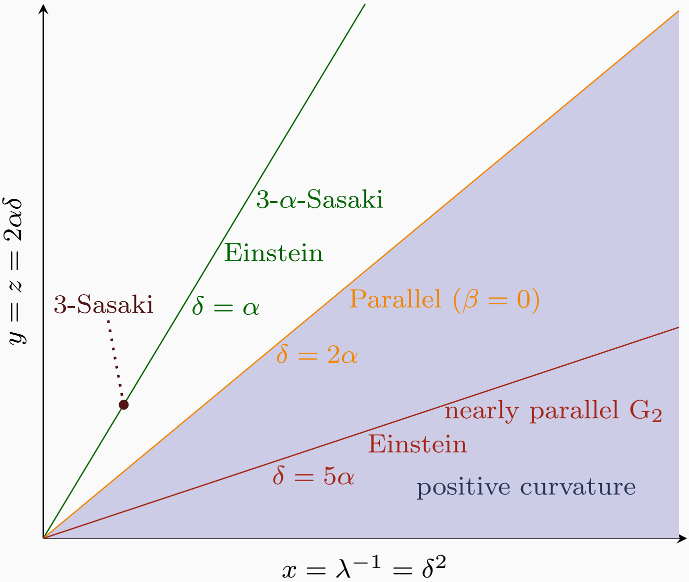
Recall: Every 7-dim. 3-(α,δ)-Sasaki manifold admits a cocalibrated G_2-structure

Corollary

The associated cocalibrated G_2-structure is given by:

$$\omega = -\omega_{145} + \omega_{167} + \omega_{246} + \omega_{257} - \omega_{347} + \omega_{356} + \omega_{123}$$

It is nearly parallel if and only if $\delta = 5\alpha$. The eigenvalue is given by $\lambda = 12\alpha$.


$y = z = 2\alpha\delta$
3-α-Sasaki
Einstein
$\delta = \alpha$
3-Sasaki
Parallel ($\beta = 0$)
$\delta = 2\alpha$
nearly parallel G_2
Einstein
$\delta = 5\alpha$
positive curvature
$x = \lambda^{-1} = \delta^2$

The Spectrum of a Normal Homogeneous Space

General Setting:

- G is a compact Lie group.
- $(G/K, g)$ is a reductive Riemannian homogeneous space.
- G/K is normal hom., i.e. g is the restriction of a biinvariant metric on G.
- $\mathrm{D}(G)$ is the set of dominant G-integral weights.

K-spherical representations

A unitary, irred. G-rep. $(\varrho_\lambda, V_\lambda)$ associated to $\lambda \in \mathrm{D}(G)$ of dimension $d(\lambda)$ is called K-*spherical* if

$$m(\lambda) = \dim \{v \in V_\lambda \mid \varrho_\lambda(K)v = v\} \neq 0.$$

The set of dominant integral weights corresponding to K-spherical rep. is denoted by $\mathrm{D}(G, K)$.

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Frobenius reciprocity theorem [Wallach, 1973]

$L^2(G/K)$ decomposes into K-spherical representations:

$$L^2(G/K) \cong \bigoplus_{\lambda \in \mathrm{D}(G,K)} \underbrace{V_\lambda \oplus \ldots \oplus V_\lambda}_{m(\lambda) \text{ times}}$$

$\rightsquigarrow$ decomposition of the eigenspaces of the Laplacian. Each ϱ_λ occurs precisely in one eigenspace.

Theorem [Urakawa 1984]

The spectrum of the Laplacian on $(G/K, g)$ is given by:

$$\Sigma(G/K, g) = \{g(\lambda + 2\delta, \lambda) \mid \lambda \in \mathrm{D}(G, K)\},$$

where $\delta = \frac{1}{2} \sum_{\mu \in R^+} \mu$. The multiplicity corresponding to $\lambda \in \mathrm{D}(G, K)$ can be computed by: $\mathrm{mult}(\lambda) = m(\lambda) \dim(V_\lambda) =: m(\lambda) d(\lambda)$.

Normal Homogeneous Realizations of $(W^{k,l}, g)$

The Realization $SU(3)/S^1_{k,l}$

Recall: The Aloff-Wallach spaces are $W^{k,l} = SU(3)/S^1_{k,l}$ and

$$g = \lambda \cdot B\big|_{\mathfrak{m}_0} + \frac{1}{x} \cdot B\big|_{\mathfrak{m}_1} + \frac{1}{y} \cdot B\big|_{\mathfrak{m}_2} + \frac{1}{z} \cdot B\big|_{\mathfrak{m}_3}.$$

Lemma

$(W^{k,l}, g)$ is a normal hom. space with resp. to $\mathrm{SU}(3)$ iff

$$\lambda = \frac{1}{x} = \frac{1}{y} = \frac{1}{z}.$$

- Urakawa (1984) computed the spectrum in this case.

THE METRIC g_{t_1,t_2}

Notation

We introduce the parameters $t_1 = \lambda = \frac{1}{x}$, $t_2 = \frac{1}{y} = \frac{1}{z}$ and denote $g_{t_1,t_2} = g$.

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- Aloff and Wallach (1975) proved: If $kl > 0$ and $0 < t_1 < t_2$ then $(W^{k,l}, g_{t_1,t_2})$ has positive curvature.

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- Wilking (1999) found a different realization (V_3, h_{r_1,r_2}) of $(W^{1,1}, g_{t_1,t_2})$ which is normal hom..

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  - $\rightsquigarrow$ He closed a gap in Berger's classification from 1961 of simply connected, normal hom. spaces with positive curvature.

Consider the inclusion

$$\iota : \mathrm{U}(2) \to \mathrm{SU}(3), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}$$

and the projection $\pi : \mathrm{U}(2) \to \mathrm{U}(2)/S^1 \cong \mathrm{SO}(3)$.

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The space V_3

$$V_3 := (\mathrm{SU}(3) \times \mathrm{SO}(3)) / U^\bullet(2), \quad U^\bullet(2) = (\iota, \pi)((\mathrm{U}(2)))$$

For $B = -\frac{1}{2}\,\mathrm{tr}$, V_3 is equipped with the restriction of the biinvariant metrics on $\mathrm{SU}(3) \times \mathrm{SO}(3)$:

$$h_{r_1,r_2} := r_1 \cdot B\big|_{\mathfrak{so}(3)} + r_2 \cdot B\big|_{\mathfrak{su}(3)}, \quad r_1, r_2 \in (0, \infty).$$

Proposition [Wilking, 1999]

For each $r_1, r_2 \in \mathbb{R}^+$ the $\mathrm{SU}(3) \times \mathrm{SO}(3)$-normal hom. space (V_3, h_{r_1,r_2}) is isometric to the non-$\mathrm{SU}(3)$-normal hom. space $(W^{1,1}, g_{t_1,t_2})$. The machting of coeff. is $t_1 = \frac{4r_1r_2}{4r_1+r_2} < r_2$ and $t_2 = r_2$. All metrics on $W^{1,1}$ with $0 < t_1 < t_2$ are covered!

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⇝ Spectrum can be calculated

- $t_1 = t_2$ is not covered by the proposition! (would correspond to $r_1 \to \infty$)

The Spectrum on $(W^{1,1}, g_{t_1,t_2})$

- The dominant integral weights $\lambda \in \mathrm{D}(\mathrm{SU}(3) \times \mathrm{SO}(3))$ are parameterized by $z_1, z_2, z_3 \in \mathbb{N}_0$ with $z_1 \geq z_2$.

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Theorem

The spectrum on (V_3, h_{r_1,r_2}) is obtained by

$$\Sigma(V_3, h_{r_1,r_2}) = \left\{ \frac{{z_3}^2 + z_3}{r_1} + \frac{4({z_1}^2 + {z_2}^2 - z_1(z_2 - 3))}{3r_2} \,\middle|\, m(z_1, z_2, z_3) > 0 \right\}.$$

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Theorem

The spectrum on $(W^{1,1}, g_{2,2})$ has been calculated by Urakawa (1984):

$$\Sigma(W^{1,1}, g_{2,2}) = \left\{ \frac{2({z_1}^2 + {z_2}^2 - z_1(z_2 - 3))}{3} \,\middle|\, m(z_1, z_2) > 0 \right\}.$$

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- This can be done using the branching rules.
  - $\rightsquigarrow$ Complicated combinatorial problem. Can be solved using a computer.

Theorem

First $\lambda \in \mathrm{D}(\mathrm{SU}(3) \times \mathrm{SO}(3), \mathrm{U}^{\bullet}(2))$ with multiplicities are:

z_1	0	2	2	3	3	4	4	4	5	5	5	5	6
z_2	0	1	1	0	3	2	2	2	1	1	4	4	0
z_3	0	0	1	1	1	0	1	2	1	2	1	2	2
mult	1	8	24	30	30	27	81	135	105	175	105	175	140

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Theorem [Urakawa 1984]

First $\lambda \in \mathrm{D}(\mathrm{SU}(3), S^1_{1,1})$ with multiplicities are:

z_1	0	2	3	3	4	5	5	6
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Observations

The first eigenvalues and their multiplicities suggest:

- Multiplicities of $\Sigma(W^{1,1}, g_{2,2})$ split into those of $\Sigma(W^{1,1}, g_{t_1,2})$.

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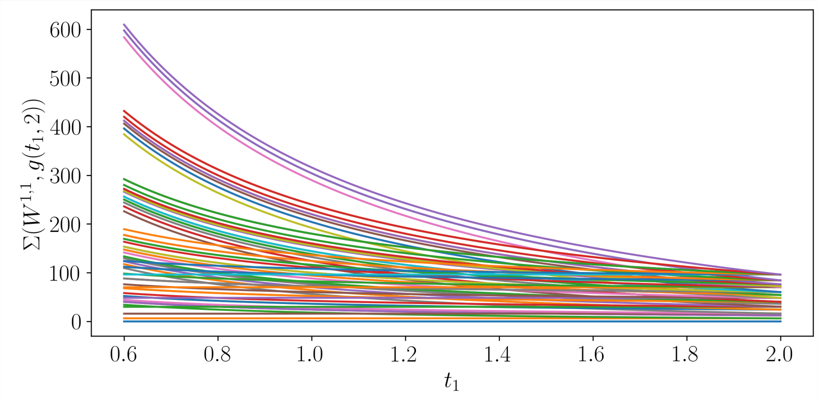
- $\Sigma(W^{1,1}, g_{t_1,2}) \neq \Sigma(W^{1,1}, g_{\tilde{t}_1,2})$ for $t_1, \tilde{t}_1 \leq 1$ and $t_1 \neq \tilde{t}_1$.
- In case $z_3 = 0$, the eigenvalue does not depend on t_1. This is precisely the spectrum of $\mathbb{C}\mathbb{P}^2$ as the canonical projection has base space $\mathbb{C}\mathbb{P}^2$.



600
500
400
300
200
100
0
$\Sigma(W^{1,1}, g(t_1, 2))$
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
t_1

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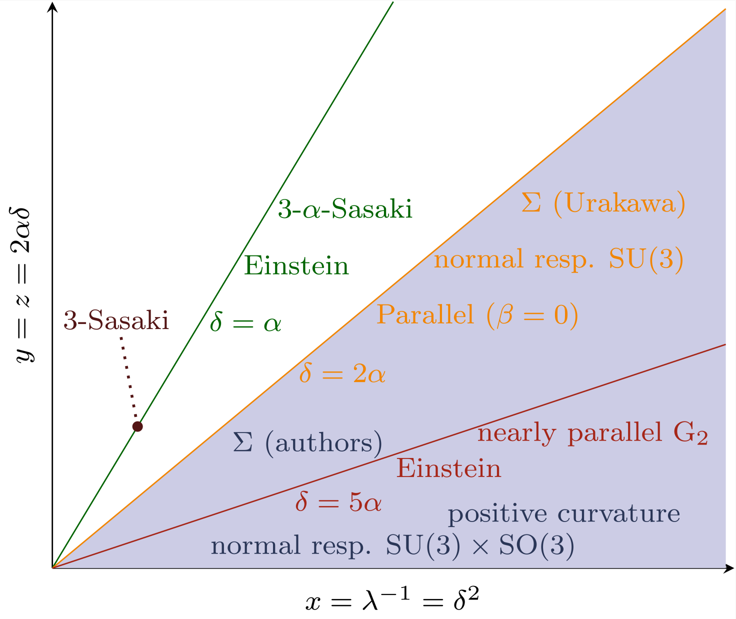
- $(W^{1,1}, g_{2,2})$ (i.e., $\beta = 0$) appears most symmetrical.
- The nearly parallel G_2 case $(t_1 = 0.8)$ can not be seen in the spectrum.



$y = z = 2\alpha\delta$
$x = \lambda^{-1} = \delta^2$
3-α-Sasaki
Einstein
$\delta = \alpha$
3-Sasaki
Σ (Urakawa)
normal resp. SU(3)
Parallel ($\beta = 0$)
$\delta = 2\alpha$
nearly parallel G_2
Σ (authors)
Einstein
$\delta = 5\alpha$
positive curvature
normal resp. SU(3) $\times$ SO(3)

References I

- Aloff, S., Wallach, N.R. (1975). *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures.* Bull. Am. Math. Soc. 81, 93–97 (Zbl 0362.53033)
- Bando, S., Urakawa, H. (1983). *Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds.* Tohoku Math. J. (2) 35(2): 155-172 (1983). DOI: 10.2748/tmj/1178229047
- Berger, M. (1961). *Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive.* Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 15.3 (1961): 179-246.
- Baum, H., Friedrich, T., Grunewald, R., Kath, I. (1991). *Twistors and Killing Spinors on Riemannian Manifolds*

Goodman, R., Wallach, N. R. (2009). *Symmetry, representations, and invariants.* Based on the book ”Representations and invariants of the classical groups” originally published by Cambridge University Press, 1998. New York, NY: Springer (Zbl 1173.22001).

Urakawa, H. (1984). *Numerical Computations of the Spectra of the Laplacian on 7-Dimensional Homogeneous Manifolds $SU(3)/T(k,l)$.* SIAM Journal on Mathematical Analysis, 15(5) 979-987. DOI: 10.1137/0515074

Wallach, N. R. (2018). *Harmonic analysis on homogeneous spaces.* 2nd edition, revised and updated republication of the 1973 original published by Marcel Dekker. Mineola, NY: Dover Publications (Zbl 1406.22001)

Wilking, B. (1999). *The Normal Homogeneous Space* $(SU(3) \times SO(3))/U^{\bullet}(2)$ *Has Positive Sectional Curvature.* Proceedings of the American Mathematical Society, 127(4), 1191–1194. http://www.jstor.org/stable/119243

THE DIFFEOMORPHISM $W^{1,1} \to V_3$

We denote by $\mathrm{U}^{\blacktriangle}(2) = (\iota, \mathrm{Id})(\mathrm{U}(2))$

$$\begin{aligned}
\mathrm{SU}(3) \big/ S^1_{1,1} &\cong \left(\mathrm{SU}(3)/S^1_{1,1} \times \mathrm{U}(2)\right) \big/ \mathrm{U}^{\blacktriangle}(2) \\
&= \left(\mathrm{SU}(3) \times \mathrm{U}(2)/S^1\right) \big/ \mathrm{U}^{\blacktriangle}(2) \\
&\cong \left(\mathrm{SU}(3) \times \mathrm{SO}(3)\right) \big/ \mathrm{U}^{\bullet}(2)
\end{aligned}$$

Berger's Classification corrected by Wilking

Berger [Berger, 1961] asserted that any simply connected, normal homogeneous space with positive sectional curvature is diffeomorphic to a CROSS $S^n, \mathbb{C}\mathbb{P}^n, \mathbb{H}\mathbb{P}^n, Ca\mathbb{P}^2$ or

1. $V_1 = Sp(2) \big/ SU(2)$
2. $V_2 = SU(5)/H$, where H is given by

$$H = \left\{ \begin{bmatrix} zA & 0 \\ 0 & \bar{z}^4 \end{bmatrix} \middle| A \in Sp(2) \subset SU(4), z \in S^1 \subset \mathbb{C} \right\} \subset U(4) \subset SU(5).$$

Berger's theorem is not correct. He missed a third exceptional space which has been found by Wilking [Wilking, 1999]:

3. (V_3, h_{r_1,r_2})

Wilking proved that this completes the classification.

COMPUTATION OF $m(\lambda)$?

- In general, one can use the Branching rules [Goodman, Wallach 1998].
- Those are combinatorially challenging:

$$m(\lambda) = \sum_{w \in W} \mathrm{sgn}(w) \wp(w(\lambda + \delta)\big|_{\mathfrak{t}(\mathfrak{k})} - \delta\big|_{\mathfrak{t}(\mathfrak{u}(2))}).$$

- $\wp$ denotes the Kostant partition function, W is the Weyl group and $\mathfrak{t}(\mathfrak{u}(2))$ the maximal torus of the complexification of $\mathfrak{u}(2) \subset \mathfrak{su}(3) \times \mathfrak{so}(3)$

GENERIC PROPERTIES OF THE SPECTRUM

- M is a manifold without boundary of dimension n
- $\mathcal{M}$ is the set of all Riemannian metrics on M.

Theorem [Bando, Urakawa 1983]

1. The spectrum counted with multiplicities depends uniformly continuous on the metric $g \in \mathcal{M}$ with resp. to the $\mathcal{C}^\infty$ topology.
2. The multiplicities $m_k(g) = \#\{i \mid \eta_i(g) = \eta_k(g)\}$ of each eigenvalue $\eta_k(g)$ depends upper semi-continuously on $g \in \mathcal{M}$: For each $g \in \mathcal{M}$ and $k = 0, 1, 2, \ldots$ there exists a $\delta > 0$ such that $d(g, g') < \delta$ implies $m_k(g') \leq m_k(g)$.