Geometry of Aloff-Wallach Spaces

Ilka Agricola Jonas Henkel Henrik Naujoks

Philipps-Universität Marburg

1. Goals & Motivation

Since their first description in 1975, Aloff-Wallach spaces have turned out to be a rich source of examples in differential geometry:

- They admit a 1-parameter family of metrics of positive curvature,
- They are prominent examples of Sasaki, 3-Sasaki, and G_2-manifolds,
- In 1999, Wilking showed that some Aloff-Wallach metrics are normal homogeneous and are thus missing in the 1961 classification of Berger.

The goal of our project is:

- Give a *complete* description of the properties of AW metrics, in particular in the newer context of G-structures with torsion & special spinors,
- Compute the spectrum of the Laplacian – this heavily relies on Wilking's alternative description.

2. The Aloff-Wallach Spaces $W^{k,l}$ as Riemannian Homogeneous Space

Definition. For $k, l \in \mathbb{Z}$ relatively prime the Aloff-Wallach spaces are defined by $W^{k,l} = \mathrm{SU}(3)/S^1_{k,l}$ where we embed $S^1_{k,l}$ via

$$\iota\colon S^1_{k,l} \hookrightarrow \mathrm{SU}(3), \quad z \mapsto \mathrm{diag}\left(z^k, z^l, z^{-k+l}\right).$$

Lemma. Let $k_1, l_1, k_2, l_2 \in \mathbb{Z}$ such that k_i, l_i are relatively prime. Then W^{k_1,l_1} is diffeomorphic to W^{k_2,l_2} if and only if

$$(k_2, l_2) \text{ or } (l_2, k_2) \in \{(k_1, l_1), (-(k_1+l_1), k_1), (-(k_1+l_1), l_1)\}.$$

We have the Lie algebra decomposition: $\mathfrak{su}(3) = \mathfrak{s}^1 \oplus \mathfrak{m}$. Moreover, $\mathfrak{m}$ splits as $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, where

$$\mathfrak{m}_0 = \mathrm{span}\{\mathrm{i}\,\mathrm{diag}[2l+k, -2k-l, k-l]\},$$
$$\dim(\mathfrak{m}_i) = 2, \quad [\mathfrak{m}_0, \mathfrak{m}_i] \subset \mathfrak{m}_i, \quad [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{s}^1 \oplus \mathfrak{m}_0, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{m}_k,$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. Using the negative re-scaled Killing form $B = -\frac{1}{2}\mathrm{tr}$ we get a family of metrics g on $W^{k,l}$, parametrized by $\lambda, x, y, z \in \mathbb{R}^+$:

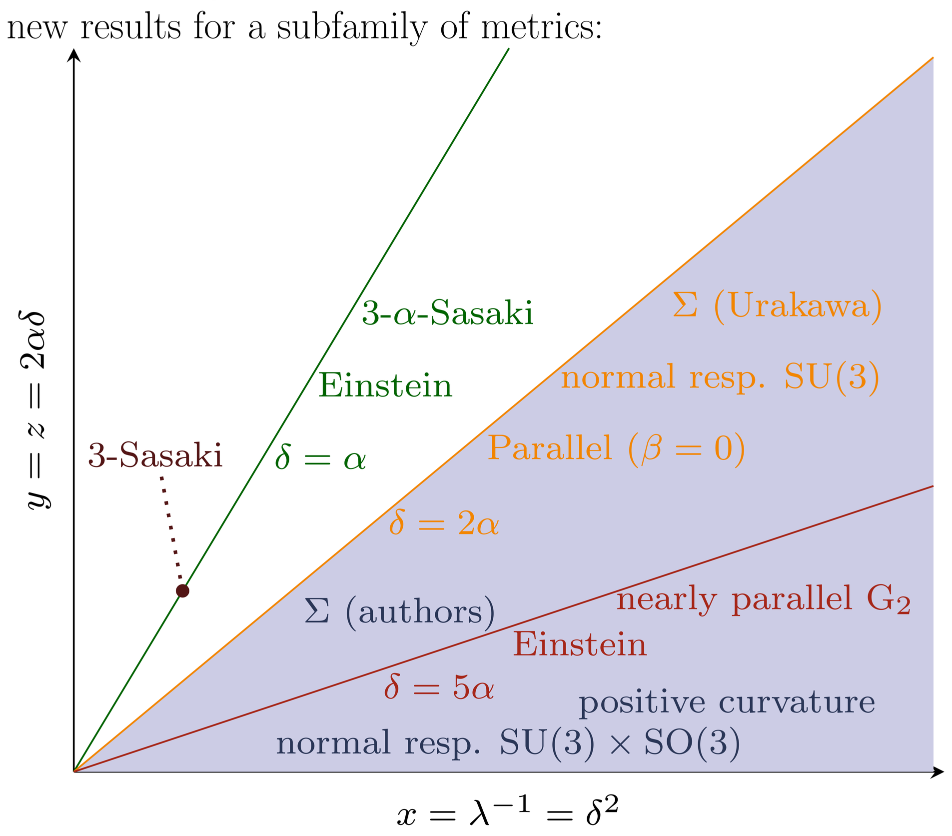
$$g = g_{\mathfrak{m}} = \lambda \cdot B\big|_{\mathfrak{m}_0} + \frac{1}{x} \cdot B\big|_{\mathfrak{m}_1} + \frac{1}{y} \cdot B\big|_{\mathfrak{m}_2} + \frac{1}{z} \cdot B\big|_{\mathfrak{m}_3}.$$

In an appropriate ONB $X_1 \in \mathfrak{m}_0$, $X_i, X_{i+1} \in \mathfrak{m}_{i+1}$, the isotropy representation $\mathrm{Ad}_{S^1_{k,l}}$ on $W^{k,l}$ is given by

$$\mathrm{Ad}_{k,l}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R[(k-l)\theta] & 0 & 0 \\ 0 & 0 & R[(2k+l)\theta] & 0 \\ 0 & 0 & 0 & R[(k+2l)\theta] \end{bmatrix},$$

where $R[\alpha] = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$. Obviously: X_1 is always $\mathrm{Ad}_{S^1_{k,l}}$-invariant. If $k = l (= 1)$, then X_2, X_3 are also $\mathrm{Ad}_{S^1_{k,l}}$-invariant.

Here is a rough overview in the case $k = l = 1$ of the known and of our new results for a subfamily of metrics:



3. G_2 and 3-(α,δ)-Sasaki Manifolds

Dfn. A Riemannian manifold (M^7, g) admits a G_2 structure $0 \neq \omega \in \Omega^3(M^7)$ with torsion if it is of Fernandez-Gray class $W_1 \oplus W_3 \oplus W_4$. This is equivalent to:

- $d * \omega = \theta \wedge *\omega$ for some 1-form θ,
- There exists a characteristic connection ∇, i.e. a metric conn. with skew torsion s. t. $\nabla\omega = 0$, and hence $\mathrm{Hol}(\nabla) \subset G_2$.

Important subclasses are:

G_2 with torsion	$\supset$	cocalibrated G_2	$\supset$	nearly parallel G_2
$W_1 \oplus W_3 \oplus W_4$		$W_1 \oplus W_3 \Leftrightarrow \delta\omega = 0$		$W_1 \Leftrightarrow d\omega = \lambda * \omega$

Dfn. An almost contact metric structure (φ, ξ, η, g) on a Riemannian manifold (M^{2n+1}, g) is given by a $(1,1)$-tensor field φ, a Reeb vector field ξ, and a 1-form η such that

$$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The fundamental 2-form is defined by $\Phi(X, Y) = g(X, \varphi Y)$.
The structure is called normal if $N_\varphi := [\varphi, \varphi] + d\eta \otimes \xi = 0$.
Important subclasses are:

α-contact metric	$\supset$	α-K-contact	$\supset$	α-Sasaki
$d\eta = 2\alpha\,\Phi,\ \alpha \in \mathbb{R}^*$		add ξ Killing		add $N_\varphi = 0$

An almost 3-contact metric manifold is a manifold M endowed with three almost contact metric structures $(\varphi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$, such that for any even permutation (i, j, k) of $(1, 2, 3)$:

$$\varphi_k = \varphi_i\varphi_j - \eta_j \otimes \xi_i, \quad \xi_k = \varphi_i\xi_j = -\varphi_j\xi_i, \quad \eta_k = \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i.$$

Then $\dim M = 4n + 3$, $n \geq 1$ and

$$TM = \mathcal{H} \oplus \mathcal{V}, \quad \mathcal{H} := \bigcap_{i=1,2,3} \ker \eta_i, \quad \mathcal{V} := \mathrm{span}\{\xi_1, \xi_2, \xi_3\}.$$

The manifold is said to be hypernormal if $N_{\varphi_i} = 0$, $i \in \{1, 2, 3\}$.

Definition. A 3-(α,δ)-Sasaki manifold is an almost 3-contact metric manifold such that $d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k$ for some $\alpha \in \mathbb{R}^*, \delta \in \mathbb{R}$, (i, j, k) any even permutation of $(1, 2, 3)$.

For $\alpha = \delta$, this reduces to a 3-α-Sasaki manifold, and for $\alpha = \delta = 1$ to a 3-Sasaki manifold.

Facts (Agricola, Dileo, 2020):

- Every 3-(α,δ)-Sasaki manifold M is hypernormal, all ξ_i are Killing.
- Its Ricci curvature in dimension 7 is given by

$$\mathrm{Ric}^g = 2\alpha\big(6\delta - 3\alpha\big)g + 2(\alpha - \delta)\big(5\alpha - \delta\big)\sum_{i=1}^{3} \eta_i \otimes \eta_i.$$

  In particular, it is ∇^g-Einstein iff $\delta = \alpha$ or $\delta = 5\alpha$.
- It admits an adapted metric connection with skew torsion, the canonical connection. It satisfies for $\beta := 2(\delta - 2\alpha)$

$$\nabla_X\varphi_i = \beta\left(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k\right), \quad \nabla_X\xi_i = \beta\left(\eta_k(X)\xi_j - \eta_j(X)\xi_k\right).$$

  Thus, we call the 3-(α,δ)-Sasaki manifold parallel if $\beta = 0$.
- Every 7-dim. 3-(α,δ)-Sasaki manifold admits a cocalibrated G_2 structure whoose characteristic conn. coincides with the canonical one.

4. G_2 and Sasaki Structures on $W^{k,l}$

The isotropy representation leads to two cases:
Case I: $k \neq l$ (one invariant vector field, $X_1 \in \mathfrak{m}_0$)

Theorem. The Aloff-Wallach spaces $\left(W^{k,l}, g\right)$ admit an α-Sasaki structure if

$$0 < 2\alpha\sqrt{\frac{k^2 + kl + l^2}{3\lambda}} = x(k + l) = ky = lz,$$

where $\xi := X_1$, $\eta := g(\cdot, X_1)$, $\varphi := -\frac{1}{\alpha}\nabla X_1$.

Case II: $k = l = 1$ (3 invariant vector fields, $X_1 \in \mathfrak{m}_0$, $X_2, X_3 \in \mathfrak{m}_1$)

Theorem. The Aloff-Wallach space $(W^{1,1}, g)$ admits a 3-(α,δ)-Sasaki structure if $\alpha > 0$, $\delta > 0$ as well as

$$x = \delta^2 = \frac{1}{\lambda}, \quad y = z = 2\alpha\delta,$$

where $\xi_i := X_i$, $\eta_i := g(\cdot, X_i)$ and

$$\varphi_i := -\frac{1}{\alpha}\nabla X_i - \frac{\alpha - \delta}{\alpha}\left[\eta_k \otimes \xi_j - \eta_j \otimes \xi_k\right].$$

We have some special subcases:

- $\delta = \alpha$: Reduction to 3-α-Sasaki structure which is ∇^g-Einstein
- $\delta = 2\alpha$: Parallel case ($\beta = 0$); implies $\frac{1}{\lambda} = x = y = z = 4\alpha^2$
  $\Rightarrow$ The metric scales all $\mathfrak{m}_i$ equally
  $\Rightarrow$ $W^{1,1}$ becomes normal homogeneous with respect to SU(3)

Corollary. The associated cocalibrated G_2 structure is given by:

$$\omega = -\omega_{145} + \omega_{167} + \omega_{246} + \omega_{257} - \omega_{347} + \omega_{356} + \omega_{123}.$$

It is nearly parallel if and only if $\delta = 5\alpha$; eigenvalue: $\lambda = 12\alpha$.

5. The Aloff-Wallach Space and Its Realizations

For this special case we introduce new parameters: $t_1 = \lambda = \frac{1}{x}$, $t_2 = \frac{1}{y} = \frac{1}{z}$ and denote the metric by g_{t_1,t_2}. Aloff and Wallach proved:

Thm. Assume that k, l, t_1, t_2 satisfy $kl > 0$ and $0 < t_1 < t_2$. Then $(W^{k,l}, g_{t_1,t_2})$ has positive curvature.

- The manifolds $(W^{k,l}, g_{t_1,t_2})$ are normal homogeneous (with respect to SU(3)) iff $t_1 = t_2$. Urakawa described how the spectrum of the Laplacian can be computed on normal homogeneous spaces and did this for $t_1 = t_2$, both in 1984.
- In 1999, Wilking found a diffferent realization of the Aloff-Wallach space $(W^{1,1}, g_{t_1,t_2})$ which is normal homogeneous, thus closing a gap in Berger's classification of simply connected, normal homogeneous spaces with positive curvature (1961).

He defined the space $V_3 := {}^{(\mathrm{SU}(3) \times \mathrm{SO}(3))}\!/_{\mathrm{U}^\bullet(2)}$, where $\mathrm{U}^\bullet(2) = (\iota, \pi)(\mathrm{U}(2))$ and

$$\pi : \mathrm{U}(2) \to {}^{\mathrm{U}(2)}\!/_{S^1} \cong \mathrm{SO}(3).$$

For $B = -\frac{1}{2}\mathrm{tr}$, V_3 becomes a normal homogeneous space for the biinvariant metrics on $\mathrm{SU}(3) \times \mathrm{SO}(3)$:

$$h_{r_1,r_2} := r_1 \cdot B\big|_{\mathfrak{so}(3)} + r_2 \cdot B\big|_{\mathfrak{su}(3)}, \quad r_1, r_2 \in (0, \infty).$$

Proposition. (Wilking, 1999) For each $r_1, r_2 \in \mathbb{R}^+$, $0 < t_1 < t_2$ the normal homogeneous space (V_3, h_{r_1,r_2}) is isometric to the Aloff-Wallach space $(W^{1,1}, g_{t_1,t_2})$ for $t_2 = r_2$ and $t_1 = \frac{4r_1r_2}{4r_1 + r_2} < r_2 = t_2$.

6. The Spectrum on Normal Homogeneous Spaces

Let G be a compact Lie group, g a biinvariant metric and $K \subset G$.
Dfn. A unitary, irred. G-rep. $(\varrho_\lambda, V_\lambda)$ of dimension $d(\lambda)$ is called *K-spherical* if $m(\lambda) = \dim\{v \in V_\lambda \mid \varrho_\lambda(K)v = v\} \neq 0$.
The set of of highest weights corresponding to K-spherical representations is denoted by $\mathrm{D}(G, K)$.
Thm. The Δ-spectrum $\Sigma(G/K, g)$ on $\mathrm{D}(G/K, g)$ is :

$$\Sigma(G/K, g) = \{g(\lambda + 2\delta, \lambda) \mid \lambda \in \mathrm{D}(G, K)\}.$$

The multiplicity associated to $\lambda \in \mathrm{D}(G, K)$ is given by $m(\lambda)d(\lambda)$.

Calculation of the spectrum on (V_3, h_{r_1,r_2}):
Lemma. The dominant integral weights are given by $\lambda = z_1\lambda_1 + z_2\lambda_2 + z_3\mu_1 \in \mathrm{D}(\mathrm{SU}(3) \times \mathrm{SO}(3))$ where $z_i \in \mathbb{N}_0 : z_1 \geq z_2 \geq 0$, $z_3 \geq 0$.
The main difficulty is to obtain the subset of all spherical representations $\lambda(z_1, z_2, z_3)$. With them one can compute the spectrum directly:

Theorem. The spectrum on (V_3, h_{r_1,r_2}) is obtained by

$$\Sigma(V_3, h_{r_1,r_2}) = \left\{\frac{z_3^2 + z_3}{r_1} + \frac{4\left(z_1^2 + z_2^2 - z_1(z_2 - 3)\right)}{3r_2} \,\middle|\, m(z_1, z_2, z_3) > 0\right\}.$$

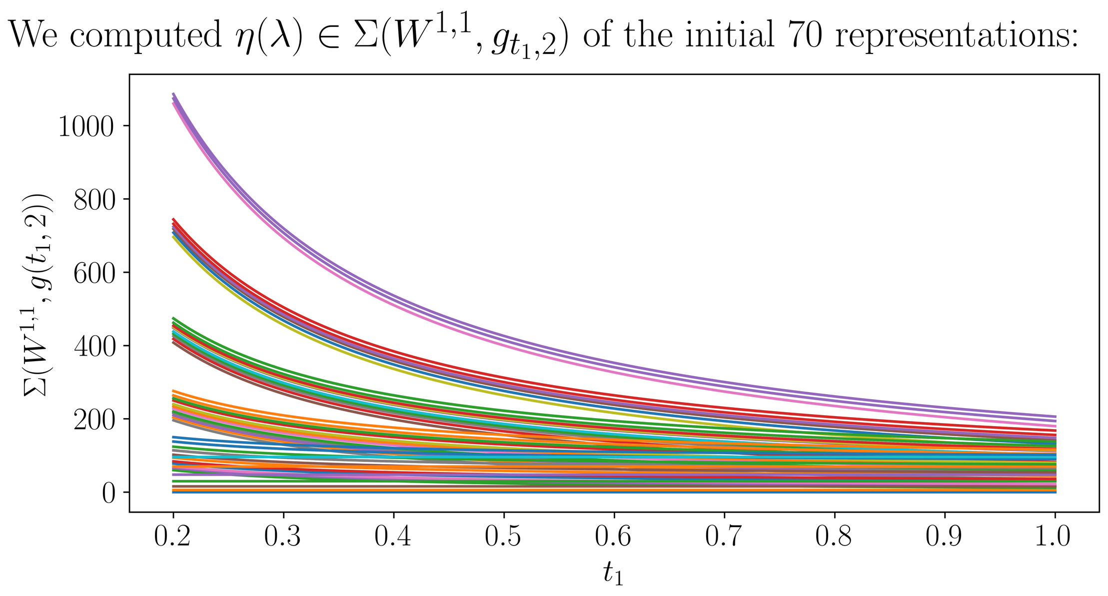
The multiplicity associated to λ is given by:

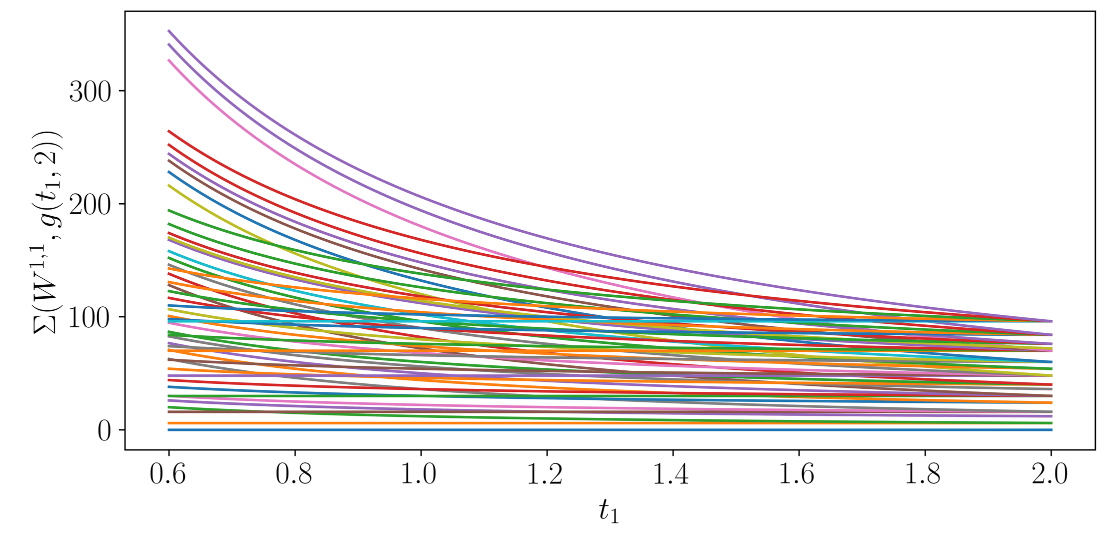
$$\mathrm{mult}(\lambda) = m(\lambda)d(\lambda) = m(\lambda)\frac{(z_1 - z_2 + 1)(z_1 + 2)(z_2 + 1)(z_3 + 1)}{2}.$$

Using the branching rules for $G = \mathrm{SU}(3) \times \mathrm{SO}(3)$ and $K = \mathrm{U}^\bullet(2)$, the spherical representations can be calculated with a computer:
Lemma. First $\lambda \in \mathrm{D}(\mathrm{SU}(3) \times \mathrm{SO}(3), \mathrm{U}^\bullet(2))$ with multiplicities are:

z_1	0	2	2	3	3	4	4	4	5	5	5	5	6	6	6	6
z_2	0	1	1	0	3	2	2	2	1	1	4	4	3	3	3	3
z_3	0	0	1	1	1	0	1	2	1	2	1	2	0	1	2	3
mult	1	8	24	30	30	27	81	135	105	175	105	175	64	192	320	448

We computed $\eta(\lambda) \in \Sigma(W^{1,1}, g_{t_1,2})$ of the initial 70 representations:





Observations. The calculated $\eta \in \Sigma(W^{1,1}, g_{t_1,2})$ suggest:

- $\Sigma(V_3, g_{t_1,2}) \longrightarrow \Sigma(W^{1,1}, g_{2,2})$ for $t_1 \to 2$, where $\Sigma(W^{1,1}, g_{2,2})$ has been computed by Urakawa in 1984.
- Multiplicities of $\Sigma(W^{1,1}, g_{2,2})$ split into those of $\Sigma(V_3, g_{t_1,2})$. For any $\mu \in \mathrm{D}(\mathrm{SU}(3), S^1_{1,1})$, $\mathrm{mult}(\mu) = \sum \mathrm{mult}(\lambda_j)$ where $\lambda_j \in \mathrm{D}(\mathrm{SU}(3) \times \mathrm{SO}(3), \mathrm{U}^\bullet(2))$ and $\eta_2(\lambda_j) \to \eta(\mu)$.
- $(W^{1,1}, g_{2,2})$ (i.e., $\beta = 0$) appears most symmetrical.
- $\Sigma(W^{1,1}, g_{t_1,2}) \neq \Sigma(W^{1,1}, g_{\tilde{t}_1,2})$ for $t_1, \tilde{t}_1 \leq 1$ and $t_1 \neq \tilde{t}_1$.
- The nearly parallel G_2 case ($t_1 = 0.8$) can not be seen in the spectrum.
- $\eta(t_1)$ is constant $\Leftrightarrow \eta(t_1) \in \Sigma(\mathbb{C}\mathbb{P}^2)$ as the canonical proj. is a Riem. submersion with totally geodesic fibers over $\mathbb{C}\mathbb{P}^2$.