DOUBLE DISK BUNDLES

[joint with J. DeVito (Tennessee)]

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Double disk bundle: A smooth, closed manifold

$$M \cong DB_- \cup_L DB_+$$

where

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- CROSSes: $\mathbf{S}^n$, $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$, $\mathbf{CaP}^2$

DeVito, K– '23

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- $n = 2m+1$: Every $\mathbf{S}^m$-bundle over $\mathbf{S}^{m+1}$ is a double disk bundle.

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- Codimension-1 singular Riemannian foliations

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  - inhomogeneous nearly Kähler structures on $\mathbf{S}^6$ and $\mathbf{S}^3 \times \mathbf{S}^3$ [Foscolo, Haskins '17]

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    - Family of 2-conn. 7-manifolds [$\supseteq$ all exotic 7-spheres] [Goette, K, Shankar '20]

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      $\therefore$ Classification of space forms $\implies$ infinitely many $\mathbf{S}^3/\Gamma$ not double disk bundles, e.g. $\mathbf{S}^3/I^*$, $\mathbf{S}^3/O^*$, $\mathbf{S}^3/T^*$

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      $\implies$ $\exists$ closed, flat manifold M such that

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      [Bieberbach '11, Auslander, Kuranishi '57]

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- [Grove] Are there general obstructions to being a double disk bundle?

DeVito, K– '23

Let M^n be a double disk bundle such that $\pi_1(M^n) = 0$ and

$$H_*(M^n; \mathbf{Q}) \cong H_*(\mathbf{S}^n; \mathbf{Q}) .$$

Then

- n even $\implies$ M^n homeo to $\mathbf{S}^n$.
- $n = 2m + 1$, M^n $(m-1)$-connected $\implies$ $H_m(M^n; \mathbf{Z})$ finite cyclic.

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[Grove, Halperin '87] Studied F in very general setting.

TOPOLOGY OF HOMOTOPY FIBRE

<table>
<tr><th>$\{\alpha,\beta\}=\{\ell_\pm\}$</th><th>Orientability of $\mathbf{S}^{\ell_\pm}$-bundles</th><th>$\pi_1(F)$</th><th>$\mathbf{Q}$-model for F</th></tr>
<tr><td rowspan="3">$1=\alpha=\beta$</td><td>Both</td><td>$\mathbf{Z}^2$</td><td>$\mathbf{S}^1\times\mathbf{S}^1\times\Omega\mathbf{S}^3$</td></tr>
<tr><td>One</td><td>$\mathbf{Z}\oplus\mathbf{Z}_2$</td><td>$\mathbf{S}^1\times\mathbf{S}^3\times\Omega\mathbf{S}^5$</td></tr>
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<tr><td>$1=\alpha<\beta$</td><td>Both</td><td rowspan="2">$\mathbf{Z}$</td><td>$\mathbf{S}^1\times\mathbf{S}^\beta\times\Omega\mathbf{S}^{\beta+2}$</td></tr>
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TOPOLOGY OF HOMOTOPY FIBRE

$\{\alpha, \beta\} = \{\ell_\pm\}$	Orientability of $\mathbf{S}^{\ell_\pm}$-bundles	$H^j(F; \mathbf{Z})$	
$\alpha \neq \beta$	Both	$\mathbf{Z}$,	$j = 0$, or $j \in \{\alpha, \beta\}$ mod $\alpha + \beta$
		$\mathbf{Z}^2$,	$j > 0$ and $j \equiv 0$ mod $\alpha + \beta$
$\alpha = \beta$	Both	$\mathbf{Z}$,	$j = 0$
		$\mathbf{Z}^2$,	$j > 0$ and $j \equiv 0$ mod α
$1 = \alpha < \beta$	$\mathbf{S}^1$-bundle	$\mathbf{Z}$,	$j = 0$, or $j \equiv \pm 1$ mod $2\beta + 2$
		$\mathbf{Z}^2$,	$j > 0$ and $j \equiv 0$ mod $2\beta + 2$
		$\mathbf{Z}_2$,	$j \in \{\beta + 1, \beta + 2\}$ mod $2\beta + 2$
$1 = \alpha = \beta$	One	$\mathbf{Z}$,	$j = 0$, or $j \equiv 1$ mod 4
		$\mathbf{Z}_2$,	$j \equiv 2$ mod 4
		$\mathbf{Z} \oplus \mathbf{Z}_2$,	$j \equiv 3$ mod 4
		$\mathbf{Z}^2$,	$j > 0$ and $j \equiv 0$ mod 4
$1 = \alpha = \beta$	Neither	$\mathbf{Z}$,	$j = 0$
		$\mathbf{Z}^2$,	$j > 0$ and $j \equiv 0$ mod 3
		$\mathbf{Z}_2^2$,	$j \equiv 2$ mod 3

[DeVito, Galaz-García, K– '20] If $F \simeq_{\mathbf{Q}} N \times \Omega \mathbf{S}^k$, then $\exists$ non-trivial homomorphism

$$\pi_{2s+1}(M) \otimes \mathbf{Q} \to \pi_{2s}(F) \otimes \mathbf{Q}$$

where $2s+1 = \begin{cases} k, & \text{if } k \text{ odd} \\ 2k-1, & \text{if } k \text{ even.} \end{cases}$

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- $H^m(F;\mathbf{Z})$ has torsion $\implies H^m(F;\mathbf{Z}) = \mathbf{Z}_2, \quad n = 2m+1 \equiv 1 \mod 4$

RANK THEOREM

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- $M^{2m+1} \simeq_{\mathbf{Q}} \mathbf{S}^{2m+1} \implies k = 2m + 1$ or $k = m + 1 \equiv 0 \bmod 2$
- $H^m(F; \mathbf{Z}) = \mathbf{Z}^2 \implies H^*(F; \mathbf{Z})$ torsion free OR $n = 7$, $\ell_\pm = 1$ and

$$H^j(F; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & j = 0 \\ \mathbf{Z}^2, & j > 0 \text{ and } j \equiv 0 \bmod 3 \\ \mathbf{Z}_2^2, & j \equiv 2 \bmod 3 \end{cases}$$

Spectral sequence for $F \to L \to M^{2m+1}$:

$$
\begin{array}{cc|ccccccc}
F & & * & 0 & \cdots & 0 & 0 & & \\
 & m & H^m(F) & 0 & \cdots & 0 & 0 & & \\
 & m-1 & H^{m-1}(F) & 0 & \cdots & 0 & 0 & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
 & 2 & H^2(F) & 0 & \cdots & 0 & 0 & * & * \\
 & 1 & H^1(F) & 0 & \cdots & 0 & 0 & 0 & * \\
 & 0 & \mathbf{Z} & 0 & \cdots & 0 & 0 & 0 & H^{m+1}(M) \\
\hline
 & & 0 & 1 & \cdots & m-2 & m-1 & m & m+1 & M^{2m+1}
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SPECTRAL SEQUENCE FOR HOMOTOPY FIBRATION

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0 & \mathbf{Z} & 0 & \cdots & 0 & 0 & 0 & \color{red}{H^{m+1}(M)} & \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
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1 & H^1(F) & 0 & \cdots & 0 & {\color{red}0} \arrow[drr, red, "d_2"'] & 0 & * & \\
0 & \mathbf{Z} & 0 & \cdots & 0 & 0 & 0 & {\color{red}H^{m+1}(M)} & \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
2 & H^2(F) & 0 & \cdots & 0 \arrow[rrrdd, "d_3"] & 0 & * & * & \\
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Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

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- $H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

SPECTRAL SEQUENCE

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

- $H^m(L; \mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F; \mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z}) / H^m(L; \mathbf{Z}) \quad \subseteq H^{m+1}(M; \mathbf{Z})$
- $H^{m+1}(M; \mathbf{Z}) / \operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

PD & UCT $\implies$ $H^m(L; \mathbf{Z})$, $H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

SPECTRAL SEQUENCE

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

- $H^m(L; \mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F; \mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z})/H^m(L; \mathbf{Z}) \quad \subseteq H^{m+1}(M; \mathbf{Z})$
- $H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

PD & UCT $\implies$ $H^m(L; \mathbf{Z})$, $H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L; \mathbf{Z})$ has torsion $\implies$ $H^m(F; \mathbf{Z})$ has torsion

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

- $H^m(L; \mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F; \mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z})/H^m(L; \mathbf{Z}) \quad \subseteq H^{m+1}(M; \mathbf{Z})$
- $H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

PD & UCT $\implies$ $H^m(L; \mathbf{Z})$, $H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L; \mathbf{Z})$ has torsion $\implies$ $H^m(F; \mathbf{Z}) = \mathbf{Z}_2$

Differential $d_{m+1} : H^m(F;\mathbf{Z}) \to H^{m+1}(M;\mathbf{Z})$ yields

- $H^m(L;\mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F;\mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F;\mathbf{Z})/H^m(L;\mathbf{Z}) \quad \subseteq H^{m+1}(M;\mathbf{Z})$
- $H^{m+1}(M;\mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L;\mathbf{Z})$

PD & UCT $\implies H^m(L;\mathbf{Z}),\ H^{m+1}(L;\mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L;\mathbf{Z})$ has torsion $\implies H^m(F;\mathbf{Z}) = \mathbf{Z}_2$

$\implies H^m(L;\mathbf{Z}) = H^m(F;\mathbf{Z}) = \mathbf{Z}_2$ and $\mathbf{Z}_2 = T^{m+1}(L;\mathbf{Z})$

SPECTRAL SEQUENCE

Differential $d_{m+1} : H^m(F;\mathbf{Z}) \to H^{m+1}(M;\mathbf{Z})$ yields

- $H^m(L;\mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F;\mathbf{Z})$
- $\mathrm{im}(d_{m+1}) \cong H^m(F;\mathbf{Z})/H^m(L;\mathbf{Z}) \quad \subseteq H^{m+1}(M;\mathbf{Z})$
- $H^{m+1}(M;\mathbf{Z})/\,\mathrm{im}(d_{m+1}) \hookrightarrow H^{m+1}(L;\mathbf{Z})$

PD & UCT $\implies$ $H^m(L;\mathbf{Z})$, $H^{m+1}(L;\mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L;\mathbf{Z})$ has torsion $\implies H^m(F;\mathbf{Z}) = \mathbf{Z}_2$

$\implies H^m(L;\mathbf{Z}) = H^m(F;\mathbf{Z}) = \mathbf{Z}_2$ and $\mathbf{Z}_2 = T^{m+1}(L;\mathbf{Z})$

$\implies \mathrm{im}(d_{m+1}) = 0$

SPECTRAL SEQUENCE

Differential $d_{m+1} : H^m(F;\mathbf{Z}) \to H^{m+1}(M;\mathbf{Z})$ yields

- $H^m(L;\mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F;\mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F;\mathbf{Z})/H^m(L;\mathbf{Z}) \quad \subseteq H^{m+1}(M;\mathbf{Z})$
- $H^{m+1}(M;\mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L;\mathbf{Z})$

PD & UCT $\implies$ $H^m(L;\mathbf{Z})$, $H^{m+1}(L;\mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L;\mathbf{Z})$ has torsion $\implies$ $H^m(F;\mathbf{Z}) = \mathbf{Z}_2$

$\implies H^m(L;\mathbf{Z}) = H^m(F;\mathbf{Z}) = \mathbf{Z}_2$ and $\mathbf{Z}_2 = T^{m+1}(L;\mathbf{Z})$

$\implies \operatorname{im}(d_{m+1}) = 0$

$\implies H^{m+1}(M;\mathbf{Z}) \hookrightarrow \mathbf{Z}_2 \subseteq H^{m+1}(L;\mathbf{Z})$

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

- $H^m(L; \mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F; \mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z}) / H^m(L; \mathbf{Z}) \quad \subseteq H^{m+1}(M; \mathbf{Z})$
- $H^{m+1}(M; \mathbf{Z}) / \operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

PD & UCT $\implies$ $H^m(L; \mathbf{Z})$, $H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L; \mathbf{Z})$ free abelian $\implies$ $H^{m+1}(M; \mathbf{Z}) / \operatorname{im}(d_{m+1}) = 0$

SPECTRAL SEQUENCE

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

- $H^m(L; \mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F; \mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z})/H^m(L; \mathbf{Z}) \quad \subseteq H^{m+1}(M; \mathbf{Z})$
- $H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

PD & UCT $\implies$ $H^m(L; \mathbf{Z})$, $H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L; \mathbf{Z})$ free abelian $\implies$ $H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) = 0$

$\implies$ $H^{m+1}(M; \mathbf{Z}) = \operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z})/H^m(L; \mathbf{Z})$

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \to H^{m+1}(M; \mathbf{Z})$ yields

- $H^m(L; \mathbf{Z}) \cong \ker(d_{m+1}) \quad \subseteq H^m(F; \mathbf{Z})$
- $\operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z})/H^m(L; \mathbf{Z}) \quad \subseteq H^{m+1}(M; \mathbf{Z})$
- $H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

PD & UCT $\implies$ $H^m(L; \mathbf{Z})$, $H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

$\therefore$ $H^{m+1}(L; \mathbf{Z})$ free abelian $\implies H^{m+1}(M; \mathbf{Z})/\operatorname{im}(d_{m+1}) = 0$

$\implies H^{m+1}(M; \mathbf{Z}) = \operatorname{im}(d_{m+1}) \cong H^m(F; \mathbf{Z})/H^m(L; \mathbf{Z})$

$\implies H^{m+1}(M; \mathbf{Z})$ generated by $\leqslant 2$ elements

and $\operatorname{rank}(H^m(L; \mathbf{Z})) = \operatorname{rank}(H^m(F; \mathbf{Z}))$

BRAID DIAGRAM

Assume WLOG $H^m(L; \mathbf{Z}) \cong H^m(F; \mathbf{Z}) \cong \mathbf{Z}^2$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}[column sep=small]
{} \arrow[dr] \arrow[rr, bend left] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(M,B_+) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
& H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] & & H^{m+1}(M) \arrow[ur] \arrow[dr] & \\
{} \arrow[ur] \arrow[rr, bend right] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] & & H^{m+1}(M,B_-) \arrow[ur] \arrow[rr, bend right] & & H^{m+1}(B_+)
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
{} \arrow[rr, bend left] \arrow[dr] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
& H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] & & H^{m+1}(M) \arrow[ur] \arrow[dr] & \\
{} \arrow[rr, bend right] \arrow[ur] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] & & H_m(B_+) \arrow[ur] \arrow[rr, bend right] & & H^{m+1}(B_+)
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur] \arrow[dr] \arrow[r, equal] & \text{finite} \\
 & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right] & & H^{m+1}(B_+)
\end{tikzcd}
$$

BRAID DIAGRAM

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
{} \arrow[dr] \arrow[rr, bend left, "0"] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur] \arrow[dr] \arrow[r, equal] & \text{finite} \\
{} \arrow[ur] \arrow[rr, bend right, "0"'] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right] & & H^{m+1}(B_+)
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z} \oplus T_m(B_-) \arrow[d, equal] & & \\
{} \arrow[rr, bend left, "0"] \arrow[dr] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur] \arrow[dr] \arrow[r, equal] & \text{finite} \\
{} \arrow[ur] \arrow[rr, bend right, "0"'] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & H^{m+1}(B_+) \\
 & & \mathbf{Z} & & \mathbf{Z} \oplus T_m(B_+) & &
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z}\oplus T_m(B_-) \arrow[d, equal] & & \mathbf{Z}^{b_{m+1}(B_-)} \arrow[d, equal] & \\
{} \arrow[dr] \arrow[rr, bend left, "0"] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) & \\
0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur] \arrow[dr] \arrow[rr, equal] & & \text{finite} \\
{} \arrow[ur] \arrow[rr, bend right, "0"'] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & H^{m+1}(B_+) \arrow[d, equal] & \\
 & & \mathbf{Z} & & \mathbf{Z}\oplus T_m(B_+) & & \mathbf{Z}^{b_{m+1}(B_+)} &
\end{tikzcd}
$$

BRAID DIAGRAM

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & \color{red}{\mathbf{Z}} \arrow[d, equal, red] & & \color{red}{\mathbf{Z} \oplus \cancel{T_m(B_-)}} \arrow[d, equal, red] & & \color{red}{\mathbf{Z}^{b_{m+1}(B_-)}} \arrow[d, equal, red] \\
 {} \arrow[dr] \arrow[rr, bend left, red, "0"] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
 \color{red}{0} \arrow[r, equal, red] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal, red] & & H^{m+1}(M) \arrow[ur] \arrow[dr] \arrow[r, equal, red] & \color{red}{\text{finite}} \\
 {} \arrow[ur] \arrow[rr, bend right, red, "0"'] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal, red] & \color{red}{\mathbf{Z}^2} & H_m(B_+) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal, red] & & H^{m+1}(B_+) \arrow[d, equal, red] \\
 & & \color{red}{\mathbf{Z}} & & \color{red}{\mathbf{Z} \oplus \cancel{T_m(B_+)}} & & \color{red}{\mathbf{Z}^{b_{m+1}(B_+)}}
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z}^{b_{m+1}(B_-)} \arrow[d, equal] \\
{} \arrow[dr] \arrow[rr, bend left, "0"] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left] & & H^{m+1}(B_-) \\
0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur] \arrow[dr] \arrow[r, equal] & \text{finite} \\
{} \arrow[ur] \arrow[rr, bend right, "0"'] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & H^{m+1}(B_+) \arrow[d, equal] \\
 & & \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z}^{b_{m+1}(B_+)}
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z}^{b_{m+1}(B_-)} \arrow[d, equal] \\
 {} \arrow[rr, bend left, "0"] \arrow[dr] & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left, "0"] & & H^{m+1}(B_-) \\
 0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur, "0"] \arrow[dr, "0"'] \arrow[r, equal] & \text{finite} \\
 {} \arrow[ur] \arrow[rr, bend right, "0"'] & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right, "0"'] \arrow[d, equal] & & H^{m+1}(B_+) \arrow[d, equal] \\
 & & \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z}^{b_{m+1}(B_+)}
\end{tikzcd}
$$

Assume WLOG $H^m(L;\mathbf{Z}) \cong H^m(F;\mathbf{Z}) \cong \mathbf{Z}^2$

$\exists$ commutative braid of exact sequences associated to $M = DB_- \cup_L DB_+$

$$
\begin{tikzcd}
 & & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z} \arrow[d, equal] & & \mathbf{Z}^{b_{m+1}(B_-)} \arrow[d, equal] \\
{} \arrow[drr] \arrow[rrr, bend left, "0"] & & & H^m(B_+) \arrow[dr] \arrow[rr, bend left] & & H_m(B_-) \arrow[dr] \arrow[rr, bend left, "0"] & & H^{m+1}(B_-) \\
 & 0 \arrow[r, equal] & H^m(M) \arrow[ur] \arrow[dr] & & H^m(L) \arrow[ur] \arrow[dr] \arrow[d, equal] & & H^{m+1}(M) \arrow[ur, "0"] \arrow[dr, "0"'] \arrow[r, equal] & \text{finite} \\
{} \arrow[urr] \arrow[rrr, bend right, "0"'] & & & H^m(B_-) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & \mathbf{Z}^2 & H_m(B_+) \arrow[ur] \arrow[rr, bend right, "0"'] \arrow[d, equal] & & H^{m+1}(B_+) \arrow[d, equal] \\
 & & & \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z}^{b_{m+1}(B_+)}
\end{tikzcd}
$$

$\implies$ SES $0 \to \mathbf{Z} \to \mathbf{Z} \to H^{m+1}(M;\mathbf{Z}) \to 0$

Thanks for your attention!