

DOUBLE DISK BUNDLES

[joint with J. DeVito (Tennessee)]

Martin Kerin (Durham)

24 August 2023

Prospects in Geometry and Global Analysis
Schloss Rauischholzhausen, Marburg

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Double disk bundle: A smooth, closed manifold

$$M \cong DB_- \cup_L DB_+$$

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- CROSSes: $\mathbf{S}^n$, $\mathbf{RP}^n$, $\mathbf{CP}^n$, $\mathbf{HP}^n$, $\mathbf{CaP}^2$

DeVito, K– '23

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Theorem optimal because:

- n even: $\exists$ exotic spheres for $n \geq 8$
- $n = 2m + 1$: Every $\mathbf{S}^m$ -bundle over $\mathbf{S}^{m+1}$ is a double disk bundle.

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- Codimension-1 singular Riemannian foliations

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- Cohomogeneity-one manifolds:

$$G \curvearrowright M \text{ such that } M/G \cong [-1, 1]$$

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- inhomogeneous nearly Kähler structures on $\mathbf{S}^6$ and $\mathbf{S}^3 \times \mathbf{S}^3$
[Foscolo, Haskins '17]

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 - Family of 2-conn. 7-manifolds $[\supseteq]$ all exotic 7-spheres [Goette, K, Shankar '20]

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- $\therefore$ Classification of space forms $\implies$ infinitely many $\mathbf{S}^3/\Gamma$ not double disk bundles, e.g. $\mathbf{S}^3/I^*$, $\mathbf{S}^3/O^*$, $\mathbf{S}^3/T^*$

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$\implies \exists$ closed, flat manifold M such that

$$\text{Hol}(M) = \Gamma, \pi_1(M) = G \text{ and } H_1(M) = 0.$$

[Bieberbach '11, Auslander, Kuranishi '57]

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- [Grove] Are there general obstructions to being a double disk bundle?

DeVito, K– '23

Let M^n be a double disk bundle such that $\pi_1(M^n) = 0$ and

$$H_*(M^n; \mathbf{Q}) \cong H_*(\mathbf{S}^n; \mathbf{Q}).$$

Then

- n even $\implies M^n$ homeo to $\mathbf{S}^n$.
- $n = 2m + 1$, M^n $(m - 1)$ -connected $\implies H_m(M^n; \mathbf{Z})$ finite cyclic.

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$$\mathcal{P} \rightarrow M; (x, \alpha) \mapsto \alpha(1)$$

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THE HOMOTOPY FIBRATION

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[Grove, Halperin '87] Studied F in very general setting.

TOPOLOGY OF HOMOTOPY FIBRE

$\{\alpha, \beta\} = \{\ell_{\pm}\}$	Orientability of $\mathbf{S}^{\ell_{\pm}}$ -bundles	$\pi_1(F)$	Q-model for F
$1 = \alpha = \beta$	Both	$\mathbf{Z}^2$	$\mathbf{S}^1 \times \mathbf{S}^1 \times \Omega \mathbf{S}^3$
	One	$\mathbf{Z} \oplus \mathbf{Z}_2$	$\mathbf{S}^1 \times \mathbf{S}^3 \times \Omega \mathbf{S}^5$
	Neither	Q_8	$\mathbf{S}^3 \times \mathbf{S}^3 \times \Omega \mathbf{S}^7$
$1 = \alpha < \beta$	Both	$\mathbf{Z}$	$\mathbf{S}^1 \times \mathbf{S}^{\beta} \times \Omega \mathbf{S}^{\beta+2}$
$1 = \alpha < \beta, \beta \text{ odd}$	$\mathbf{S}^1$ -bundle		$\mathbf{S}^1 \times \mathbf{S}^{2\beta+1} \times \Omega \mathbf{S}^{2\beta+3}$
$1 < \alpha \leq \beta$	Both	0	$\mathbf{S}^{\alpha} \times \mathbf{S}^{\beta} \times \Omega \mathbf{S}^{\alpha+\beta+1}$
$1 < \alpha = \beta$			$\mathbf{S}^{\alpha} \times \Omega \mathbf{S}^{\alpha+1}$
$2 = \alpha = \beta$			$\mathrm{SU}(3)/T^2 \times \Omega \mathbf{S}^7$
			$\mathrm{Sp}(2)/T^2 \times \Omega \mathbf{S}^9$
			$\mathrm{G}_2/T^2 \times \Omega \mathbf{S}^{13}$
$4 = \alpha = \beta$			$\mathrm{Sp}(3)/\mathrm{Sp}(1)^3 \times \Omega \mathbf{S}^{13}$
			$A_4(4) \times \Omega \mathbf{S}^{17}$
			$A_6(4) \times \Omega \mathbf{S}^{25}$
$8 = \alpha = \beta$			$F_4/\mathrm{Spin}(8) \times \Omega \mathbf{S}^{25}$

TOPOLOGY OF HOMOTOPY FIBRE

$\{\alpha, \beta\} = \{\ell_{\pm}\}$	Orientability of $\mathbf{S}^{\ell_{\pm}}$ -bundles	$H^j(F; \mathbf{Z})$
$\alpha \neq \beta$	Both	$\mathbf{Z}$, $j = 0$, or $j \in \{\alpha, \beta\} \bmod \alpha + \beta$ $\mathbf{Z}^2$, $j > 0$ and $j \equiv 0 \bmod \alpha + \beta$
$\alpha = \beta$	Both	$\mathbf{Z}$, $j = 0$ $\mathbf{Z}^2$, $j > 0$ and $j \equiv 0 \bmod \alpha$
$1 = \alpha < \beta$	$\mathbf{S}^1$ -bundle	$\mathbf{Z}$, $j = 0$, or $j \equiv \pm 1 \bmod 2\beta + 2$ $\mathbf{Z}^2$, $j > 0$ and $j \equiv 0 \bmod 2\beta + 2$ $\mathbf{Z}_2$, $j \in \{\beta + 1, \beta + 2\} \bmod 2\beta + 2$
$1 = \alpha = \beta$	One	$\mathbf{Z}$, $j = 0$, or $j \equiv 1 \bmod 4$ $\mathbf{Z}_2$, $j \equiv 2 \bmod 4$ $\mathbf{Z} \oplus \mathbf{Z}_2$, $j \equiv 3 \bmod 4$ $\mathbf{Z}^2$, $j > 0$ and $j \equiv 0 \bmod 4$
$1 = \alpha = \beta$	Neither	$\mathbf{Z}$, $j = 0$ $\mathbf{Z}^2$, $j > 0$ and $j \equiv 0 \bmod 3$ $\mathbf{Z}_2^2$, $j \equiv 2 \bmod 3$

[DeVito, Galaz-García, K– '20] If $F \simeq_{\mathbf{Q}} N \times \Omega \mathbf{S}^k$, then $\exists$ non-trivial homomorphism

$$\pi_{2s+1}(M) \otimes \mathbf{Q} \rightarrow \pi_{2s}(F) \otimes \mathbf{Q}$$

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· $H^m(F; \mathbf{Z}) = \mathbf{Z}^2 \implies H^*(F; \mathbf{Z}) \text{ torsion free OR } n = 7, \ell_{\pm} = 1 \text{ and}$

$$H^j(F; \mathbf{Z}) = \begin{cases} \mathbf{Z}, & j = 0 \\ \mathbf{Z}^2, & j > 0 \text{ and } j \equiv 0 \pmod{3} \\ \mathbf{Z}_2^2, & j \equiv 2 \pmod{3} \end{cases}$$

SPECTRAL SEQUENCE FOR HOMOTOPY FIBRATION

Spectral sequence for $F \rightarrow L \rightarrow M^{2m+1}$:

F		*	0	$\dots$	0	0		
	m	$H^m(F)$	0	$\dots$	0	0		
	$m-1$	$H^{m-1}(F)$	0	$\dots$	0	0		
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	2	$H^2(F)$	0	$\dots$	0	0	*	*
	1	$H^1(F)$	0	$\dots$	0	0	0	*
	0	$\mathbf{Z}$	0	$\dots$	0	0	0	$H^{m+1}(M)$
		0	1	$\dots$	$m-2$	$m-1$	m	$m+1$
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	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
	2	$H^2(F)$	0	$\dots$	0	0	*	*
	1	$H^1(F)$	0	$\dots$	0	0	0	*
	0	$\mathbf{Z}$	0	$\dots$	0	0	0	$H^{m+1}(M)$
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$m-1$	$H^{m-1}(F)$	0	$\cdots$	0	0			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
2	$H^2(F)$	0	$\cdots$	0	0	*	*	
1	$H^1(F)$	0	$\cdots$	0	0	0	0	*
0	$\mathbf{Z}$	0	$\cdots$	0	0	0	0	$H^{m+1}(M)$
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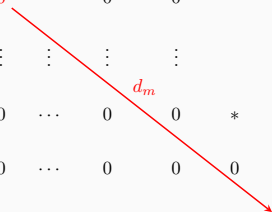
F		*	0	$\dots$	0	0		
m	$H^m(F)$	0	$\dots$	0	0			
$m-1$	$H^{m-1}(F)$	0	$\dots$	0	0			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
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0	$\mathbf{Z}$	0	$\dots$	0	0	0	$H^{m+1}(M)$	
		0	1	$\dots$	$m-2$	$m-1$	m	$m+1$
								M^{2m+1}

A red arrow labeled d_3 points from the entry 0 at row 2, column $m-2$ to the entry $H^{m+1}(M)$ at row 0, column $m+1$.

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Spectral sequence for $F \rightarrow L \rightarrow M^{2m+1}$:

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
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Differential $d_{m+1} : H^m(F; \mathbf{Z}) \rightarrow H^{m+1}(M; \mathbf{Z})$ yields

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- $H^{m+1}(M; \mathbf{Z}) / \operatorname{im}(d_{m+1}) \hookrightarrow H^{m+1}(L; \mathbf{Z})$

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PD & UCT $\implies H^m(L; \mathbf{Z}), H^{m+1}(L; \mathbf{Z})$ isomorphic torsion subgroups

SPECTRAL SEQUENCE

Differential $d_{m+1} : H^m(F; \mathbf{Z}) \rightarrow H^{m+1}(M; \mathbf{Z})$ yields

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$\therefore H^{m+1}(L; \mathbf{Z})$ has torsion $\implies H^m(F; \mathbf{Z}) = \mathbf{Z}_2$

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SPECTRAL SEQUENCE

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$\implies H^{m+1}(M; \mathbf{Z})$ generated by ≤ 2 elements

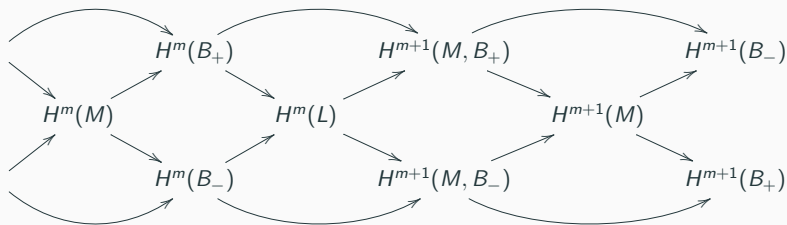
and $\operatorname{rank}(H^m(L; \mathbf{Z})) = \operatorname{rank}(H^m(F; \mathbf{Z}))$

Assume WLOG $H^m(L; \mathbf{Z}) \cong H^m(F; \mathbf{Z}) \cong \mathbf{Z}^2$

BRAID DIAGRAM

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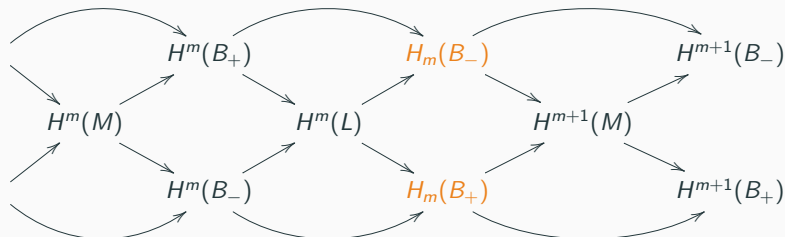
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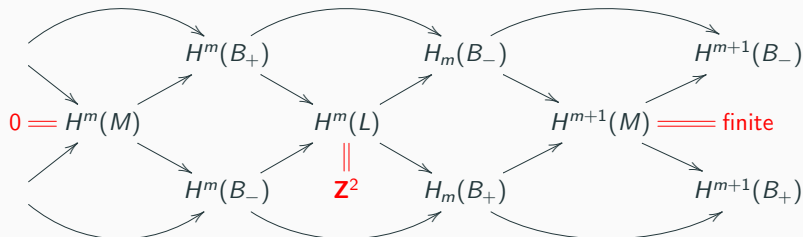
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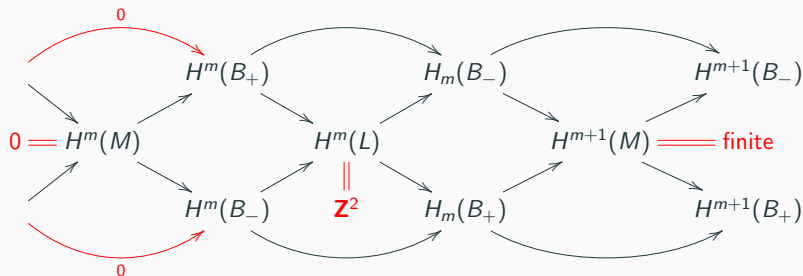
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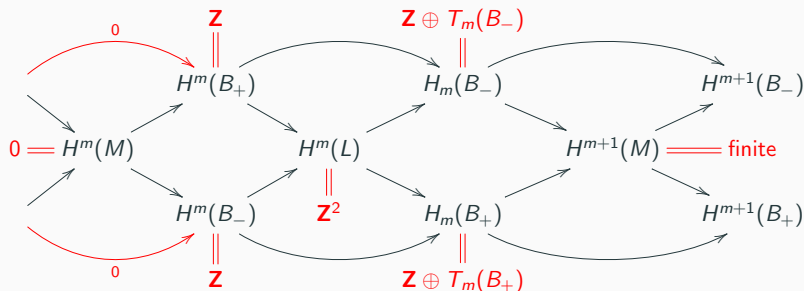
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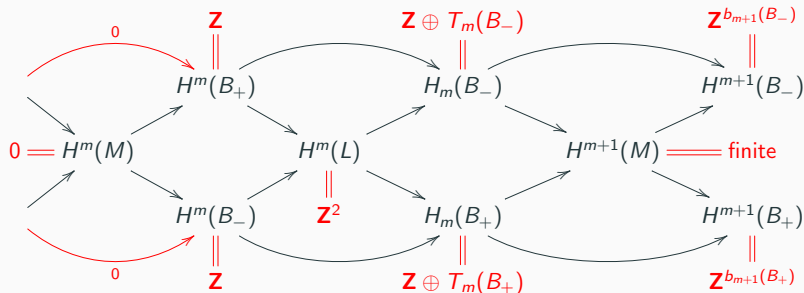
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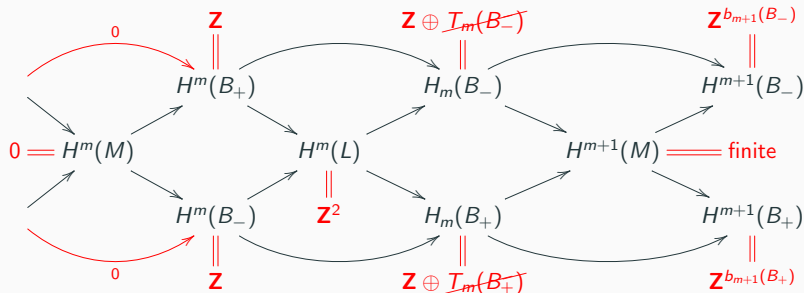
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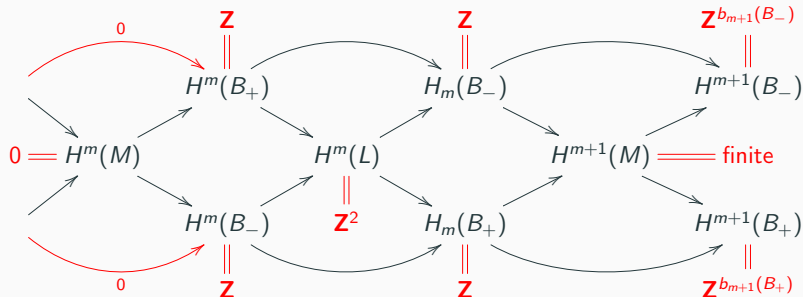
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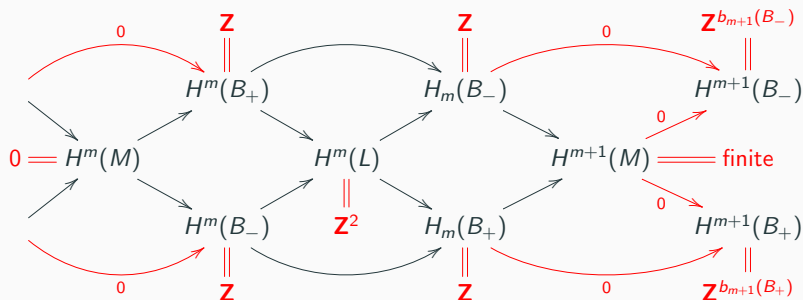
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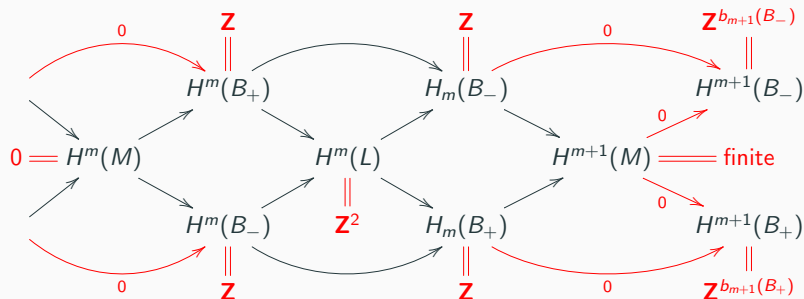
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$$\implies \text{SES } 0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow H^{m+1}(M; \mathbf{Z}) \rightarrow 0$$

Thanks for your attention!