Eta Invariant and localization

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August 21 2023
Prospects in Geometry and Global Analysis
Castle Rauischholzhausen

Contents

Dirac Operator on Spin^c Manifold

- (Y, g^{TY}) : oriented Riemannian manifold of dim n.
- Existence of spin^c structure on $Y \iff$ topological condition : second Stiefel-Whitney class of TY : $w_2(TY) = c_1(L) \in H^2(Y, \mathbb{Z}_2)$, with L complex line bundle on Y.
- Assume Y is spin^c, the spinor bundle over Y formally is

$$\mathcal{S} = \mathcal{S}(TY, L) = S \otimes L^{1/2},$$

  and both classical spinor S and $L^{1/2}$ need not exist.
- Let (E, h^E) be a Hermitian complex vector bundle over Y, with Hermitian connection ∇^E.
- Dirac operator : $\{e_i\}$ any orthonormal frame of TY,

$$D^Y \otimes E := \sum_i c(e_i) \nabla^{\mathcal{S}\otimes E}_{e_i} : \mathscr{C}^\infty(Y, \mathcal{S} \otimes E) \to \mathscr{C}^\infty(Y, \mathcal{S} \otimes E).$$

  $D^Y \otimes E$ is a 1^{st} order, self-adjoint, elliptic differential operator.

Atiyah-Singer Index Theorem

- X : even-dimensional compact Spin^c manifold. $\underline{E} := (E, h^E, \nabla^E)$.

$$D_{\pm}^X \otimes E : \mathscr{C}^\infty(X, \mathcal{S}_\pm \otimes E) \to \mathscr{C}^\infty(X, \mathcal{S}_\mp \otimes E).$$

- $\operatorname{Ind}(D_+^X \otimes E) := \operatorname{Tr}|_{\ker(D_+^X \otimes E)}[1] - \operatorname{Tr}|_{\ker(D_-^X \otimes E)}[1]$.
- Atiyah-Singer index theorem (1963) :

$$\underbrace{\operatorname{Ind}(D_+^X \otimes E)}_{\text{Analysis}} = \underbrace{\int_X \operatorname{Td}(TX, L)\operatorname{ch}(\underline{E})}_{\text{Topological}} \in \mathbb{Z}.$$

Here

$\operatorname{ch}(\underline{E}) = \operatorname{Tr}\left[\exp(\frac{i}{2\pi} R^E)\right]$, $c_1(L, \nabla^L) = \frac{i}{2\pi} R^L$,

$\widehat{\mathrm{A}}(TX, \nabla^{TX}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{i}{4\pi}R^{TX}}{\sinh\left(\frac{i}{4\pi}R^{TX}\right)}\right)$.

$\operatorname{Td}(TX, L) := \widehat{\mathrm{A}}(TX, \nabla^{TX}) e^{c_1(L, \nabla^L)/2} \in \Omega^*(X)$.

Atiyah-Patodi-Singer Index Theorem

- X : even-dimensional compact Spin^c manifold with boundary Y. Metrics have product structures near the boundary.
- Let $P_{\geqslant 0,+}$ be the orthogonal projection from $L^2(Y,(\mathcal{S}_+\otimes E)|_Y)$ onto $\oplus_{\lambda\geqslant 0}\mathscr{E}_{\lambda,+}$, eigenspace of positive eigenvalues of the Dirac operator $D^Y\otimes E$ on $\mathscr{C}^\infty(Y,(\mathcal{S}_+\otimes E)|_Y)$.
- Global (APS) boundary condition $D^X_+\otimes E$:

$$s\in\mathscr{C}^\infty(X,\mathcal{S}_+\otimes E) \text{ s. t. } P_{\geqslant 0,+}(s|_Y)=0.$$

- Atiyah-Patodi-Singer index theorem (1975) :

$$\mathrm{Ind}_{\mathrm{APS}}(D^X_+\otimes E)=\int_X \mathrm{Td}(TX,L)\,\mathrm{ch}(\underline{E})-\bar{\eta}\in\mathbb{Z}.$$

here $\bar{\eta}=\frac{1}{2}(\eta(0)+\dim\mathrm{Ker}(D^Y\otimes E))$ is the reduced (Atiyah-Patodi-Singer) eta invariant of the boundary Y.

Eta Invariant

- Y : odd-dim. compact Spin^c manifold. $\underline{E} = (E, h^E, \nabla^E)$ on Y.
- $D^Y \otimes E$: Dirac op. w. eigenvalues $\{\lambda_i\}$, eigenspaces $\mathscr{E}_{\lambda_i}$.
- $\eta(s) := \sum_{\lambda_i > 0} \frac{\mathrm{Tr}|_{\mathscr{E}_{\lambda_i}}[1]}{|\lambda_i|^s} - \sum_{\lambda_j < 0} \frac{\mathrm{Tr}|_{\mathscr{E}_{\lambda_j}}[1]}{|\lambda_j|^s}, \quad s \in \mathbb{C}.$
- (Atiyah-Patodi-Singer '75) $\eta(s)$ is holomorphic for $\mathrm{Re}(s) \gg 0$ and holomorphic at 0 after meromorphic continuation.
- Reduced APS eta invariant :

$$\bar{\eta}(Y, \underline{E}) := \frac{1}{2}\eta(0) + \frac{1}{2}\,\mathrm{Tr}|_{\ker(D^Y \otimes E)}[1] \in \mathbb{R}.$$

- The eta invariant cannot be computable in a local way and not a topological invariant as the index. Formally

$$\eta(0) = \text{Number of pos. eigen. - N. of neg. eigen. of } D^Y \otimes E.$$

APS Index Theorem : an example

- $4k$-dim. oriented compact manifold X with boundary Y.
  Symm. bilinear form on $\widehat{H}^{2k}(X) = \mathrm{Im}\{H^{2k}(X,Y) \to H^{2k}(X)\}$

$$\begin{array}{ccccc} \widehat{H}^{2k}(X) & \times & \widehat{H}^{2k}(X) \longrightarrow & \widehat{H}^{4k}(X) \simeq \mathbb{R}, \\ \alpha & & \beta & \displaystyle\int_X \alpha \wedge \beta \end{array}$$

  Its signature is called the signature Sign(X) of X.

- $L(\cdot)$ Hirzebruch L-polynomial,

$$\underbrace{\mathrm{Sign(X)}}_{\text{Topology}} = \underbrace{\int_X L(TX, \nabla^{TX})}_{\text{Geometry}} - \underbrace{\eta(0)}_{\text{Analysis}} .$$

  $\eta(0)$ eta invariant of $B\phi = (-1)^{k+p+1}(*d - d*)\phi$ for $\phi \in \Omega^{2p}(Y)$.

Chern-Simons functional and eta invariant

- Y a 3-dim. oriented compact manifold.
- (Stiefel) The tangent bundle TY is parallelizable.
  Fix a trivialization σ of TY on Y.
- *Chern-Simons functional* for connection $\nabla^{TY} = d + A$ on TY is

$$(1.1) \quad CS(\nabla^{TY}) = \frac{-1}{16\pi^2} \int_Y \mathrm{Tr} \left[A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right] \in \mathbb{R}.$$

- $CS(\nabla^{TY}) \in \mathbb{R}/\mathbb{Z}$ does not depend on the trivialization σ of TY.
- APS : Realization of $CS(\nabla^{TY})$ in $\mathbb{R}$ by eta invariant η.

$$2\, CS(\nabla^{TY}) = 3\, \eta \in \mathbb{R}/\mathbb{Z}.$$

Chern-Simons functional and eta invariant II

- More generally, any G-principal bundle on Y is trivializable if a G is a connected simply connected compact Lie group.
- $\mathscr{A}$ space of all G-connections on the G-bundle over Y. Witten's invariant of 3-manifolds : formally as the 'partition function'

$$Z_k(Y) = \int_{\mathscr{A}} \exp(ikCS(A))\mathscr{D}A \quad \text{for } k \in \mathbb{Z}. \tag{1.2}$$

Jones polynomials, Reshetikhin-Turaev-Witten invariants for knots, etc.

Extensions of eta invariants

- Families extension : Bismut-Cheeger eta form (1989) for a fibration $\pi : M \to B$ of smooth manifolds with compact fibers. The analytic counter part of the direct image in differential K-theory.
- Equivariant extension : with an group action $- - -- >$ equivariant eta invariant

Equivariant Eta Invariant

- Y : odd-dim. compact G-Spin^c manifold for compact Lie group G.
- $\underline{E} = (E, h^E, \nabla^E)$ G-equivariant on Y.
- Then $D^Y \otimes E$ commutes with the G-action.
  Its eigenspace $\mathscr{E}_{\lambda_i}$ with eigenvalue $\{\lambda_i\}$.
- $\eta_g(s) := \sum_{\lambda_i > 0} \frac{\mathrm{Tr}\,|_{\mathscr{E}_{\lambda_i}}[g]}{|\lambda_i|^s} - \sum_{\lambda_j < 0} \frac{\mathrm{Tr}\,|_{\mathscr{E}_{\lambda_j}}[g]}{|\lambda_j|^s}, \quad g \in G, s \in \mathbb{C}.$
- (Donnelly '78) $\eta_g(s)$ is holomorphic for $\mathrm{Re}(s) \gg 0$ and holomorphic at 0 after meromorphic continuation.
- Equivariant reduced APS eta invariant :

$$\bar{\eta}_g(Y, \underline{E}) := \frac{1}{2}\eta_g(0) + \frac{1}{2}\,\mathrm{Tr}\,|_{\ker(D^Y \otimes E)}[g] \in \mathbb{C},$$

is the boundary term of the equivariant APS index theorem with global (APS) boundary condition.

Localization of Eta Invariant ?

- How to undersand $\bar{\eta}_g$ as a function on G and its relation on the contribution on the fixed point set Y^g (i.e., localization formula) ?
- Example : $Y = S^1$, $\underline{E}$ trivial line bundle, $G = S^1$, $g.e^{i\theta} := e^{2\pi ikt+i\theta}$, $g = e^{2\pi it} \in S^1$, $k \in \mathbb{N}^*$.
- $Y^{S^1} := \{y \in Y : gy = y,\ \forall g \in S^1\} = \emptyset$, $A = \{g : g^k = 1\}$. Then

$$\bar{\eta}_g(Y) = \begin{cases} \frac{1}{1-g^k} & \text{if } g \in S^1 \backslash A\,; \\ \frac{1}{2} & \text{if } g \in A. \end{cases}$$

  $\bar{\eta}_g$ is not even continuous on $G = S^1$!
- Theorem (Liu-Ma 2020) Y odd-dim. compact S^1-Spin^c manifold. If $Y^{S^1} = \emptyset$, then $\bar{\eta}_g(\underline{TY}, \underline{E})$ as a function on $S^1 \backslash A$ with finite set $A = \{g \in S^1 : Y^g \neq \emptyset\}$, is the restriction of a rational function on S^1 with integral coefficients and without poles on $S^1 \backslash A$.

Localization of Eta Invariant II

- If $Y^{S^1} \neq \emptyset$, localization of eta invariants ? Much hard to explain our precise result.
- Some localization formulas :
  Berline-Vergne localization formula (1983) : (M, g^{TM}) a compact oriented Riemannian manifold. $X \in \mathscr{C}^\infty(M, TM)$ a Killing vector field and $M^X = \{x \in M : X(x) = 0\}$.
  Let $\alpha \in \Omega^\bullet(M)$ such that $(d - i_X)\alpha = 0$, then

$$\int_M \alpha = \int_{M^X} \frac{\alpha}{\mathrm{Pf}(\frac{-J^X + R^N}{2\pi})} . \tag{2.1}$$

  N be the normal bundle of M^X in M
  $J^X = \nabla^{TM}_{\cdot} X \in \mathscr{C}^\infty(M^X, \mathrm{End}(N))$.

Localization of Indices

- For simplicity, consider $G = S^1$.
- X : even-dim. compact S^1-Spin^c manifold, $\mathrm{Ind}_g(D^X \otimes E) := \mathrm{Tr}\,|_{\ker(D^X_+ \otimes E)}[g] - \mathrm{Tr}\,|_{\ker(D^X_- \otimes E)}[g]$.
- $\{X^{S^1}_\alpha\}_\alpha$ connected comp. of $X^{S^1} = \{x \in X : gx = x, \forall g \in S^1\}$.
- N_α normal bundle of $X^{S^1}_\alpha$ in X. we fix a complex structure on N_α s. t. the weights of the S^1-action on N_α are all positive. $\mathrm{Sym}(N^*_\alpha) = 1 + \sum_{k>0} \mathrm{Sym}^k(N^*_\alpha)$ symmetric power of N^*_α.

Theorem (Atiyah-Segal 1968, Under reform. of Liu-Ma-Zhang 2003)

$$\mathrm{Ind}_g(D^X \otimes E) = \sum_\alpha \mathrm{Ind}_g\left(D^{X^{S^1}_\alpha} \otimes \mathrm{Sym}(N^*_\alpha) \otimes E|_{X^{S^1}_\alpha}\right)$$

as distributions on S^1. If $X^{S^1} = \emptyset$, $\mathrm{Ind}_g(D^X \otimes E) = 0$!

- $\mathrm{Ind}_g(D^X \otimes E)$ polynomial on $g, g^{-1} \in S^1$ with integral coefficients

Localization of Eta Invariants

- Y odd-dim. compact S^1-Spin^c manifold.
- $A = \{g \in S^1 : Y^{S^1} \neq Y^g\}$ finite set.

Main result (Liu-Ma 20)

We construct explicitly $\underline{F_{\alpha,\pm}}$ as certain tensor product of the normal bundle N_α of $Y_\alpha^{S^1}$ in Y, $\chi_\alpha \in \mathbb{Z}[x]$, s.t. $\forall \underline{E}$,

$$\bar{\eta}_g(X, \underline{E}) - \Big\{ \sum_\alpha \chi_\alpha(g)^{-1} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{F_{\alpha,+}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) - \sum_\alpha \chi_\alpha(g)^{-1} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{F_{\alpha,-}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \Big\}$$

as a function on $S^1 \backslash A$ is the restriction of a rational function on S^1 with integral coefficients, and it has no poles on $S^1 \backslash A$.

Idea of proof

- For $g \in S^1$, set $\mathbb{Q}_g := \{P(g)/P'(g) \in \mathbb{C} : P, P' \in \mathbb{Z}[x], P'(g) \neq 0\}$.
- We construct explicitly $\underline{F_{\alpha,\pm}}$ as certian tensor product of the normal bundle N_α of $Y_\alpha^{S^1}$ in Y, $\chi_\alpha \in \mathbb{Z}[x]$, s.t. $\forall \underline{E}$, $g \in S^1 \setminus A$,

$$\mathcal{L}(g) := \bar{\eta}_g(Y, \underline{E}) - \Big\{ \sum_\alpha \chi_\alpha(g)^{-1} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{F_{\alpha,+}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) - \sum_\alpha \chi_\alpha(g)^{-1} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{F_{\alpha,-}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \Big\} \in \mathbb{Q}_g$$

  One key step : inverse $\lambda_{-1}(N) = \mathbb{C} + \sum_{k>0}(-1)^k \Lambda^k(N)$.
  Tools : algebraic formalism of differential K-Theory and Equivariant Bismut-Zhang embedding formula for eta invariants
- $\mathcal{L}(g) = \frac{\sum a_i g^i}{\sum b_j g^j}$. Need to show that $a_i, b_j \in \mathbb{Z}$ does not depend on g : Study carefully $\bar{\eta}_g$ as a function on S^1.
  Tools : analytic localization technique of Bismut-Lebeau

Localization of Eta Invariants

Regularity of equi. eta inv. (Liu-Ma 22)

For $g \in S^1$, $K \in \mathrm{Lie}(S^1)$, $t \in \mathbb{R}$ and $|t|$ small, $\exists\, c_j(g,K) \in \mathbb{C}$, $1 \leq j \leq (\dim Y^g + 1)/2$, s. t.

$$\bar{\eta}_{ge^{tK}}(Y, \underline{E}) - \sum_{j=1}^{(\dim Y^g+1)/2} c_j(g,K) t^{-j}$$

is an analytic function of t. If $g \in S^1 \setminus A$, then $c_j(g,K) = 0\ \forall j$.

Goette 00 : Expansion of $\bar{\eta}_{ge^{tK}}$ as $t \to 0$ if $g = 1$, $Y^{S^1} = \emptyset$.

Theorem (Liu-Ma 20)

The function $\mathcal{L}(g)$ on $S^1 \backslash A$ is the restriction of a rational function on S^1 with integral coefficients and without poles on $S^1 \backslash A$. Thus if we know $\mathcal{L}(g)$ for a transcendental $g \in S^1 \backslash A$, we know $\mathcal{L}(g)\ \forall g \in S^1 \backslash A$.

Differential K-Theory

- For compact space Y, $K^0(Y)$ is the topo. K-group of Y.
- The Chern character $\mathrm{ch} : K^0(Y) \to H^{\mathrm{even}}(Y, \mathbb{R})$.

$$\begin{array}{ccc} ? & \longrightarrow & K^0(Y) \\ \downarrow & & \downarrow \mathrm{ch} \\ \Omega^{\mathrm{even}}_{closed}(Y, \mathbb{R}) & \xrightarrow{\text{de Rham}} & H^{\mathrm{even}}(Y, \mathbb{R}). \end{array}$$

- $? = \widehat{K}^0(Y)$, differential K-group of Y.

History

- Differential K-theory is partly motivated by Witten 98 for R-R charge on D branes in type IIA/B superstring theory and introduced by Freed-Hopkins 00.
- It is the real analogue of the arithmetic K-theory in arithmetic algebraic geometry developed by Gillet, Soulé, etc in 1990s.

Equivalent Definitions and more :

- Hopkins-Singer 05,
- Freed-Klonoff-Lott 08, 10 ;
- Bunke-Schick 09 ;
- Simons-Sullivan 10 ;
- Tradler-Wilson-Zeinalian 13, 16 ;
- Hekmati-Murray-Schlegel-Vozzo 15 ;
- Gorokhovsky-Lott 16, ...

Chern-Simons Form

- $\underline{E} = (E, h^E, \nabla^E)$, curvature $R^E = \left(\nabla^E\right)^2 \in \Omega^2(Y, \mathrm{End}(E))$,

$$\mathrm{ch}(\underline{E}) = \mathrm{Tr}\left[\exp\left(-\frac{R^E}{2\pi \mathrm{i}}\right)\right] \in \Omega^*_{closed}(Y, \mathbb{R}).$$

- For $\underline{E}_1 = (E, h_1^E, \nabla_1^E)$,
Chern-Simons form $\widetilde{\mathrm{ch}}(\underline{E}, \underline{E}_1) \in \Omega^*(Y, \mathbb{R})/\mathrm{Im}\, d$ is well-defined s.t.

$$d\, \widetilde{\mathrm{ch}}(\underline{E}, \underline{E}_1) = \mathrm{ch}(\underline{E}_1) - \mathrm{ch}(\underline{E}).$$

- $\nabla^{\pi^* E}$ connection on $\pi^* E$ (for projection $\pi : Y \times \mathbb{R} \to Y$) s.t.
$\nabla^{\pi^* E}|_{Y \times \{0\}} = \nabla^E$, $\nabla^{\pi^* E}|_{Y \times \{1\}} = \nabla_1^E$, then

$$\widetilde{\mathrm{ch}}(\underline{E}, \underline{E}_1) = \int_0^1 \{\mathrm{ch}(\underline{\pi^* E})\}^{ds} ds.$$

Definition of Differential K-Group (by Freed-Lott)

- $\underline{E} = (E, h^E, \nabla^E)$, $\phi \in \Omega^{\mathrm{odd}}(Y, \mathbb{R})/\mathrm{Im}d$.
- $\widehat{K}^0(Y)$, group generated by semigroup $(\underline{E}, \phi)/\sim$.
- $(\underline{E_1}, \phi_1) \sim (\underline{E_2}, \phi_2) \iff \exists \underline{E_3}$ and $\exists$ bundle isomorphism $\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3$, s.t.

$$\phi_2 - \phi_1 = \widetilde{\mathrm{ch}}\left(\underline{E_1} \oplus \underline{E_3}, \Phi^*(\underline{E_2} \oplus \underline{E_3})\right).$$

- $\widehat{K}^0(Y)$ is a ring :

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] := [\underline{E} \otimes \underline{F}, \mathrm{ch}(\underline{E}) \wedge \psi + \mathrm{ch}(\underline{F}) \wedge \phi - d\phi \wedge \psi].$$

- Let $\underline{\mathbb{C}}$ be the trivial complex line bundle over Y with the trivial metric and connection, $1 := [\underline{\mathbb{C}}, 0]$ is a unity for the product $\cup$.

Chern-Weil Theory

$$\begin{array}{ccc} \widehat{K}^0(Y) & \longrightarrow & K^0(Y) \\ \downarrow \mathrm{ch}(\underline{E}) - d\phi & & \downarrow \mathrm{ch}([E]) \\ \Omega^{\mathrm{even}}_{closed}(Y,\mathbb{R}) & \xrightarrow{\text{de Rham}} & H^{\mathrm{even}}(Y,\mathbb{R}). \end{array}$$

- Diagram commutes $\Longleftrightarrow$ Chern-Weil theory.

g-equivariant Differential K-Theory (Liu-Ma)

- Y is S^1-compact manifold, $\underline{E}$ is S^1-equi. v. b. on Y.
- $g \in S^1$, $\phi \in \Omega^{\rm odd}(Y^g, \mathbb{R})/{\rm Im} d$.
- $\widehat{K}_g^0(Y)$ is the group generated by semigroup $(\underline{E}, \phi)/\sim$:
- $(\underline{E_1}, \phi_1) \sim (\underline{E_2}, \phi_2)$: $\exists$ S^1-equi. $\underline{E_3}$ and $\exists$ S^1-equi. bundle isomorphism $\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3$, s.t.

$$\phi_2 - \phi_1 = \widetilde{\rm ch}_g \left(\underline{E_1} \oplus \underline{E_3}, \Phi^*(\underline{E_2} \oplus \underline{E_3}) \right).$$

- $\widehat{K}_g^0(Y)$ is a ring :

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] := [\underline{E} \otimes \underline{F}, {\rm ch}_g(\underline{E}) \wedge \psi + {\rm ch}_g(\underline{F}) \wedge \phi - d\phi \wedge \psi].$$

- Let $\underline{\mathbb{C}}$ be the trivial complex line bundle over Y with the trivial metric and connection, $1 := [\underline{\mathbb{C}}, 0]$ is a unity for the product.

The Home of $\underline{\lambda_{-1}(N_\alpha^*)}^{-1}$

- $R(S^1)$ is the representation ring of S^1. $\widehat{K}_g^0(Y)$ is $R(S^1)$-module.
- $\widehat{K}_g^0(Y)_{I(g)}$ localization of $\widehat{K}_g^0(Y)$ at $I(g) = \{\chi \in R(S^1) : \chi(g) = 0\}$.
  elements of $\widehat{K}_g^0(Y)_{I(g)}$: $\left([\underline{E}, \phi] - [\underline{E'}, \phi']\right)/\chi$, where $\chi(g) \neq 0$.
- $A := \{g \in S^1 : Y^g \neq Y^{S^1}\}$ finite set.

Theorem 1 (Liu-Ma 20)

For $g \in S^1 \backslash A$, $\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right]$ is invertible in $\widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}$.

Now we sketch a proof of Theorem 1, which also gives a realization of $\underline{\lambda_{-1}(N_\alpha^*)}^{-1} = (\underline{F}_{\alpha,+} - \underline{F}_{\alpha,-})/\chi$.

γ-filtration

- Pre-λ-ring Structure on $\widehat{K}^0(Y)$ s.t. $\lambda_t(\underline{E}) = \sum_{j\geq 0} \Lambda^j(\underline{E})t^j$.
- Set $\gamma_t(x) = \sum_{j\geq 0} \gamma^j(x)t^j := \lambda_{\frac{t}{1-t}}(x)$.
- For $r = \mathrm{rk}(E)$, $\gamma^k(\underline{E} - r)$ is a finite dim. virtual vector bundle

$$\gamma^k(\underline{E} - r) = \begin{cases} \sum_{i=0}^k (-1)^{k-i} \begin{pmatrix} r-i \\ k-i \end{pmatrix} \Lambda^i(\underline{E}) & \text{if } 0 \leq k \leq r; \\ 0 & \text{if } k > r. \end{cases}$$

$$\lambda_t(\underline{E})^{-1} = \gamma_{\frac{t}{1+t}}(\underline{E} - r)^{-1}(1+t)^{-r} = (1+t)^{-r}\Big(1 + \sum_{i=1}^r \gamma^i(\underline{E} - r)t^i(1+t)^{-i}\Big)^{-1}$$

$$= (1+t)^{-r}\left(1 + \sum_{k=1}^{\infty} t^k(1+t)^{-k} \sum_{\substack{(n_1,\cdots,n_r)\in\mathbb{N}^r, \\ \sum_{i=1}^r i\cdot n_i = k}} (-1)^{\sum_{i=1}^r n_i} \frac{(\sum_{i=1}^r n_i)!}{\prod_{i=1}^r n_i!} \prod_{i=1}^r \big(\gamma^i(\underline{E} - r)\big)^{n_i}\right).$$

To simplify the notations, we denote by

$$\lambda_t(\underline{E})^{-1} = (1+t)^{-r}\left(1 + \sum_{k=1}^{\infty} t^k(1+t)^{-k}\big(P_{k,+}(\underline{E}) - P_{k,-}(\underline{E})\big)\right).$$

γ-filtration

$$\lambda_t(\underline{E})^{-1} = (1+t)^{-r}\left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k} \big(P_{k,+}(\underline{E}) - P_{k,-}(\underline{E})\big)\right),$$

Lemma (Liu-Ma 20)

There exists $\mathcal{N}_{r,m} > 0$, only depending on $r = \mathrm{rk}(E)$, $m = \dim Y$, such that for any geometric triple $\underline{E}$ on Y and $k > \mathcal{N}_{r,m}$, we have $[P_{k,+}(\underline{E}), 0] = [P_{k,-}(\underline{E}), 0] \in \widehat{K}^0(Y)$.

$\Longrightarrow \mathrm{ch}(P_{k,+}(\underline{E})) = \mathrm{ch}(P_{k,-}(\underline{E})) \in \Omega^{2*}(Y, \mathbb{R})$.

Realization of $\underline{\lambda_{-1}(N_\alpha^*)}^{-1}$

We denote by $\underline{'E}$ when forgetting the group action. In the sense of $K^0_{S^1}(Y_\alpha^{S^1}) \simeq R(S^1) \otimes K^0(Y_\alpha^{S^1})$, $\underline{\lambda_{-1}(N_\alpha^*)} \simeq \bigotimes_v \lambda_{-h^{-v}} \left(\underline{'N^*_{\alpha,v}}\right)$. Set $r_{\alpha,v} = \mathrm{rk} N_{\alpha,v}, \quad m_\alpha = \dim Y_\alpha^{S^1}$, formally,

$$\lambda_{-h^{-v}} \left(\underline{'N^*_{\alpha,v}}\right)^{-1} = \frac{h^{v r_{\alpha,v}}}{(h^v - 1)^{r_{\alpha,v}}} \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(h^v - 1)^k} \left(P_{k,+}\left(\underline{'N^*_{\alpha,v}}\right) - P_{k,-}\left(\underline{'N^*_{\alpha,v}}\right)\right)\right).$$

Set

$$\underline{\lambda_{-1}(N_\alpha^*)^{-1}_{\mathcal{N}}} = \bigotimes_v \frac{h^{v r_{\alpha,v}}}{(h^v - 1)^{r_{\alpha,v}}} \left(1 + \sum_{k=1}^{\mathcal{N}} \frac{(-1)^k}{(h^v - 1)^k} \left(P_{k,+}\left(\underline{'N^*_{\alpha,v}}\right) - P_{k,-}\left(\underline{'N^*_{\alpha,v}}\right)\right)\right).$$

Theorem 2 (Liu-Ma 20)

For any $\mathcal{N} > \sup_{\alpha,v} \mathcal{N}_{r_{\alpha,v}, m_\alpha}$, we have

$$\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right]^{-1} = \left[\underline{\lambda_{-1}(N_\alpha^*)^{-1}_{\mathcal{N}}}, 0\right] \in \widehat{K}^0_g(Y_\alpha^{S^1})_{I(g)}.$$

Localization of Eta Invariant ?

Question

For $\underline{\lambda_{-1}(N_\alpha^*)} := \Lambda^{\text{even}}(\underline{N_\alpha^*}) - \Lambda^{\text{odd}}(\underline{N_\alpha^*})$, if $Y^g = Y^{S^1}$, $\forall \underline{E}$,

$$\bar{\eta}_g(Y, \underline{E}) \overset{?}{=} \sum_\alpha \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)}^{-1} \otimes \underline{E}|_{Y_\alpha^{S^1}})$$

$$\bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)^{-1}_{\mathcal{N}_1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \overset{?}{=} \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)^{-1}_{\mathcal{N}_2}} \otimes \underline{E}|_{Y_\alpha^{S^1}}).$$

Push-forward in Differential K-Theory

Variation formula of eta invariants

If Y is odd-dim. compact S^1-equi. Spin^c, $\exists\ \chi \in R(S^1)$ s. t. $\forall\ g \in S^1$,

$$\bar{\eta}_g(Y, \underline{E_1}) - \bar{\eta}_g(Y, \underline{E_0}) = \int_{Y^g} \mathrm{Td}_g(TY, L)\widetilde{\mathrm{ch}}_g(\underline{E_0}, \underline{E_1}) + \chi(g).$$

Proposition (Liu-Ma 20)

Let $\mathbb{Q}_g := \{P(g)/P'(g) \in \mathbb{C} : P, P' \in \mathbb{Z}[x], P'(g) \neq 0\}$. If Y is odd-dim. S^1-equi. Spin^c, the following push-forward map is well-defined :

$$\widehat{f_{Y\,!}} : \widehat{K}^0_g(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g,$$

$$[\underline{E}, \phi]/\chi \mapsto \chi(g)^{-1} \left(-\int_{Y^g} \mathrm{Td}_g(TY, L) \wedge \phi + \bar{\eta}_g(Y, \underline{E}) \right) \mod \mathbb{Q}_g.$$

$$\Longrightarrow\ \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)^{-1}_{\mathcal{N}_1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) - \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)^{-1}_{\mathcal{N}_2}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \in \mathbb{Q}_g.$$

Localization of Eta Invariant ?

Question

If $Y^g = Y^{S^1}$, i.e., $g \in S^1 \backslash A$, for $\mathcal{N}$ large, $\forall \underline{E}$,

$$\bar{\eta}_g(Y, \underline{E}) - \sum_\alpha \bar{\eta}_g(Y_\alpha^{S^1}, \underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}) \in \mathbb{Q}_g?$$

Differential Atiyah-Hirzebruch Direct Image

Equivariant Bismut-Zhang embedding formula (B-Z 93, Liu 18)

Consider the embedding $\iota : Y^{S^1} \to Y$. For S^1-equi. $\underline{\mu}$ over Y^{S^1}, we can constr. the geometric version of equi. A-H direct image $\underline{\xi_\pm} = \hat{\iota}_!(\underline{\mu})$. Moreover, there exists $\chi' \in R(S^1)$, s.t. for any $g \in S^1 \backslash A$,

$$\sum_\alpha \bar{\eta}_g(\underline{Y_\alpha^{S^1}}, \underline{\mu}) = \bar{\eta}_g(Y, \underline{\xi_+}) - \bar{\eta}_g(Y, \underline{\xi_-}) + \chi'(g).$$

Theorem 3 (Liu-Ma 20)

$\iota : Y^{S^1} \to Y$ induces a pullback map $\hat{\iota}^* : \widehat{K}_g^0(Y)_{I(g)} \to \widehat{K}_g^0(Y^{S^1})_{I(g)}$, which is an isomorphism. Moreover, the following diagram commutes :

$$\begin{array}{ccc} \widehat{K}_g^0(Y^{S^1})_{I(g)} & \xleftarrow{\;[\underline{\lambda_{-1}(N^*)}, 0]^{-1} \cup \hat{\iota}^*\;} & \widehat{K}_g^0(Y)_{I(g)} \\ & {\scriptstyle \widehat{f_{Y^{S^1}!}}} \searrow \quad \swarrow {\scriptstyle \widehat{f_{Y!}}} & \\ & \mathbb{C}/\mathbb{Q}_g. & \end{array}$$

Localization of Eta Invariants

- Recall that $\widehat{f_{Y}}_!([\underline{E},\phi]) = -\int_{Y^g} \mathrm{Td}_g(TY,L)\wedge\phi + \bar{\eta}_g(Y,\underline{E})$.

$$\begin{array}{ccc}
\widehat{K}^0_g(Y^{S^1})_{I(g)} & \xleftarrow{\;[\underline{\lambda_{-1}(N^*)},\,0]^{-1}\cdot\,\hat{\iota}^*\;} & \widehat{K}^0_g(Y)_{I(g)} \\
 & {\scriptstyle \widehat{f_{Y^{S^1}}}_!}\searrow \quad \swarrow{\scriptstyle \widehat{f_{Y}}_!} & \\
 & \mathbb{C}/\mathbb{Q}_g . &
\end{array}$$

$$\left[\underline{\lambda_{-1}(N^*_\alpha)},0\right]^{-1} = \left[\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}},0\right] = [,0]\,.$$

Theorem (Liu-Ma 20)

If Y is odd-dim. S^1-equi. Spin^c, for $g\in S^1\backslash A$, for $\mathcal{N}$ large, $\forall\underline{E}$,

$$\mathcal{L}(g) := \bar{\eta}_g(Y,\underline{E}) - \sum_\alpha \bar{\eta}_g(Y^{S^1}_\alpha, \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}\otimes\underline{E}|_{Y^{S^1}_\alpha}) \in \mathbb{Q}_g .$$

Reference

- B. Liu, *Real embedding and equivariant eta forms*, Math. Z. **292** (2019), 849–878.
- B. Liu and X. Ma, *Differential K-theory, η-invariants and localization*, C. R. Math. Acad. Sci. Paris, **357** (2019), no. 10, 803–813.
- B. Liu and X. Ma, *Differential K-theory and localization formula for η-invariants*, Invent. Math. **222** (2020), 545–613.
- B. Liu and X. Ma, *Comparison of two equivariant eta forms*, Adv. Math., **404** (2022), 108163, 76pp.