Moment polytopes in real symplectic geometry

Paul-Emile Paradan

University of Montpellier, France

Prospects in Geometry and Global Analysis
Castle Rauischholzhausen, 2023

Plan

The talk is devoted to the description of convex polyhedral cones which are associated to some representations of compact Lie groups.

- Eigenvalues and singular values
- Description of several geometric cones:
  - The Horn cone
  - A cone of eigenvalues
  - The singular Horn cone
- Convexity in Hamitonian geometry
- Convexity in *real* Hamitonian geometry: O'Shea-Sjamaar's Theorem
- General results for the cones associated to isotropy representations of Riemannian symmetric spaces
- Key point in the proof : Kirwan-Ness stratification and Ressayre's pairs

Eigenvalues and singular values

Let $\mathrm{e}(X) = (\mathrm{e}_1 \geq \cdots \geq \mathrm{e}_n) \in \mathbb{R}^n_+$ be the **eigenvalues** of a Hermitian (or real symmetric) $n \times n$ matrix.

Fact : two isomorphisms $\boxed{Herm(n)/U(n) \xrightarrow{\mathrm{e}} \mathbb{R}^n_+}$ and $\boxed{Sym(n)/SO(n) \xrightarrow{\mathrm{e}} \mathbb{R}^n_+}$

Let $\mathrm{s}(X) = (\mathrm{s}_1 \geq \cdots \geq \mathrm{s}_q \geq 0) \in \mathbb{R}^q_{++}$ be the **singular values** of a complex $p \times q$ matrix.

Fact : an isomorphism $\boxed{M_{p,q}(\mathbb{C})/U(p) \times U(q) \xrightarrow{\mathrm{s}} \mathbb{R}^q_{++}}$

Basic questions: what are the relations between

1. $\mathrm{e}(X)$, $\mathrm{e}(Y)$ and $\mathrm{e}(X+Y)$ for $X, Y \in Herm(n)$.
2. $\mathrm{s}(X)$, $\mathrm{s}(Y)$ and $\mathrm{s}(X+Y)$ for $X, Y \in M_{p,q}(\mathbb{C})$.
3. $\mathrm{e}(X)$ and $\mathrm{e}(\mathfrak{R}(X))$ where $\mathfrak{R}(X) \in Sym(n)$ is the real part of $X \in Herm(n)$.
4. $\mathrm{e}(X)$ and $\mathrm{s}(X_{12})$ where X_{12} is the off-diagonal bloc of $X \in Herm(n)$.
5. $\mathrm{s}(X)$, $\mathrm{s}(X_{12})$ and $\mathrm{s}(X_{21})$ for $X \in M_{n,n}(\mathbb{C})$.
6. $\mathrm{s}(X)$, $\mathrm{s}(X_{11})$ and $\mathrm{s}(X_{22})$ for $X \in M_{n,n}(\mathbb{C})$.
7. $\cdots$

The aim of this presentation is to explain the methods used to answer these kind of questions. **Keywords :** Hamiltonian action, moment map, anti-symplectic involution.

Classical geometric cone: the Horn cone

The Horn cone

$$\mathrm{Horn}(n) := \Big\{(\mathrm{e}(X), \mathrm{e}(Y), \mathrm{e}(X+Y));\ X, Y \in Herm(n)\Big\}$$

Some notations:

- $\mathbb{R}^n_+ = \{x = (x_1 \geq \cdots \geq x_n)\}$.
- $I = \{i_1 < \cdots < i_r\} \subset \mathbb{N} - \{0\} \rightsquigarrow \mu(I) = (i_r - r \geq \cdots \geq i_1 - 1 \geq 0) \in \mathbb{R}^r_+$.
- If $x \in \mathbb{R}^n$ and $I \subset \{1, \ldots, n\}$, we write $|x| = \sum_{k=1}^n x_k$ and $|x|_I = \sum_{i \in I} x_i$.

Schubert Calculus : cohomology of the Grassmannian $\mathbb{G}(r, n)$

- $\mathbb{G}(r, n) := \{E \subset \mathbb{C}^n,\ \dim E = r\}$
- $H^*(\mathbb{G}(r, n)) = \bigoplus_{I \subset [n], \sharp I = r} \mathbb{Z}\Theta_I$
- $H^{\max}(\mathbb{G}(r, n)) = \mathbb{Z}\Theta_{[r]}$
- $\Theta_I \cdot \Theta_{I^o} = \Theta_{[r]}$ when $I^o = \{n + 1 - i, i \in I\}$
- $\Theta_{I^o} \cdot \Theta_{J^o} \cdot \Theta_L = \ell\Theta_{[r]}, \ell \neq 0 \Longleftrightarrow \left(V_{\mu(I)} \otimes V_{\mu(J)} \otimes V^*_{\mu(L)}\right)^{U(n)} \neq 0$

Classical geometric cone: the Horn cone

The study of the cone $\mathrm{Horn}(n)$ started long ago: Weyl (1932), Ky Fan (1949), Lidskii (1950), Thompson-Freede (1971).

Horn conjecture (1962)

An element $(x, y, z) \in (\mathbb{R}^n_+)^3$ belongs to $\mathrm{Horn}(n)$ if and only if

- $|x| + |y| = |z|$ (trace condition)
- $|x|_I + |y|_J \geq |z|_L$ for any subsets $I, J, L \subset \{1, \ldots, n\}$ of cardinal $r < n$ satisfying :

$$\mathit{Condition}_{(I,J,L)} : \qquad (\mu(I), \mu(J), \mu(L)) \in \mathrm{Horn}(r)$$

Proof of the Horn conjecture

- Klyachko (1998): Horn conjecture holds with $\mathit{Condition}_{(I,J,L)}$ replaced by

$$\mathit{Condition}'_{(I,J,L)} : \qquad \Theta_{I^o} \cdot \Theta_{J^o} \cdot \Theta_L = \ell\Theta_{[r]}, \ell \neq 0, \quad in \quad H^*(\mathbb{G}(r,n))$$

- Saturation Theorem of Knutson-Tao (1999):

$$\mathit{Condition}'_{(I,J,L)} \iff \mathit{Condition}_{(I,J,L)}$$

Final improvements by Belkale (2001) and Knutson-Tao-Woodward (2004) : equations for $\ell = 1$ are sufficient and not redundant.

A cone of eigenvalues

Consider the map $\Re : Herm(n) \to Sym(n)$ which associates to a Hermitian matrix its real part. We are interested in the following cone:

$$\mathcal{E}(n) := \left\{ (\mathrm{e}(X), \mathrm{e}(\Re(X))) \, ; \; X \in Herm(n) \right\}$$

First description: an application of the O'Shea-Sjamaar theorem

An element $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ belongs to $\mathcal{E}(n)$ if and only if $(x, x, 2y) \in \mathrm{Horn}(n)$.

A refinement:

Theorem (PEP, 2022)

An element $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ belongs to $\mathcal{E}(n)$ if and only if

$$|x| = |y| \qquad \text{and} \qquad |x|_I \geq |y|_J$$

holds for any subsets $I, J \subset [n]$ of cardinal $r < n$ such that $(2\mu(I), \mu(J)) \in \mathcal{E}(r)$.

Example

- $\mathcal{E}(1)$, $\mathcal{E}(2)$, $\mathcal{E}(3)$ and $\mathcal{E}(4)$ are defined by 1, 2, 7 and 16 inequalities.
- $\mathrm{Horn}(1)$, $\mathrm{Horn}(2)$, $\mathrm{Horn}(3)$ and $\mathrm{Horn}(4)$ are defined by 1, 7, 19 and 51 inequalities.

The singular Horn cone

If $p \geq q \geq 1$, the singular value map $\mathrm{s} : M_{p,q}(\mathbb{C}) \to \mathbb{R}^q_{++}$ is defined by $\mathrm{s}(A) = \sqrt{\mathrm{e}(A^* A)}$.

Singular Horn cone

$$\mathrm{Singular}(p,q) := \Big\{ (\mathrm{s}(A), \mathrm{s}(B), \mathrm{s}(A+B)),\ A, B \in M_{p,q}(\mathbb{C}) \Big\}$$

Map: $x \in \mathbb{R}^q \quad \longmapsto \quad \widehat{x} = (x_1, \ldots, x_q, 0, \ldots, 0, -x_q, \ldots, -x_1) \in \mathbb{R}^n$

First description: an application of the O'Shea-Sjamaar theorem

$(x, y, z) \in (\mathbb{R}^q_{++})^3$ belongs to $\mathrm{Singular}(p,q)$ if and only if $(\widehat{x}, \widehat{y}, \widehat{z}) \in \mathrm{Horn}(p+q)$.

Example

We will see that $\mathrm{Singular}(3,3)$ is determined by **96 inequalities** whereas $\mathrm{Horn}(6)$ needs **536 inequalities.**

Some notations:

- Polarized sets $X_\bullet = X_+ \coprod X_-$ of $[q] := \{1, \ldots, q\}$
- Signature function: $\epsilon : X_\bullet \to \{\pm\}$
- Signed inequalities:

$$(\star)_{I_\bullet, J_\bullet, L_\bullet} \qquad \sum_{i \in I_\bullet} \epsilon_i \mathrm{s}_i(A) + \sum_{j \in J_\bullet} \epsilon_j \mathrm{s}_j(B) + \sum_{\ell \in L_\bullet} \epsilon_\ell \mathrm{s}_\ell(A+B) \leq 0$$

The singular Horn cone : inequalities

- To a polarized subset $X_\bullet \subset [q]$ we associate two subsets of cardinal $\sharp X_\bullet$:
  - $X_\bullet^p = X_+ \cup \{p+q+1-\ell, \ell \in X_-\} \subset [p+q]$,
  - $\widetilde{X}_\bullet^p \subset [p+q-r]$ (more complicated definition).
- Involution on $\mathbb{R}^q$: $x = (x_1, \cdots, x_q) \mapsto x^* = (-x_q, \cdots, -x_1)$.

Theorem (PEP, 2021)

$\mathrm{Singular}(p,q)$ is determined by the inequalities $(\star)_{I_\bullet, J_\bullet, L_\bullet}$ where $I_\bullet, J_\bullet, L_\bullet$ satisfy the following conditions: $\sharp I_\bullet = \sharp J_\bullet = \sharp L_\bullet = r < q$ and

1. $(\mu(I_\bullet^p), \mu(J_\bullet^p), \mu(L_\bullet^p)^* + 2(p+q-r)1^r) \in \mathrm{Horn}(r)$,
2. $(\mu(\widetilde{I}_\bullet^p), \mu(\widetilde{J}_\bullet^p), \mu(\widetilde{L}_\bullet^p)^* + 2(p+q-2r)1^r) \in \mathrm{Horn}(r)$.

Why two conditions? In fact they are equivalent to the cohomological condition

$$\Theta_{I_\bullet^n} \odot \Theta_{J_\bullet^n} \odot \Theta_{L_\bullet^n} = \ell[pt],\ \ell \neq 0 \quad in \quad H^*(\mathbb{F}(r, n-r, n)),$$

where $\mathbb{F}(r, n-r, n)$ denotes the two-steps flag variety parameterizing nested sequences of linear subspaces $E \subset F \subset \mathbb{C}^n$ where $\dim E = r$ and $\dim F = n - r$.

Singular(3, 3)

$(a, b, c) \in (\mathbb{R}^3_{++})^3$ belongs to $\mathrm{Singular}(3,3)$ if and only if, **modulo permutation**, we have

1. 18 Weyl inequalities
   - $a_1 + b_1 \geq c_1$
   - $a_1 + b_3 \geq c_3$
   - $a_1 + b_2 \geq c_2$
   - $a_2 + b_2 \geq c_3$
2. 18 Lidskii inequalities
   - $a_1 + a_2 + b_1 + b_2 \geq c_1 + c_2$
   - $a_1 + a_2 + b_1 + b_3 \geq c_1 + c_3$
   - $a_1 + a_2 + b_2 + b_3 \geq c_2 + c_3$
   - $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 \geq c_1 + c_2 + c_3$
3. 36 signed Lidskii inequalities
   - $a_1 + a_2 + b_1 - b_2 \geq c_1 - c_2$
   - $a_1 + a_2 + b_1 - b_3 \geq c_1 - c_3$
   - $a_1 + a_2 + b_2 - b_3 \geq c_2 - c_3$
   - $a_1 + a_2 + a_3 + b_1 + b_2 - b_3 \geq c_1 + c_2 - c_3$
   - $a_1 + a_2 + a_3 + b_1 - b_2 + b_3 \geq c_1 - c_2 + c_3$
   - $a_1 + a_2 + a_3 - b_1 + b_2 + b_3 \geq -c_1 + c_2 + c_3$
4. 15 others inequalities
   - $a_1 + a_3 + b_1 + b_3 \geq c_2 + c_3$
   - $a_1 + a_3 + b_1 - b_3 \geq c_2 - c_3$
   - $(a_1 + a_2 - a_3) + (b_1 - b_2 + b_3) + (-c_1 + c_2 + c_3) \geq 0$

Convexity in Hamiltonian geometry

Kähler manifold (M, Ω) acted on by a compact Lie group U:

- Holomorphic action of $U_{\mathbb{C}} \circlearrowleft M$.
- The action $U \circlearrowleft (M, \Omega)$ is Hamiltonian, with **proper** moment map $\Phi_{\mathfrak{u}} : M \to \mathfrak{u}^*$.

Theorem (Kirwan, 1984)

$\Delta_{\mathfrak{u}}(M) = \Phi_{\mathfrak{u}}(M) \cap \mathfrak{t}^*_+$ is a closed convex locally polyhedral subset.

Basic question

Determine the inequalities defining the Kirwan polytope $\Delta_{\mathfrak{u}}(M)$.

Example

- Compact Lie groups $U \hookrightarrow \tilde{U}$ with Lie algebras $\mathfrak{u} \hookrightarrow \tilde{\mathfrak{u}}$ and projection $\pi : \tilde{\mathfrak{u}}^* \to \mathfrak{u}^*$.
- Kähler manifold: $\tilde{U}_{\mathbb{C}} \simeq T\tilde{U} \simeq T^*\tilde{U}$
- Hamiltonian action $\tilde{U} \times U \circlearrowleft \tilde{U}_{\mathbb{C}}$: $\boxed{(\tilde{g}, g) \cdot m = \tilde{g}\, m\, g^{-1}}$
- Moment map $\Phi : \tilde{U}_{\mathbb{C}} \to \tilde{\mathfrak{u}} \times \mathfrak{u}$: $\boxed{\tilde{g} e^{i\tilde{X}} \longmapsto (-\tilde{g}\tilde{X}, \pi(\tilde{X}))}$
- Kirwan polytope : $\boxed{\mathrm{Horn}(\tilde{U}, U) = \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathfrak{t}}_+ \times \mathfrak{t}_+,\ U\xi \subset \pi(\tilde{U}\tilde{\xi}) \right\}}$

Convexity in *real* Hamiltonian geometry

We suppose that (M, Ω, U, Φ) is equipped with **involutions**:

1. an involution σ on U
2. an anti-holomorphic involution τ on M such that $\tau^*(\Omega) = -\Omega$
3. compatibility conditions: $\boxed{\tau(g \cdot x) = \sigma(g) \cdot \tau(x)}$ and $\boxed{\Phi(\tau(x)) = -\sigma(\Phi(x))}$

Example ($U(n)$ with the involution $\sigma(g) = \overline{g}$)

- Any adjoint orbit $\mathcal{O}_\lambda = U(n) \cdot \mathrm{diag}(i\lambda_1, \ldots, i\lambda_n)$ is stable under $\tau(x) = -\overline{x}$.
- $(\mathcal{O}_\lambda)^\tau = i\mathcal{O}_\lambda^{\mathbb{R}}$ with $\mathcal{O}_\lambda^{\mathbb{R}} := \{X \in Sym(n), \mathrm{e}(X) = \lambda\}$.
- $\boxed{\mathcal{O}_\nu \subset \mathcal{O}_\lambda + \mathcal{O}_\mu \Longleftrightarrow \mathcal{O}_\nu^{\mathbb{R}} \subset \mathcal{O}_\lambda^{\mathbb{R}} + \mathcal{O}_\mu^{\mathbb{R}}}$

Map: $a \in \mathbb{R}^q \longmapsto \widehat{a} = (a_1, \ldots, a_q, 0, \ldots, 0, -a_q, \ldots, -a_1) \in \mathbb{R}^n$

Example ($U(n)$ with the involution $\sigma(g) = I_{p,q}\, g\, I_{p,q}$)

- $\mathcal{O}_\lambda$ is stable under $\tau(x) = -I_{p,q}\, x\, I_{p,q}$ if and only if $\exists a \in \mathbb{R}^q_{++}$, $\lambda = \widehat{a}$
- $(\mathcal{O}_{\widehat{a}})^\tau \simeq \mathcal{V}_a$ where $\mathcal{V}_a = \{X \in M_{p,q}(\mathbb{C}), \mathrm{s}(X) = a\}$
- $\boxed{\mathcal{O}_{\widehat{c}} \subset \mathcal{O}_{\widehat{a}} + \mathcal{O}_{\widehat{b}} \Longleftrightarrow \mathcal{V}_c \subset \mathcal{V}_a + \mathcal{V}_b}$

Real moment polytopes: O'Shea-Sjamaar Theorem

Involution on U

- $K := (U^\sigma)^0$ acts on $\mathfrak{p} = i\mathfrak{u}^{-\sigma}$
- σ-invariant maximal torus $T \subset U$ and $\mathfrak{t}_+ =$ Weyl chamber for U
- $\mathfrak{a} = i\mathfrak{t}^{-\sigma}$ of maximal dimension $\rightsquigarrow$ $\mathfrak{a}_+ = i(\mathfrak{t}^{-\sigma} \cap \mathfrak{t}_+) \simeq \mathfrak{p}/K$

Anti-holomorphic involution on (M, Ω)

- $Z := M^\tau$ is a Lagrangian submanifold (that we suppose non-empty).
- Real moment map $\Phi_\mathfrak{p} : Z \to \mathfrak{p}$.
- The set $\Delta_\mathfrak{p}(Z) := \Phi_\mathfrak{p}(Z) \cap \mathfrak{a}_+$ parameterizes the K-orbits in $\Phi_\mathfrak{p}(Z)$.

Theorem (O'S-S, 2000)

$$\Delta_\mathfrak{p}(Z) \simeq \Delta_\mathfrak{u}(M) \cap \mathfrak{t}^{-\sigma}$$

$\Delta_\mathfrak{p}(Z)$ is called the **real moment polytope**.

The example of isotropic representations of symmetric spaces

Let us consider an involution σ on $U \subset \widetilde{U}$.
The involution σ extends to an **antilinear** involution $\sigma_{\mathbb{C}}$ on $U_{\mathbb{C}} \subset \widetilde{U}_{\mathbb{C}}$.

- $G = (U_{\mathbb{C}}^{\sigma_{\mathbb{C}}})^0 \subset \widetilde{G} = (\widetilde{U}_{\mathbb{C}}^{\sigma_{\mathbb{C}}})^0$: real reductive Lie groups
- Maximal compact subgroups $K = (U^{\sigma})^0 \subset \widetilde{K} = (\widetilde{U}^{\sigma})^0$
- Cartan decompositions : $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

Hamiltonian action of $\widetilde{U} \times U$ on $\widetilde{U}_C$ $\rightsquigarrow$ Kirwan polytope $\mathrm{Horn}(\widetilde{U}, U)$.

- $\widetilde{G}$ = Lagrangian submanifold of $\widetilde{U}_{\mathbb{C}}$ is equipped with an action of $\widetilde{K} \times K$
- Restriction of the moment map $\Phi : \widetilde{U}_{\mathbb{C}} \to \tilde{\mathfrak{u}} \times \mathfrak{u}$ defines $\Phi_{\mathfrak{p}} : \widetilde{G} \to \tilde{\mathfrak{p}} \times \mathfrak{p}$
- Real moment polytope:

$$\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K) = \left\{ (\tilde{\xi}, \xi) \in \tilde{\mathfrak{a}}_+ \times \mathfrak{a}_+ \mid K \cdot \xi \subset \pi(\widetilde{K} \cdot \tilde{\xi}) \right\}$$

Corollary of O'Shea-Sjamaar Theorem

$$\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K) \simeq \mathrm{Horn}(\widetilde{U}, U) \bigcap \tilde{\mathfrak{t}}^{-\sigma} \times \mathfrak{t}^{-\sigma}$$

Isotropic representations of symmetric spaces: examples

Initial question : what are the relations between

1. $\mathrm{e}(X)$, $\mathrm{e}(Y)$ and $\mathrm{e}(X+Y)$ for $X, Y \in Herm(n)$.
2. $\mathrm{s}(X)$, $\mathrm{s}(Y)$ and $\mathrm{s}(X+Y)$ for $X, Y \in M_{p,q}(\mathbb{C})$.
3. $\mathrm{e}(X)$ and $\mathrm{e}(\Re(X))$ where $\Re(X) \in Sym(n)$ is the real part of $X \in Herm(n)$.
4. $\mathrm{e}(X)$ and $\mathrm{s}(X_{12})$ where X_{12} is the off-diagonal bloc of $X \in Herm(n)$.
5. $\mathrm{s}(X)$, $\mathrm{s}(X_{12})$ and $\mathrm{s}(X_{21})$ for $X \in M_{n,n}(\mathbb{C})$.
6. $\mathrm{s}(X)$, $\mathrm{s}(X_{11})$ and $\mathrm{s}(X_{22})$ for $X \in M_{n,n}(\mathbb{C})$.

Answer : compute the real moment polytope $\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K)$ in the following cases

1. $G = GL_n(\mathbb{C})$ and $\widetilde{G} = G \times G$ $\rightsquigarrow$ $\mathrm{Horn}(n)$
2. $G = U(p,q)$ and $\widetilde{G} = G \times G$ $\rightsquigarrow$ $\mathrm{Singular}(p,q)$
3. $G = GL_n(\mathbb{R})$ and $\widetilde{G} = GL_n(\mathbb{C})$ $\rightsquigarrow$ $\mathcal{E}(n)$
4. $G = U(p,q)$ and $\widetilde{G} = GL_n(\mathbb{C})$
5. $G = U(p,q) \times U(q,p)$ and $\widetilde{G} = U(n,n)$
6. $G = U(p,p) \times U(q,q)$ and $\widetilde{G} = U(n,n)$

Determination of the inequalities of $\mathrm{Horn}(\widetilde{U}, U)$

- Maximal torus $T \subset U$ and $\widetilde{T} \subset \widetilde{U}$, such that $T \subset \widetilde{T}$
- Weyl groups W, $\widetilde{W}$ and longest element $w_o \in W$
- $\mathfrak{R} := \mathfrak{R}(\tilde{\mathfrak{u}}/\mathfrak{u}) \subset \mathfrak{t}^*$ set of roots relatively to the action $T \circlearrowleft \tilde{\mathfrak{u}}/\mathfrak{u} \otimes \mathbb{C}$
- $\gamma \in \mathfrak{t}$ is $\mathfrak{R}$-admissible if γ is rational and $\mathrm{Vect}(\mathfrak{R} \cap \gamma^{\perp}) = \mathrm{Vect}(\mathfrak{R}) \cap \gamma^{\perp}$
- Schubert classes $\Theta^{\gamma}_{w} \in H^*(U/U^{\gamma}, \mathbb{Z})$ associated to $w \in W/W^{\gamma}$
- Schubert classes $\Theta^{\gamma}_{\tilde{w}} \in H^*(\widetilde{U}/\widetilde{U}^{\gamma}, \mathbb{Z})$ associated to $\tilde{w} \in \widetilde{W}/\widetilde{W}^{\gamma}$
- Morphism $\iota^* : H^*(\widetilde{U}/\widetilde{U}^{\gamma}, \mathbb{Z}) \to H^*(U/U^{\gamma}, \mathbb{Z})$ associated to $\iota : U/U^{\gamma} \hookrightarrow \widetilde{U}/\widetilde{U}^{\gamma}$

Theorem

$(\tilde{\xi}, \xi) \in \mathrm{Horn}(\widetilde{U}, U)$ if and only if the inequality $\boxed{(\tilde{\xi}, \tilde{w}\gamma) \geq (\xi, w_o w\gamma)}$ holds for any $(\gamma, w, \tilde{w}) \in \mathfrak{t} \times W/W^{\gamma} \times \widetilde{W}/\widetilde{W}^{\gamma}$ satisfying

- γ is antidominant and $\mathfrak{R}$-admissible,
- Cohomological condition: $\Theta^{\gamma}_{w} \cdot \iota^* \left(\Theta^{\gamma}_{\tilde{w}}\right) = [pt]$ in $H^*(U/U^{\gamma}, \mathbb{Z})$,
- Numerical condition: $N(\gamma, w, \tilde{w}) = 0$.

Different versions of the theorem due to: Berenstein-Sjamaar (2000), Kapovich-Leeb-Millson (2005), Belkale-Kumar (2006), Ressayre (2010).

Determination of the inequalities of $\mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K)$

- Maximal abelian subspaces $\mathfrak{a} \subset \mathfrak{p}$ and $\tilde{\mathfrak{a}} \subset \tilde{\mathfrak{p}}$, such that $\mathfrak{a} \subset \tilde{\mathfrak{a}}$.
- Restricted Weyl group : $W_{\mathfrak{a}} = N_W(\mathfrak{a})/Z_W(\mathfrak{a})$ and $W_{\tilde{\mathfrak{a}}} = N_{\widetilde{W}}(\tilde{\mathfrak{a}})/Z_{\widetilde{W}}(\tilde{\mathfrak{a}})$.
- Restricted root system $\Sigma \subset \mathfrak{a}^*$: set of roots relatively to the action $\mathfrak{a} \circlearrowleft \tilde{\mathfrak{p}}/\mathfrak{p}$
- $\gamma \in \mathfrak{a}$ is Σ-admissible if γ is rational and $\mathrm{Vect}(\Sigma \cap \gamma^\perp) = \mathrm{Vect}(\Sigma) \cap \gamma^\perp$
- Schubert classes Θ_w^γ parameterized by $(W/W^\gamma)^\sigma \simeq W_{\mathfrak{a}}/W_{\mathfrak{a}}^\gamma$
- Schubert classes $\Theta_{\tilde{w}}^\gamma$ parameterized by $(\widetilde{W}/\widetilde{W}^\gamma)^\sigma \simeq \widetilde{W}_{\tilde{\mathfrak{a}}}/\widetilde{W}_{\tilde{\mathfrak{a}}}^\gamma$

Theorem (PEP, 2021)

$(\tilde{x}, x) \in \mathrm{Horn}_{\mathfrak{p}}(\widetilde{K}, K)$ if and only if the inequality $\boxed{(\tilde{x}, \tilde{w}\gamma) \geq (x, w_o w\gamma)}$ holds for any $(\gamma, w, \tilde{w}) \in \mathfrak{a} \times W_{\mathfrak{a}}/W_{\mathfrak{a}}^\gamma \times \widetilde{W}_{\tilde{\mathfrak{a}}}/\widetilde{W}_{\tilde{\mathfrak{a}}}^\gamma$ satisfying

- γ is antidominant and Σ-admissible,
- Cohomological condition: $\Theta_w^\gamma \cdot \iota^*\left(\Theta_{\tilde{w}}^\gamma\right) = [pt]$ in $H^*(U/U^\gamma, \mathbb{Z})$,
- Numerical condition: $N(\gamma, w, \tilde{w}) = 0$.

In 2008, Kapovich-Leeb-Millson obtained a **weaker description** of $\mathrm{Horn}_{\mathfrak{p}}(K \times K, K)$:

- Their "Cohomological condition" holds in $H^*(K/K^\gamma, \mathbb{Z}_2)$.
- They don't have a "Numerical condition".

Determination of the facets of a Kirwan polytope

First case: suppose that $0 \notin \Delta_{\mathfrak{u}}(M)$.

- Let $\gamma =$ orthogonal projection of 0 on $\Delta_{\mathfrak{u}}(M)$.
- Let $C \subset M^{\gamma}$ be the connected component containing $\Phi_{\mathfrak{u}}^{-1}(\gamma)$.
- **Białynicki-Birula's submanifold** : $C^{-} = \{m \in M, \lim_{\infty} e^{-it\gamma} m \in C\}$.

Kirwan-Ness stratification 1

- A Zariski open subset of M is diffeomorphic to a Zariski open subset of $U_{\mathbb{C}} \times_{P_{\gamma}} C^{-}$.
- $\boxed{(\xi, \gamma) \geq (\Phi_{\mathfrak{u}}(C), \gamma)}$ for all $\xi \in \Delta_{\mathfrak{u}}(M)$.

Second case: suppose that $a \in \mathfrak{t}^{*}_{+}$ is a regular element not contained in $\Delta_{\mathfrak{u}}(M)$.

- Let $\gamma_a = a' - a$ where $a' =$ orthogonal projection of a on $\Delta_{\mathfrak{u}}(M)$.
- Let $C_a \subset M^{\gamma_a}$ be the connected component containing $\Phi_{\mathfrak{u}}^{-1}(a')$.
- **Białynicki-Birula's submanifold** : $C_a^{-} = \{m \in M, \lim_{\infty} e^{-it\gamma_a} m \in C_a\}$.

Kirwan-Ness stratification 2

- A Zariski open subset of M is diffeomorphic to a Zariski open subset of $B \times_{B \cap P_{\gamma_a}} C_a^{-}$.
- $\boxed{(\xi, \gamma_a) \geq (\Phi_{\mathfrak{u}}(C_a), \gamma_a)}$ for all $\xi \in \Delta_{\mathfrak{u}}(M)$.

Ressayre's pairs

$\mathfrak{u}$**-dimension:** If $D \subset M$, we define $\dim_{\mathfrak{u}}(D) = \inf\{\dim(\mathfrak{u}_x), x \in D\}$.

Ressayre's pairs

(C, γ) is a Ressayre's pair if

- γ is rational,
- $C \subset M^{\gamma}$ and $\dim_{\mathfrak{u}}(C) - \dim_{\mathfrak{u}}(M) \in \{0, 1\}$,
- A Zariski open subset of M is diffeomorphic to a Zariski open subset of $B \times_{B \cap P_{\gamma}} C^{-}$.

Rmq: the notion of Ressayre's pair has nothing to do with the symplectic structure.

Theorem: Ressayre, 2010 (algebraic varieties) and PEP, 2020 (Kähler manifolds)

An element $\xi \in \mathfrak{t}^*_+$ belongs to $\Delta_{\mathfrak{u}}(M)$ if and only if $\boxed{(\xi, \gamma) \geq (\Phi_{\mathfrak{u}}(C), \gamma)}$ for any Ressayre's pair (C, γ).

This technique can be adapted to describe **real moment polytopes** by considering Ressayre's pair (C, γ) compatible with the involutions:

- $\sigma(\gamma) = -\gamma$,
- $\tau(C) = C$ and $C \cap Z \neq \emptyset$,
- $\dim_{\mathfrak{p}}(C \cap Z) - \dim_{\mathfrak{p}}(Z) \in \{0, 1\}$.

The End

Thank you for your attention !