The Gysin kernel and Bloch's conjecture

Rina Paucar Rojas and Claudia Schoemann

IMCA Lima and Laboratoire GAATI Polynésie française

Contact Information:

Instituto de Matemáticas y Ciencias Afines
Universidad Nacional de Ingeneria
Lima, Perú

Phone: +51 964556687

Email: `rina.paucar@imca.edu.pe`

Laboratoire GAATI
Université de la Polynésie française
BP 6570 - 98702 Faa'a, Polynésie française

Phone: +689 40 803 832
Mobile: +689 87238787 or +49 1781664421
Email: `claudia.schoemann@upf.pf`

Abstract

Let S be a smooth projective surface over $\mathbb{C}$, and let C be a smooth hyperplane section of S. Let $\mathrm{CH}_0(S)_{\deg=0}$ and $\mathrm{CH}_0(C)_{\deg=0}$ be the Chow groups of zero cycles of degree 0 of S and C, respectively. We present a result on the kernel of the Gysin homomorphism from $\mathrm{CH}_0(C)_{\deg=0}$ to $\mathrm{CH}_0(S)_{\deg=0}$ induced by the closed embedding of C into S and study the connection with Bloch's conjecture and constant cycles curves. We give some facts in positive characteristic.

Content

Background

Let S be a smooth projective connected surface over $\mathbb{C}$ and Σ the linear system of a very ample divisor D on S. Let $d := \dim(\Sigma)$ be the dimension of Σ and

$$\phi_\Sigma : S \hookrightarrow \mathbb{P}^d$$

the closed embedding of S into $\mathbb{P}^d$, induced by Σ.

For any closed point $t \in \Sigma \cong \mathbb{P}^{d^*}$, let C_t be the corresponding hyperplane section on S, and let

$$r_t : C_t \hookrightarrow S$$

be the closed embedding of the curve C_t into S.

Let Δ be the discriminant locus of Σ, that is,

$$\Delta := \{t \in \Sigma : C_t \text{ is singular}\}.$$

Then

$$U := \Sigma \setminus \Delta = \{t \in \Sigma : C_t \text{ is smooth}\}.$$

Let

$$r_{t^*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z})$$

be the Gysin homomorphism on cohomology groups induced by r_t, whose kernel $H^1(C_t, \mathbb{Z})_{\mathrm{van}}$ is called the *vanishing cohomology of C_t* (see [3], 3.2.3).

Let $J_t = J(C_t)$ be the Jacobian of the curve C_t and let B_t be the abelian subvariety of the abelian variety J_t corresponding to the Hodge substructure $H^1(C_t, \mathbb{Z})_{\mathrm{van}}$ of $H^1(C_t, \mathbb{Z})$.

Let $\mathrm{CH}_0(S)_{\deg=0}$ be the Chow group of zero cycles of degree 0 on S, and for any closed point $t \in \Sigma$, let $\mathrm{CH}_0(C_t)_{\deg=0}$ be the Chow group of zero cycles of degree 0 on C_t.

For any closed point $t \in \Sigma$, let

$$r_{t^*} : \mathrm{CH}_0(C_t)_{\deg=0} \to \mathrm{CH}_0(S)_{\deg=0}$$

be the Gysin pushforward homomorphism on the Chow groups of degree 0 zero cycles of C_t and S, respectively, induced by r_t, whose kernel

$$G_t = \mathrm{Ker}(r_{t^*})$$

is called the *Gysin kernel* associated with the hyperplane section C_t.

The Gysin kernel

Let $U = \Sigma \setminus \Delta$. For the formulation of the following theorem see also [3], pp. 304-5.

Theorem 1. *a) For each $t \in U$ there is an abelian variety $A_t \subset B_t$ such that*

$$G_t = \mathrm{Ker}(r_{t^*}) = \bigcup_{countable} \text{translates of } A_t$$

b) For a very general $t \in U$ (i.e. for every t in a c-open subset U_0 of U) either

1. $A_t = B_t$, and then $G_t = \bigcup_{countable}$ translates of B_t, or

2. $A_t = 0$, and then G_t is countable.

c) If $alb_S : \mathrm{CH}_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$ is not an isomorphism, for a very general t in U, then G_t is countable.

Connection with Bloch's Conjecture

Bloch's conjecture states (see [1])

Conjecture 1. *Let S be a smooth projective surface over $\mathbb{C}$. If $p_g(S) = 0$, then*

$$alb_S : \mathrm{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is an isomorphism.

The contrapositive form of item c) in Theorem 1 is

Corollary 1. *If G_t is uncountable, for a very general t in U, then $alb : \mathrm{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$ is an isomorphism.*

So if a surface has $p_g = 0$, in order that Bloch's conjecture holds for this surface, i.e. that alb_S is an isomorphism, it is enough to prove that for a very general t in U the Gysin kernel G_t is uncountable.

Constant cycles curves

Definition 1. *A curve $C \subset S$ is a* pointwise constant cycle curve *if and only if*

$$r_{C^*} : \mathrm{Pic}^0(\tilde{C}) = \mathrm{CH}_0(\tilde{C})_{\deg=0} \to \mathrm{CH}_0(S)$$

is the zero map. Here, $r_C : \tilde{C} \to S$ is the composition of the normalization $\tilde{C} \to C$ with the closed embedding $C \hookrightarrow S$.

Over $\mathbb{C}$, i.e. over an algebraically closed uncountable field, constant cycle curves are pointwise constant cycle curves and vice versa (see [5], Proposition 3.7).

For surfaces of general type with $p_g = 0$ and constant cycle curves the contrapositive form of item c) in Theorem 1 is

Corollary 2. *Let S be a surface of general type with $p_g(S) = 0$. If, for a very general $t \in U$, we have $r_{t*} = 0$, i.e. the curve C_t is a constant cycle curve, then S satisfies Bloch's conjecture, i.e. $\mathrm{CH}_0(S)_{\deg=0} = 0$.*

So for a surface of general type with $p_g = 0$, in order that Bloch's conjecture holds, i.e. $\mathrm{CH}_0(S)_{\deg=0} = 0$, it is enough to prove that for a very general t in U the curve C_t is a constant cycle curve.

If furthermore the irregularity $q(S) = 0$ by [5], Proposition 4.1 we have

Proposition 1. *Let S be a smooth projective surface with $p_g(S) = q(S) = 0$ over an algebraically closed field k of characteristic 0. Then S satisfies Bloch's conjecture, i.e. $\mathrm{CH}_0(S)_0 = 0$, if and only if every curve in S is a constant cycle curve.*

Facts in positive characteristic

Let S be a $K3$ surface and $q = p^r$, $p \neq 2$. Over $\overline{\mathbb{F}}_p$ every curve is a pointwise constant cycle curve, i.e. $\mathrm{CH}_0(S)_0 = 0$. Conjecturally (Bloch-Beilinson) for S over $\overline{\mathbb{F}_p(t)}$ equally $\mathrm{CH}_0(S)_0 = 0$.

By [5], Proposition 9.2 we have

Proposition 2. *Let S be a $K3$ surface over $\overline{\mathbb{F}}_p$. Then $S \times_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}_p(t)}$ satisfies the Bloch-Beilinson conjecture , i.e. $\mathrm{CH}_0(S \times_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}_p(t)})_0 = 0$, if and only if every curve $C \subset S$ is a constant cycle curve.*

Hence S satisfies $\mathrm{CH}_0(S \times_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}_p(t)})_0 = 0$, i.e. every curve is a pointwise constant cycle curve if and only if every curve is a constant cycle curve, i.e. if and only if the notions of pointwise constant cycle curve and constant cycle curve are equivalent.

By [5], Proposition 9.4 we have

Proposition 3. *Let S be a $K3$ surface over a finite field $\mathbb{F}_q$ for which Kimura-O'Sullivan finite-dimensionality holds, e.g. S is a Kummer surface. Then every curve in $S_{\overline{\mathbb{F}}_q}$ is a constant cycle curve.*

Hence pointwise constant cycle curves are constant cycle curves and vice versa and every curve in S is such a curve. This also holds for unirational and all supersingular $K3$ surfaces.

Further by ([5], Proposition 9.6) we have

Proposition 4. *Every closed point $x \in S$ in a $K3$ surface over $\overline{\mathbb{F}}_p$ is contained in a constant cycle curve $x \in C \subset S$.*

Hence, in addition to being contained in a pointwise constant cycle curve, every closed point $x \in S$ is also contained in a constant cycle curve.

In [2], Bogomolov and Tschinkel proved the existence of a rational curve through every point and explicitely geometrically constructed Jacobian Kummer surfaces.

Now let S be a $K3$ surface over $\mathbb{C}$. We have the

Corollary 3. *Let S be an algebraic $K3$ surface over $\mathbb{C}$ with regular involution ι acting without fixed points on S, so that the quotient $V = S/\iota$ is a smooth Enriques surface. Then the motive $M(S)$ is of abelian type, hence finite dimensional.*

Outlook

Apply item c) in the Theorem 1 in order to verify Bloch's conjecture for particular cases of surfaces with $p_g = q = 0$.

Consider the question by the help of constant cycle curves.

Extend the argument to fields of positive characteristic and other types of surfaces.

References

[1] Spencer Bloch *K_2 of Artinian Q-algebras, with application to algebraic cycles*, Comm. in Algebra 3.5 (1975) , pp. 405-428.

[2] F. Bogomolov, Y. Tschinkel *Rational curves and points on $K3$ surfaces*, Amer. J. Math. 127 (2005), 825-835.

[3] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II*, Volume 77. Cambridge University Press, 2003.

[4] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, Volume 76. Cambridge University Press, 2002.

[5] Daniel Huybrechts. *Curves and cycles on K3 surfaces*, Algebraic Geometry 1 (2014), no. 1, 69-106.

[6] A A Roitman. *Rational equivalence of zero-dimensional cycles*, Mat. Sb. (N.S.) 89(131) (1972) 569-585, 671.