Courant, Bézout, and topological persistence

Leonid Polterovich, Tel Aviv

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with Lev Buhovsky, Jordan Payette, Iosif Polterovich, Egor Shelukhin and Vukašin Stojisavljević

What is this lecture about?

Persistence modules and barcodes: convenient algebraic/combinatorial tool for book-keeping information on oscillation and topology of (sub)level sets of functions on manifolds.

Quantitative flavor - stability: Close functions have close barcodes

Main idea: Apply persistence to oscillation and nodal (zero) sets of (linear combinations of) eigenfunctions of Laplace-Beltrami operator on manifolds - get coarse analogues of Courant and Bézout theorems - classical themes in spectral geometry. BP^3S^2, 2022

Another application: Transcendental Bézout problem in several complex variables. BP^2S^2, 2023

Interplay: Topology and Analysis/Geometry of smooth functions.

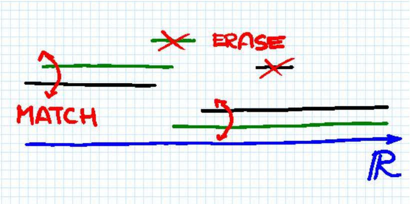
Edelsbrunner, Harer, Carlsson,... Last decade in the context of topological data analysis.

Barcode $\mathcal{B} = \{I_j, m_j\}$-finite collection of intervals I_j with multiplicities m_j, $I_j = (a_j, b_j]$, $a_j < b_j \leq +\infty$.

Bottleneck distance between barcodes: $\mathcal{B}, \mathcal{C}$ are δ-matched, $\delta > 0$ if after erasing some intervals in $\mathcal{B}$ and $\mathcal{C}$ of length $< 2\delta$ we can match the rest in 1-to-1 manner with error at most δ at each end-point.

$$d_{bot}(\mathcal{B}, \mathcal{C}) = \inf \delta \ .$$

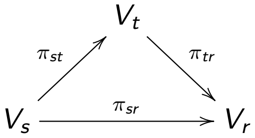
Figure: Matching



$\mathcal{F}$ – a field.

Persistence module: a pair (V, π), where V_t, $t \in \mathbb{R}$ are $\mathcal{F}$-vector spaces, $\dim V_t < \infty$, $V_s = 0$ for all $s \ll 0$.

$\pi_{st} : V_s \to V_t$, $s < t$ linear maps: $\forall s < t < r$

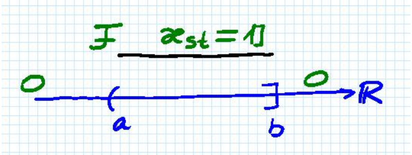


Regularity: For all but finite number of jump points $t \in \mathbb{R}$, there exists a neighborhood U of t such that π_{sr} is an isomorphism for all $s, r \in U$. Extra assumption ("semicontinuity") at jump points.

Structure theorem

Interval module $(\mathcal{F}(a,b],\kappa)$, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup +\infty$:
$\mathcal{F}(a,b]_t = \mathcal{F}$ for $t \in (a,b]$ and $\mathcal{F}(a,b]_t = 0$ otherwise;
$\kappa_{st} = \mathbb{1}$ for $s,t \in (a,b]$ and $\kappa_{st} = 0$ otherwise.

Figure: Interval module



Structure theorem: For every persistence module (V,π) there exists unique barcode $\mathcal{B}(V) = \{(I_j, m_j)\}$ such that $V = \oplus \mathcal{F}(I_j)^{m_j}$.

Persistence in Morse theory

M-compact manifold, $f : M \to \mathbb{R}$-Morse function.

Persistence module $V_t(f) := H_*(\{f < t\})$
H_*-homology with coefficients in a field.
Persistence morphisms are induced by the inclusions of sublevels
$\{f < s\} \hookrightarrow \{f < t\}, \ \ s < t$.
$\mathcal{B}(f)$ - barcode of $V(f)$

Stability Theorem (Cohen-Steiner,Edelsbrunner,Harer, 2007)
$||f|| := \max |f|$-uniform norm
$(C^\infty(X), ||\cdot||) \to (\text{Barcodes}, d_{bot}), \ \ f \mapsto \mathcal{B}(f)$ is 1-Lipshitz.

"Long" bars: $N_\delta(f)$ - number of bars in $\mathcal{B}(f)$ of length $> \delta$.
Cohen-Steiner-Edelsbrunner-Mileyko (2010)

Example: 2-sphere

Persistence module $V_t(f) := H_*(\{f < t\}, \mathcal{F})$, H_*-homology.

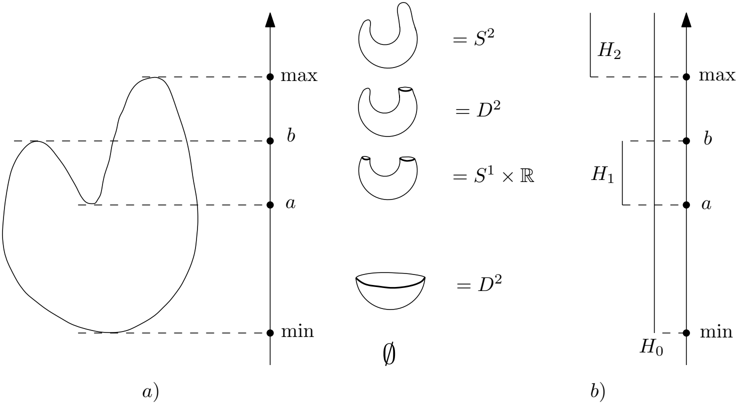


Figure: The height function on the (topological) sphere and the corresponding barcode.

Long bars vs. Sobolev norms

Sobolev norm: $||f||_{k,p}$, $k, p \geq 1$ - Sobolev norm of f, L_p-norm of the k-th derivative, $k > n/p$

Theorem[BP^3S^2], 2022
$N_\delta(f) \leq C_1 \delta^{-n/k} ||f||_{k,p}^{n/k} + C_2, \ \forall \delta > 0$

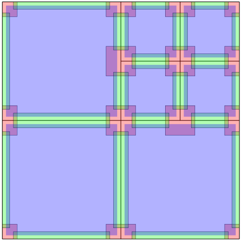
Earlier results:
$p = \infty$ (uniform derivative bounds) Kronrod, Vitushkin (50-ies), Yomdin (1985)

$k = 1, p = \infty$ Cohen-Steiner-Edelsbrunner-Mileyko (uses N_δ)

$n = 2, k = 2, p = 2$ P.-M.Sodin (2007) (geometric trick) + I.P.- P.-Stojisavljevic (2017) (uses N_δ)

Generalization: oscillation of sections of vector bundles (cf. a problem of V.Arnold, 2003)

Step 1. Approximate by polynomials on small cubes, use Milnor's bound (1964) $\sharp(\text{critical points}) \leq \text{deg}^{\text{dim}}$, and Morse theory. Cf. Yomdin, innovation: multiscale/stopping time.



Step 2. Glue the bounds by Mayer-Vietoris (à la Yomdin), albeit for persistence modules (non-existent in 1985).

Subadditivity Theorem. [BP^3S^2], 2022 Let $U \to V \to W$ be an exact sequence of persistence modules. Then $N_{2\delta}(V) \leq N_\delta(U) + N_\delta(W)$.

Uses algebraic ideas (extension of persistence modules) inspired by Skraba-Turner (2020)

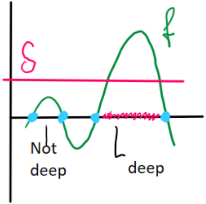
Coarse nodal count

M - compact n-dim Riemannian mfd, $f : M \to \mathbb{R}$ - smooth fn

$Z = \{f = 0\}$ - nodal set; components of $M \setminus Z$ - nodal domains

Coarse nodal count:
$m_r(f, \delta) = \dim \operatorname{Im} \left(H_r(\{|f| > \delta\}) \to H_r(M \setminus Z) \right), \ \delta > 0.$

Example: $m_0(f, \delta)$- number of nodal domains U_j with $\max_{U_j} |f| > \delta$ - deep nodal domains (P.-M.Sodin, 2007).



Theorem[BP^3S^2], 2022 $m_r(f, \delta) \leq C_1 \delta^{-n/k} ||f||_{k,p}^{n/k} + C_2, \ \forall \delta > 0,$
where C_1, C_2 do not depend on f, δ.

Laplace-Beltrami operator: M^n- compact Riemannian manifold
$\Delta f = -\mathrm{div}(\mathrm{grad}\, f)$, $f \in C^\infty(M)$
If $\partial M \neq \emptyset$, assume Dirichlet boundary conditions $f|_{\partial M} = 0$.
Discrete positive spectrum: $\Delta f_\lambda = \lambda f_\lambda$.

Notation: $m_0(f) = m_0(f, 0)$ - number of nodal domains.

Figure: credit: E.J.Heller, S.Zelditch

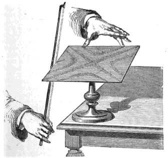


Figure: Chladni figures

Courant thm (1923)+ Weyl law (1911): $m_0(f_\lambda) \leq C(\lambda + 1)^{n/2}$.

Coarse Courant for linear combinations

$\mathcal{F}_\lambda$ - span of eigenfunctions with eigenvalues $\leq \lambda$

Theorem [BP^3S^2], 2022 For any $\delta > 0$, $k > n/2$ and any $f \in \mathcal{F}_\lambda$ with $||f||_{L^2} = 1$,
$m_r(f, \delta) \leq \frac{C_1}{\delta^{n/k}}(\lambda + 1)^{\frac{n}{2}} + C_2$

Sharpness: Sharp in λ

Historical remarks on Courant for linear combinations:
Holds in dimension 1 (Sturm, 1836).
In higher dim known as Courant-Herrmann conjecture (1932) (flawed footnote in Courant-Hilbert book).

Counterexamples: Arnold, Viro (1970ies),
Buhovsky-Logunov-Sodin (2020)- infinitely many nodal domains.

Extends: to positive elliptic operators on vector bundles.

Application: By stability, yields constraints on barcodes of functions well approximated by $\mathcal{F}_\lambda$.

Bézout theorem

Bézout theorem, 1779 d generic hypersurfaces in $\mathbb{C}P^d$ have a number of intersection points given by the product of their degrees.
Can we extend this statement in the following directions?

(i) Intersection of zero sets of Laplace-Beltrami eigenfunctions on Riemannian manifolds (inspired by Arnold, 2003 combined with Donnelly-Fefferman, 1998);

(ii) Intersection of affine submanifolds - the transcendental Bézout problem, Griffiths, Cornalba-Shiffman, 1970-es.

NO! Evidence (à la Buhovsky-M.Sodin-Logunov) for (i), famous Cornalba-Shiffman counterexample for (ii).
But YES... if one cuts small oscillations, i.e. removes intersections which do not persist after a mild perturbation.

Tool: persistence modules and barcodes

Eigenfunctions vs. polynomials

Laplace-Beltrami operator: M^n- closed Riemannian manifold
$\mathcal{F}_\lambda$ - span of eigenfunctions of Δ with eigenvalues $\leq \lambda$

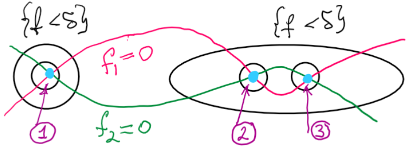
Donnelly-Fefferman philosophy (1988) : $f \in \mathcal{F}_\lambda$, $\lambda \gg 1$,
"similar" to polynomial of deg $= \sqrt{\lambda}$
$Z_f = \{f = 0\}$ - nodal set

Example: On sphere S^n with round metric,
let $f_1, \dots, f_n$ - be generic eigenfunctions with eigenvalue
$\lambda = d(d + n - 1)$
f_i - homogeneous polynomial of degree d on $\mathbb{R}^{n+1}$
Then $|\bigcap_i Z_{f_i}| \leq \text{const}(n) \cdot \lambda^{n/2}$, agrees with Bézout thm.

Random setting: Expectation $\approx \lambda^{n/2}$ (Gichev, 2009). Similar bound on certain homogeneous Riemannian manifolds (Akhiezer-Kazarnovskii, 2017).

Persistent intersection count: $Z_f := \{f = 0\}$
$z_0(f, \delta) = \dim \operatorname{Im}(H_0(Z_f) \to H_0(\{|f| < \delta\}))$

Let $f_1, \dots, f_n \in \mathcal{F}_\lambda$, $||f_j||_{L^2} = 1$, $j = 1, \dots n$,
$f = (f_1^2 + \cdots f_n^2)^{1/2}$, $Z_f = \cap_i Z_{f_i}$.



$n = 2$, $Z_{f_1} \cap Z_{f_2} = \{f = 0\} = \{1, 2, 3\}$.
$z_0 = 2$ as points $2, 3$ land in the same component of $\{f < \delta\}$.

Theorem $[BP^3S^2]$, 2022 Let $k > n/2$ be an integer, $\delta > 0$.

$$z_0(f, \delta) \le \frac{C_1}{\delta^{n/k}} (\lambda + 1)^{\frac{n}{2}} + C_2,$$

where C_1, C_2 depend on n, k and metric.

Transcendental Bézout problem

With Lev Buhovsky, Iosif Polterovich, Egor Shelukhin and Vukašin Stojisavljević

Transcendental Bézout problem: count of zeros of entire maps $\mathbb{C}^n \to \mathbb{C}^n$.

Starting point: Serre's G.A.G.A.: complex *projective* analytic geometry reduces to algebraic geometry.

Example - Chow's thm.: Every closed complex submanifold of $\mathbb{C}P^n$ is algebraic.

Fails in affine setting:
$f : \mathbb{C} \to \mathbb{C}$, $f(z) = e^z - 1 = (e^x \cos y - 1) + ie^x \sin y,\ z = x + iy$.
$Z_f = \{2\pi ki,\ k \in \mathbb{Z}\}$.
Not biholomorphically equivalent to any algebraic (and hence finite) proper subset of $\mathbb{C}$.

Resolution: replace the notion of the degree of a polynomial.

Maximum modulus

B_r-closed ball of radius r, $\mu(f,r) = \max_{z \in B_r} |f(z)|$

Degree-like features:

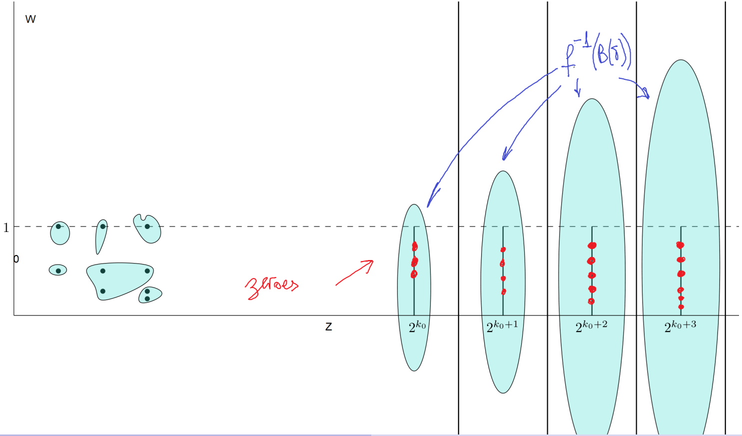
- If $\frac{\log \mu(f,r)}{\log r} < k+1$, $\forall r \gg 1$, then f is a polynomial of deg $\leq k$. (generalization of Liouville's theorem).
- Let $\zeta(f,r)$ be the number of zeros of an entire function $f : \mathbb{C} \to \mathbb{C}$ inside a ball B_r, $f(0) \neq 0$. Then, for $a > 1$, $\zeta(f,r) \leq C \log \mu(f, ar)$ $\forall r > 0$, where C- positive constant depending on a and $f(0)$.

In Example above ζ and $\log \mu$ grow linearly in r.

Cornalba-Shiffman Example (1972)

$n \geq 2$. There exists entire map f with $\log \mu(f, r) \leq Cr^{\epsilon}$ for every $\epsilon > 0$ with $\zeta(f, r)$ growing arbitrarily fast.

Griffiths: *"This is the first instance known to this author when the analogue of a general result in algebraic geometry fails to hold in analytic geometry."*



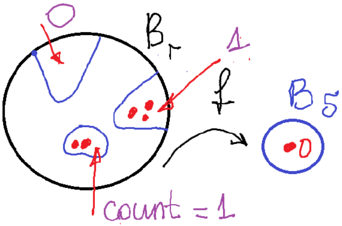
Coarse zero count

$f : \mathbb{C}^n \to \mathbb{C}^n$-analytic, $\delta, r > 0$

Coarse zero count:

$\zeta(f, r, \delta) = \dim Im(H_0(\{f = 0\} \cap B_r) \to H_0(\{|f| < \delta\} \cap B_r)$

This is the number of connected components of the set $f^{-1}(B_\delta) \cap B_r$ which contain zeros of f.



$\zeta(f,r,\delta) = \dim Im(H_0(\{f=0\}\cap B_r) \to H_0(\{|f|<\delta\}\cap B_r)$

Theorem. (BP^2S^2, 2023) For $a>1$, $\delta \in (0, \frac{\mu(f,ar)}{e})$
$\zeta(f,r,\delta) \le C\left(\log\left(\frac{\mu(f,ar)}{\delta}\right)\right)^{2n-1}$,
where C depends on a and n, but not on r or δ.

Example: $f:\mathbb{C}^n \to C^n$, $f(z_1,\dots,z_n) = (e^{z_1}-1,\dots,e^{z_n}-1)$.
Then $\log\mu(f,r) \approx r$, $\zeta(f,r) \approx r^n$, $r\to\infty$.

CS Example, $n=2$**:** $\log\mu(f,r) \approx (\log r)^2$, $\zeta(f,r,\delta) \approx \log r$.
Our results state $\zeta(f,r,\delta) \lessapprox \left(\log\left(\frac{\mu(f,ar)}{\delta}\right)\right)^3$.
Thus our estimate on log-scale (for $\log\zeta$) is sharp.

Question: Is the power $2n-1$ at log in Theorem sharp?

THANK YOU!