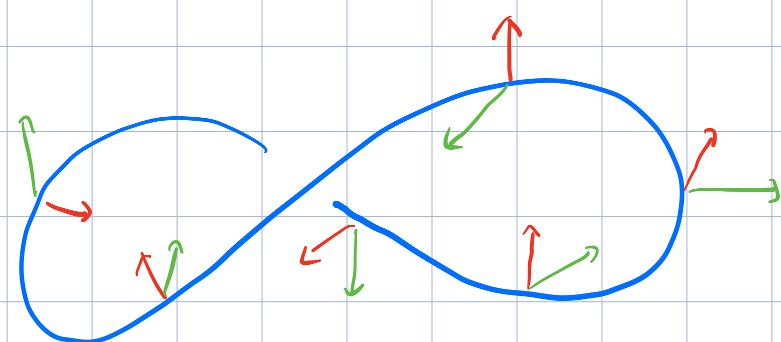
Non-linear proper Fredholm maps and the stable homotopy groups of spheres.

joint with Lauran Toussaint
& Alberto Abbondandolo



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Prospects in geometry and global analysis
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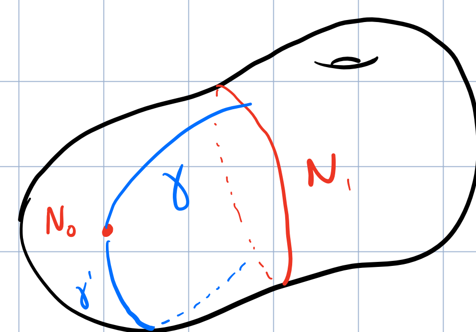
Rauischolzhausen

Motivation: geometric ODE's / PDE's

(M, g) Riemannian mfd. N_0, N_1 submanifolds.

$$P = \{ \gamma : I \longrightarrow M \mid \gamma(0) \in N_0 \quad \gamma(1) \in N_1 \}$$ path space



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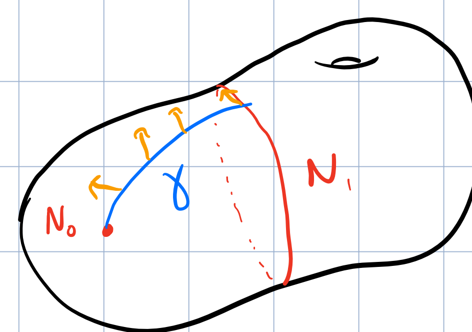
$$T_\gamma P = \{ X \in \Gamma(\gamma^* TM) \mid X(0) \in TN_0 \quad X(1) \in TN_1 \}$$ vector fields along γ



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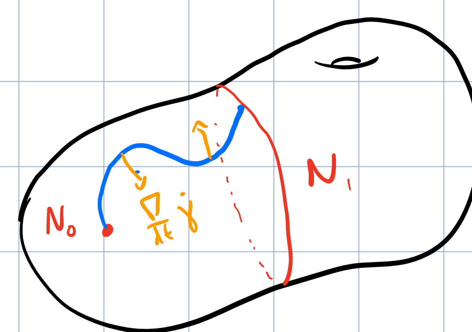

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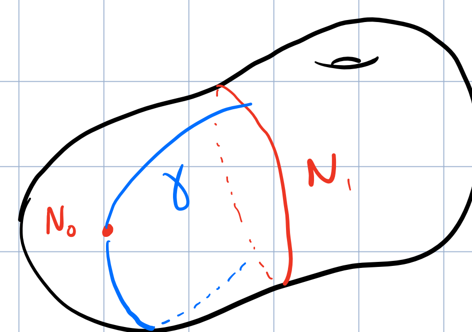

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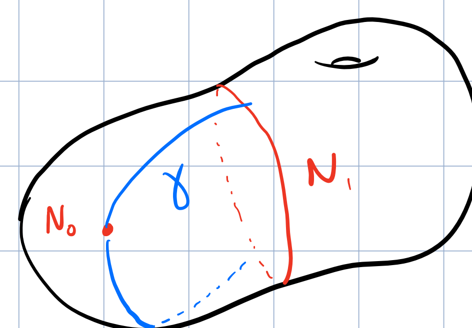

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Questions
- What behavior persists if we vary g?
- Always solutions?
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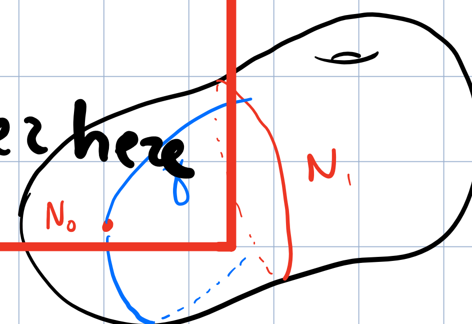
Warning!

Toy example

Morse theory is better here



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Classify non linear proper Fredholm maps

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in terms of finite dimensional invariants

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understand the solution sets $f^{-1}(0)$.

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- $f : P \longrightarrow TP$
- $\tilde{f} : P \longrightarrow \mathbb{H}$

Notation:

$$\mathbb{H} = \ell^2(\mathbb{N}, \mathbb{R}) = \left\{ (x_1, x_2, \ldots) \mid \sum_{i=1}^{\infty} x_i^2 < \infty \right\}.$$

canonical inclusions $\mathbb{R}^n \subset \mathbb{H}$

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$$\Phi(\mathbb{H}) = \left\{ L \in L(\mathbb{H}, \mathbb{H}) \;\middle|\; \begin{array}{l} \dim \ker L < \infty \\ \dim \operatorname{coker} L < \infty \end{array} \right\}$$

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index L = dim ker − dim coker.

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linear Fredholm operators. "$\mathbb{H}/\operatorname{im} L$"

Positive index = "high dimensional to low dimensional"

zero index = "same dimensions"

negative index = "low dimensional to high dimensional"

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$$\mathcal{F}(\mathbb{H}) = \left\{ f : \mathbb{H} \longrightarrow \mathbb{H} \;\middle|\; df : \mathbb{H} \longrightarrow \Phi(\mathbb{H}) \right\}$$

non-linear Fredholm mappings

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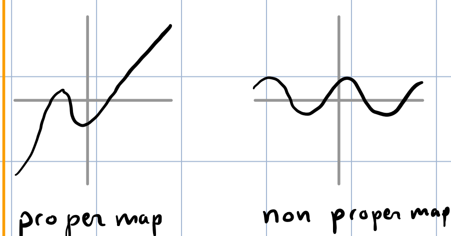
$$F^{prop}(\mathbb{H}) = \left\{ f : \mathbb{H} \longrightarrow \mathbb{H} \;\middle|\; \begin{array}{l} f \text{ proper} \\ df : \mathbb{H} \longrightarrow \Phi(\mathbb{H}) \end{array} \right\}$$

non-linear proper Fredholm mappings

f is proper if $f^{-1}(C)$ is compact for all C compact



" ∞ goes to ∞ ".

square brackets denote homotopy classes.

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$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$
$$Sg : \mathbb{R}^{n+k} \oplus H \longrightarrow \mathbb{R}^k \oplus H$$
$$(x, y) \longmapsto (g(x), y)$$

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$$g : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$$

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$$C^\infty(\mathbb{R}^{n+k}, \mathbb{R}^k) \longrightarrow \mathcal{F}_n(H)$$

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$$(x, y) \longmapsto (g(x), y)$$

$$C^\infty_{prop}(\mathbb{R}^{n+k}, \mathbb{R}^k) \longrightarrow F_n^{prop}(\mathbb{H})$$

$$g \longmapsto S_g$$

what is $[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}$?

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what is $[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}$?

A proper map extends to one point compactification.

A diagram

$$[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \xrightarrow{S} \mathcal{F}_n^{prop}[\mathbb{H}]$$

A diagram

$$[S^{n+k-1}, S^{k-1}] \xrightarrow[\text{radial extension}]{\cong} [\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop} \xrightarrow{S} \mathcal{F}_n^{prop}[\mathbb{H}]$$

A diagram

$$
\begin{tikzcd}
{[S^{n+k-1}, S^{k-1}]} \arrow[r, "\cong", "\text{radial extension}"'] \arrow[d, "\text{Suspension}"'] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}} \arrow[r, "S"] \arrow[d, "\times \mathbb{R}"] & \mathcal{F}_n^{prop}[\mathbb{H}] \\
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\end{tikzcd}
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\cdots & \cdots &
\end{tikzcd}
$$

radial extension

A diagram

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\cdots \arrow[d] & \cdots \arrow[d] & \\
\pi_n^S \arrow[r, "\cong"] & {[\mathbb{R}^{n+k}, \mathbb{R}^{k}]^{prop}_S} \arrow[uuur, "S"'] &
\end{tikzcd}$$

stable homotopy group of spheres

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\end{tikzcd}
$$

The Theorems:

Thm (Toussaint-R) $S : \Pi^S_n \longrightarrow \mathcal{F}^{prop}_n[\mathbb{H}]$

is surjective but <u>not</u> injective.

$$S(a) = S(b) \text{ iff } a = \pm b.$$

The Theorems:

Previous work e.g. Švarc, Geba, Bergez, Banez, Furuta, ... did not consider the fully nonlinear problem

Thm (Toussaint-R) $S : \pi_n^S \longrightarrow F_n^{prop}[H]$

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	FO_n^{prop}	$\# F^{prop}$
$k<0$	0	0
$k=0$	$\mathbb{Z}$	$\mathbb{N}$
$k=1$	$\mathbb{Z}/2$	2
$k=2$	$\mathbb{Z}/2$	2
$k=3$	$\mathbb{Z}/24$	13

Note

$$F^{prop}[H] = \bigcup_n F_n^{prop}[H] \cong \pi^S/\sim$$

does not have a natural group structure. ($FO^{prop}[H]$ does)

however the multiplicative structure remains.

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Thm (Toussaint-R): let $FO_n^{prop}[H]$ oriented maps
$S : \Pi_n^S \longrightarrow FO_n^{prop}[H]$ is a bijection.

Thm (Toussaint-R) $n \neq 0$
let $f \in F_n^{prop}(H)$. Then there exists a $k \in \mathbb{N}$ s.t.

$$f^k = \underbrace{f \circ \cdots \circ f}_{k \text{ times}}$$

is proper Fredholm homotopic to a non surjective proper Fredholm map.

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

Where does the identification come from?

$f_i : \mathbb{R}^2 \to \mathbb{R}^2$

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$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{degree } 1$$

$$f_{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \qquad \text{degree } -1.$$

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$$\begin{pmatrix}\cos\pi t & \sin\pi t\\ -\sin\pi t & \cos\pi t\end{pmatrix} \qquad \begin{array}{c} t=0 \\ \begin{pmatrix}1&0\\0&1\end{pmatrix}\end{array} \qquad \begin{array}{c} t=1 \\ \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\end{array}$$

$$\begin{pmatrix}-1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots\end{pmatrix} \sim \begin{pmatrix}-1 & & \\ & \begin{pmatrix}1 & \\ & 1\end{pmatrix} & & \\ & & \begin{pmatrix}1 & \\ & 1\end{pmatrix} & \\ & & & \ddots\end{pmatrix} \sim \begin{pmatrix}-1 & & \\ & \begin{pmatrix}1 & \\ & -1\end{pmatrix} & & \\ & & \begin{pmatrix}1 & \\ & -1\end{pmatrix} & \\ & & & \ddots\end{pmatrix} \sim \begin{pmatrix}\begin{pmatrix}-1 & \\ & -1\end{pmatrix} & & \\ & \begin{pmatrix}1 & \\ & -1\end{pmatrix} & & \\ & & \begin{pmatrix}1 & \\ & \cdot\end{pmatrix} & \\ & & & \ddots\end{pmatrix}$$

Where does the identification come from?

In finite dimensions degree 1 map is not proper homotopic to degree -1 map.

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \qquad \text{degree } 1$$

$$f_{-1}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-x\\y\end{pmatrix} \qquad \text{degree } -1.$$

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$$\begin{pmatrix}-1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&\ddots\end{pmatrix} \sim \begin{pmatrix}-1&&\\&\begin{pmatrix}1&\\&1\end{pmatrix}&\\&&\begin{pmatrix}1&\\&1\end{pmatrix}\\&&&\ddots\end{pmatrix} \sim \begin{pmatrix}-1&&\\&\begin{pmatrix}1&\\&-1\end{pmatrix}&\\&&\begin{pmatrix}1&\\&-1\end{pmatrix}\\&&&\ddots\end{pmatrix} \sim \begin{pmatrix}\begin{pmatrix}-1&\\&-1\end{pmatrix}&&\\&\begin{pmatrix}1&\\&-1\end{pmatrix}&\\&&\begin{pmatrix}1&\\&-1\end{pmatrix}\\&&&\ddots\end{pmatrix} \sim \begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&\ddots\end{pmatrix}$$

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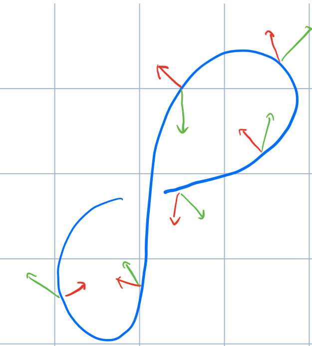
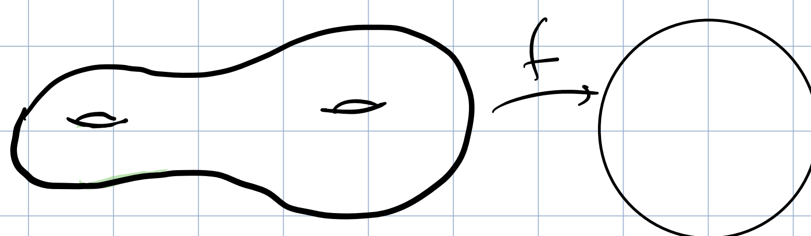
The inverse of $f : S^{n+k} \longrightarrow S^k$ is $T\circ f$ (T coordinate flip)

$$S(T\circ f) = S(T)\,S(f) = S(f)$$

One more ingredient: Framed cobordism.



$f: \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$. proper



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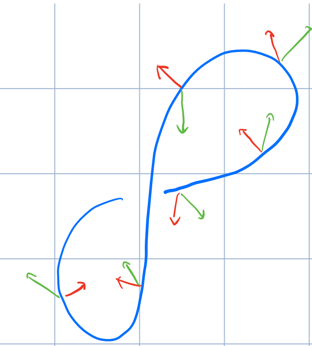
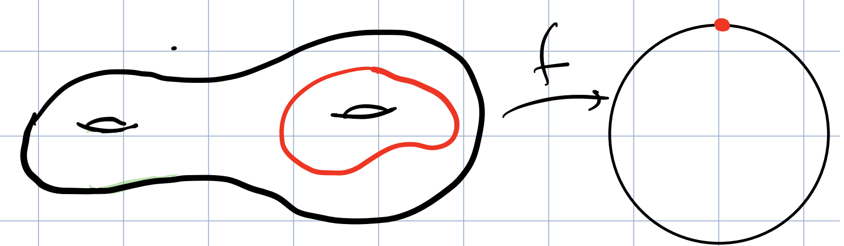


$$f : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k . \qquad \text{proper}$$

y regular value : $X = f^{-1}(y)$ closed n-dim mfd.



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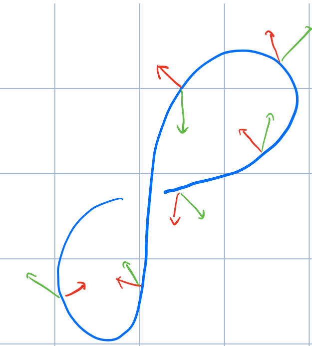
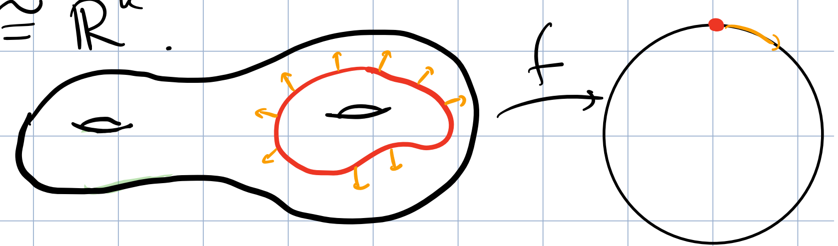


y regular value : $X = f^{-1}(y)$ closed n-dim mfd. Framed by df:

$$0 \to T_x X \to \mathbb{R}^{n+k} \xrightarrow{df} \mathbb{R}^k \to 0 \qquad \forall x \in X.$$

df induces iso $N_x X \cong \mathbb{R}^k$.

$\ker df|_X = TX$



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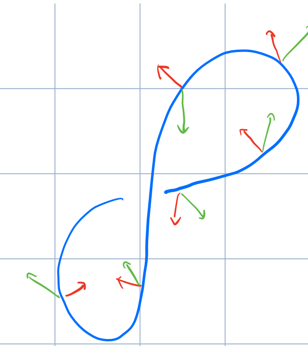

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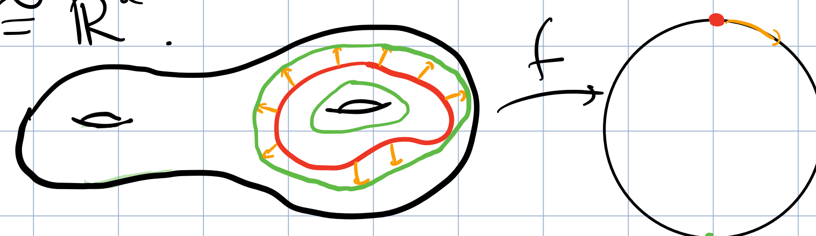

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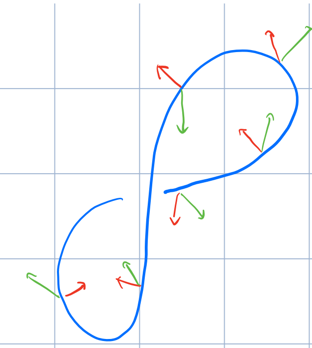

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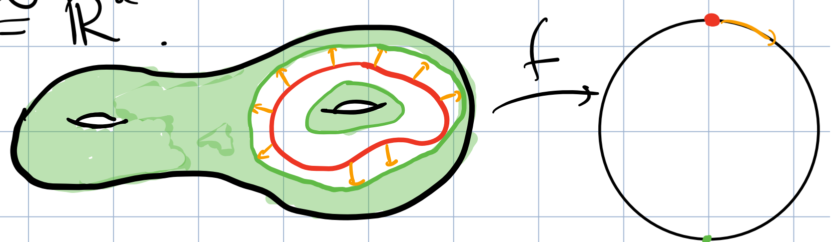

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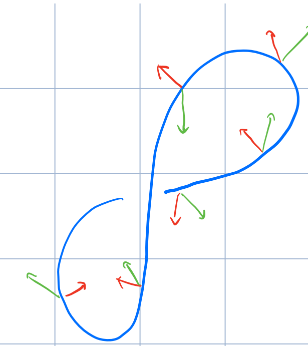

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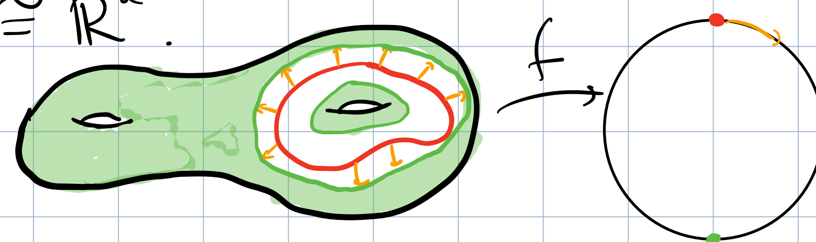
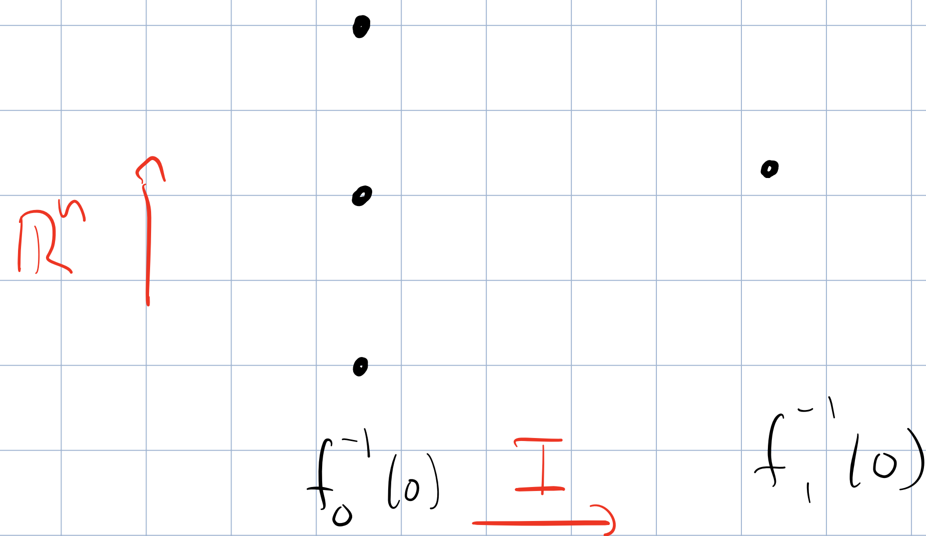


Thm (Csepai): for k sufficiently large

$$[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \cong \Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{matrix} X^n \subset \mathbb{R}^{n+k} \\ NX \xrightarrow{\cong} \mathbb{R}^k \end{matrix} \right\} \Big/ \text{cobordism}$$

Basic example $\mathbb{R}^n \to \mathbb{R}^n$



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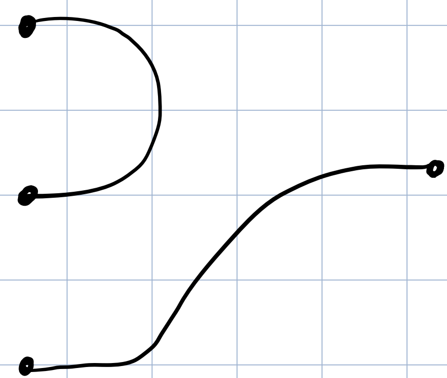
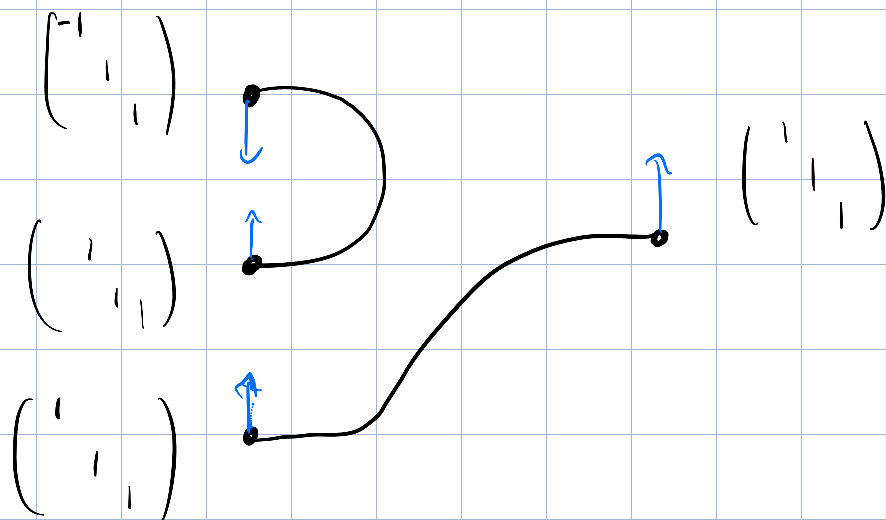


Basic example $\mathbb{R}^n \to \mathbb{R}^n$



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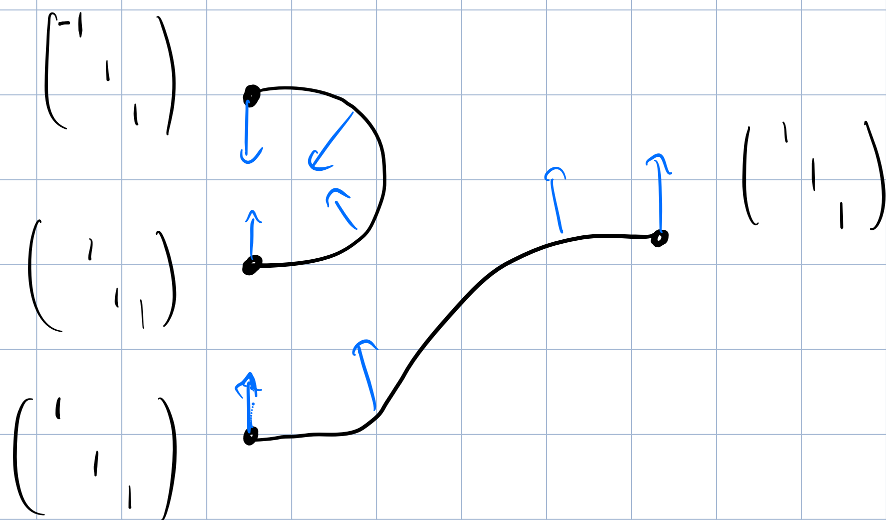

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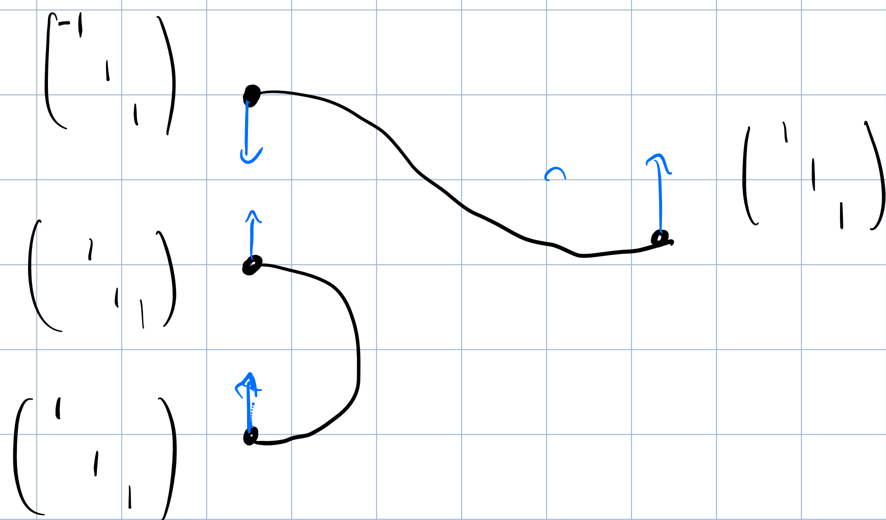

This does <u>not</u> work.

$$[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop} \xrightarrow{\cong} \Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \mathrm{Ker}\, A_x = TX \end{array} \right. \Big/ \text{cobordism}$$

$$\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \ker A_x = T_x X \end{array} \right. \Big/ \text{cobordism} \\
F_n^{prop}[H] &
\end{tikzcd}$$

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) \arrow[d, "S"] \\
F_n^{prop}[\mathbb{H}] \arrow[r, "\cong"', "\text{Abbondandolo-R.}"'] & \Omega_n^{fr}(\mathbb{H})
\end{tikzcd}
$$

$$\Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism}$$

$$\Omega_n^{fr}(\mathbb{H}) = \left\{ \begin{array}{l} X \subset \mathbb{H} \\ A : \mathbb{H} \longrightarrow \Phi_n(\mathbb{H}) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism.}$$

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) \arrow[d, "S"] \\
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\end{tikzcd}
$$

Abbondandolo-R.

$$\Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A: X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \mathrm{Ker}\, A_x = T_x X \end{array} \right. \Big/ \text{cobordism}$$

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- $\mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ contractible.
- $GL(\mathbb{R}^k)$ non-trivial topology

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) \arrow[d, "S"] \\
F_n^{prop}[\mathbb{H}] \arrow[r, "\cong"'] & \Omega_n^{fr}(\mathbb{H})
\end{tikzcd}
$$

Abbondandolo-R.

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$$

- $\mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k)$ contractible.
- $GL(\mathbb{R}^k)$ non-trivial topology
- $\Phi_n(\mathbb{H}) \simeq BO$ non trivial topology
- $GL(\mathbb{H})$ contractible
- need framing defined on $\mathbb{H}$.

$$
\begin{tikzcd}
{[\mathbb{R}^{n+k}, \mathbb{R}^k]^{prop}} \arrow[r, "\cong"] \arrow[d, "\int S"'] & \Omega_n^{fr}(\mathbb{R}^{n+k}) \arrow[d, "\int S"] \\
F_n^{prop}[\mathbb{H}] \arrow[r, "\cong"', "\text{Abbondandolo-R.}"'] & \Omega_n^{fr}(\mathbb{H})
\end{tikzcd}
$$

$$
\Omega_n^{fr}(\mathbb{R}^{n+k}) = \cancel{\left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : X \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \operatorname{Ker} A_x = T_x X \end{array} \right. \Big/ \text{cobordism}}
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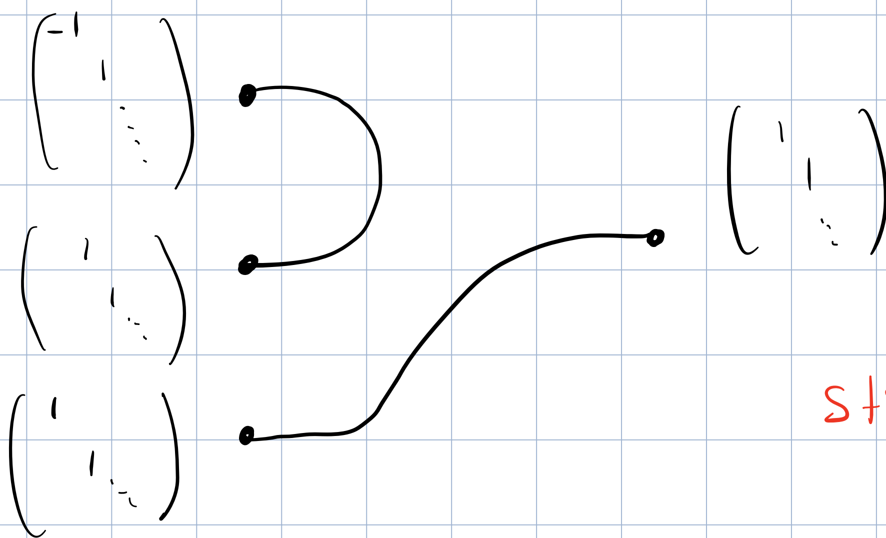
Equivalent definition

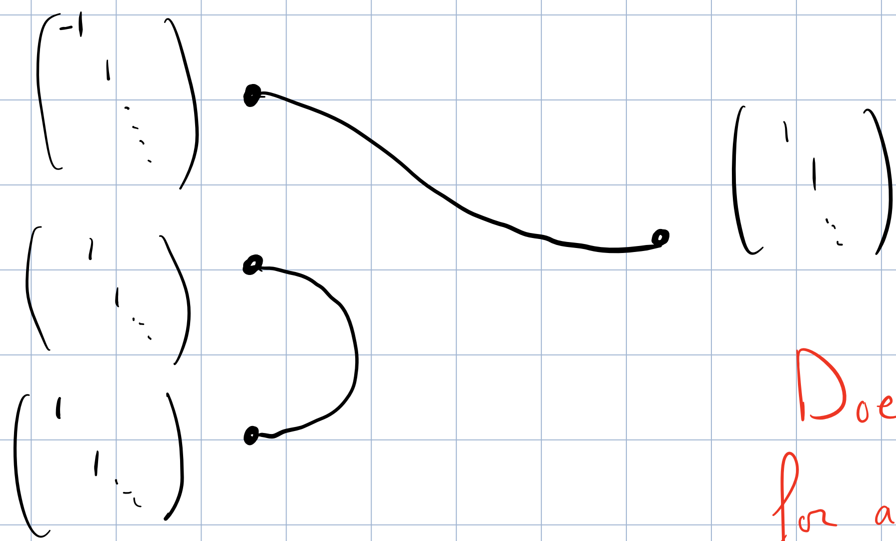
$$
\Omega_n^{fr}(\mathbb{R}^{n+k}) = \left\{ \begin{array}{l} X \subset \mathbb{R}^{n+k} \\ A : \mathbb{R}^{n+k} \longrightarrow \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k) \\ \operatorname{Ker} A_x = T_x X \end{array} \right. \Big/ \text{cobordism}
$$

The suspended version $Sf : \mathbb{H} \to \mathbb{H}$



Still works

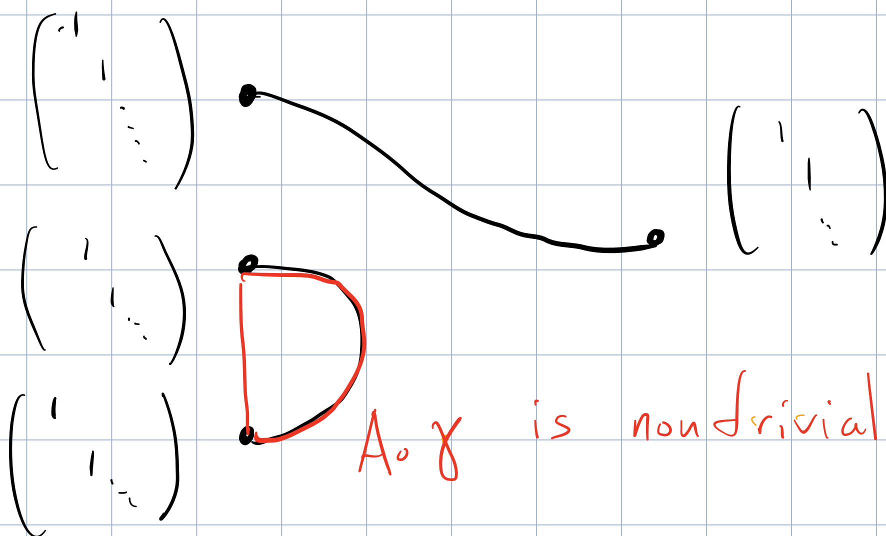
The suspended version $Sf : \mathbb{H} \to \mathbb{H}$



Does not work for a different reason as

$$\begin{pmatrix}-1&&\\&1&\\&&\ddots\end{pmatrix} \sim \begin{pmatrix}1&&\\&1&\\&&\ddots\end{pmatrix}$$

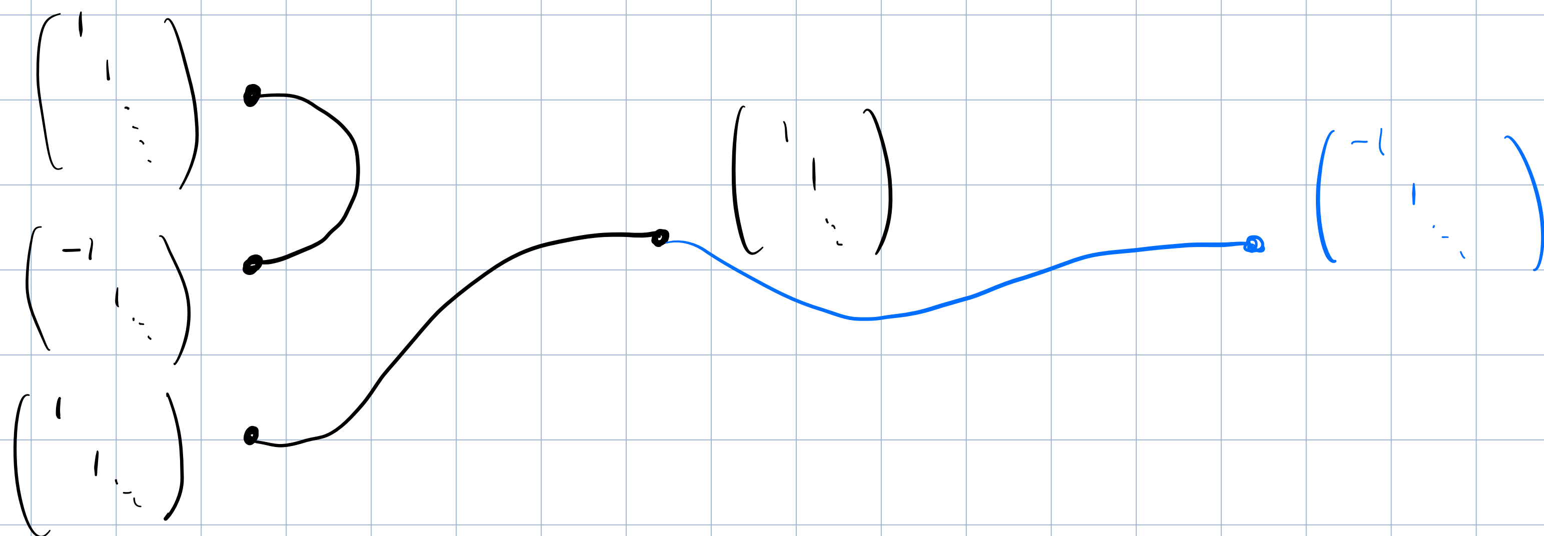
The suspended version $Sf : \mathbb{H} \to \mathbb{H}$



$A_0 \gamma$ is nontrivial loop in $\pi_1(\Phi(\mathbb{H}))$

can't fill in.

The suspended version $Sf : \mathbb{H} \to \mathbb{H}$



Thus degree k map equals degree $-k$ map

$$\mathcal{F}_0^{prop}[\mathbb{H}] = \mathbb{Z}/_{x \sim -x} \cong \mathbb{N}_0$$

The specialstructure of a suspended framed submanifold.

The special structure of a suspended framed submanifold.

- $X \subset \mathbb{R}^{n+k} \subset \mathbb{H}$ embedded in (first $n+k$ coordinates)

The special structure of a suspended framed submanifold.

- $X \subset \mathbb{R}^{n+k} \subset \mathbb{H}$
- framing $A : \mathbb{H} \longrightarrow \Phi_n(\mathbb{H})$ has the form

$$A = \begin{array}{c|cc} & \mathbb{R}^{n+k} & (\mathbb{R}^{n+k})^{\perp} \\ \hline \mathbb{R}^{k} & \tilde{A} & 0 \\ (\mathbb{R}^{k})^{\perp} & 0 & S_n \end{array}$$

The special structure of a suspended framed submanifold.

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$\tilde{A} \in \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^{k})$

$S_n(x_1, x_2, \dots) = (x_n, x_{n+1}, \dots)$

The special structure of a suspended framed submanifold.

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only depends on first $n+k$ coordinates

$$\tilde{A} \in \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^{k})$$

$$S_n(x_1, x_2, \dots) = (x_n, x_{n+1}, \dots)$$

left n-shift

The special structure of a suspended framed submanifold.

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$$A = \begin{array}{ccc} & \mathbb{R}^{n+k} & (\mathbb{R}^{n+k})^\perp \\ \begin{array}{c} \mathbb{R}^k \\ (\mathbb{R}^n)^\perp \end{array} & \multicolumn{2}{c}{\begin{pmatrix} \tilde{A} & 0 \\ 0 & S_n \end{pmatrix}} \end{array} \qquad \tilde{A} \in \mathrm{Hom}(\mathbb{R}^{n+k}, \mathbb{R}^k)$$

$$S_n(x_1, x_2, \dots) = (x_n, x_{n+1}, \dots)$$

Define Ω_s^{fr} as the set of these up to cobordism

The surjectivity proof

Thm (Toussaint-R)

$$S : \Pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}]$$

is surjective

The surjectivity proof

Thm (Toussaint-R)

$$S : \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}] \iff S : \Omega_S^{fr} \longrightarrow \Omega^{f,}(\mathbb{H})$$

is surjective $\qquad\qquad$ is surjective

The surjectivity proof

Thm (Toussaint-R)

$$S : \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}] \iff S : \Omega_S^{fr} \longrightarrow \Omega^{fr}(\mathbb{H})$$

is surjective $\qquad\qquad$ is surjective

Step 1 manifolds up to cobordism in $\mathbb{R}^{\ell}$ $\iff$ manifolds up to cobordism in $\mathbb{H}$.

ℓ suf

The surjectivity proof

Thm (Toussaint-R)

$S: \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}]$ is surjective $\Leftrightarrow$ $S: \Omega^{fr}_S \longrightarrow \Omega^{fr}(\mathbb{H})$ is surjective

Step 1 manifolds up to cobordism in $\mathbb{R}^{\ell}$, ℓ suf $\Leftrightarrow$ manifolds up to cobordism in $\mathbb{H}$.

This is easy by Whitney embedding theorem (same reason classifying spaces exist).

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Step 1 manifolds up to cobordism in $\mathbb{R}^\ell$, ℓ suf $\iff$ manifolds up to cobordism in $\mathbb{H}$.

This is easy by Whitney embedding theorem (same reason classifying spaces exist).

Step 2: Goal: Deform framing to be of finite type.

The surjectivity proof

Thm (Toussaint-R)

$$S: \pi_n^S \longrightarrow F_n^{prop}[\mathbb{H}] \iff S: \Omega^{fr}_S \longrightarrow \Omega^{fr}(\mathbb{H})$$

is surjective $\qquad\qquad\qquad\qquad$ is surjective

Step 1 manifolds up to cobordism in $\mathbb{R}^\ell$ (ℓ suf) $\iff$ manifolds up to cobordism in $\mathbb{H}$.

This is easy by Whitney embedding theorem (same reason classifying spaces exist).

Step 2: Goal: Deform framing to be of finite type.

$$A = \begin{array}{cc} & \begin{array}{cc} \mathbb{R}^{n+k} & (\mathbb{R}^{n+k})^\perp \end{array} \\ \begin{array}{c} \mathbb{R}^k \\ (\mathbb{R}^k)^\perp \end{array} & \begin{pmatrix} \tilde{A} & 0 \\ 0 & S_n \end{pmatrix} \end{array}$$

$W = I \times X$

$B = A_t$

while preserving $\operatorname{Ker} A|_X = TX$

Linear algebra

Let $A \in \Phi(\mathbb{H})$

$\exists$ $(n+k)$, $V = (\mathbb{R}^{n+k})^{\perp}$ s.t.

Goal

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$T: A(V) \longrightarrow (\mathbb{R}^{k})^{\perp}$

Linear algebra

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choose $\tau \in GL(\mathbb{H})$ s.t. $\tau : A(V)^\perp \longrightarrow \mathbb{R}^k$

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$$\tau A = \begin{array}{c c} & \begin{array}{cc} \mathbb{R}^{n+k} & V \end{array} \\ \begin{array}{c} \mathbb{R}^k \\ (\mathbb{R}^k)^\perp \end{array} & \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \end{array}$$

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$GL(\mathbb{H})$ is connected so there exists path τ_t from $id_{\mathbb{H}}$ to τ.

hence $A \sim \tau A$ while preserving $\operatorname{Ker} A$

Then

$$\tau A = \begin{array}{cc} & \begin{array}{cc} \mathbb{R}^{n+k} & V \end{array} \\ \begin{array}{c} \mathbb{R}^{k} \\ (\mathbb{R}^{k})^{\perp} \end{array} & \begin{pmatrix} B & 0 \\ C & D \end{pmatrix} \end{array}$$

$GL(V, (\mathbb{R}^k)^\perp)$ is connected (contractible)

hence there is a path D_t
$D \sim S_n$

$$\begin{pmatrix} B & 0 \\ (1-t)C & D_t \end{pmatrix} \longrightarrow \quad \tau A \sim \begin{pmatrix} B & 0 \\ 0 & S_n \end{pmatrix}$$

Crucial property: if $\ker A \subset \mathbb{R}^{n+k}$ then

$$\ker A_t = \ker A \qquad \forall t$$

Pointwise we can solve the problem.

Now: parametrically

K-theory to the rescue!

Let $A: M \longrightarrow \Phi(H)$ $\quad M$ reasonable.

$\exists\ (n+k)$, $V = (\mathbb{R}^{n+k})^{\perp}$ s.t.

- $\operatorname{Ker} A \cap V = 0$
- $\operatorname{coker}_V A := (A(V))^{\perp}$ is a k dim vector bundle.

(Atiyah-Jänich: $[\mathbb{R}^{n+k}] - [\operatorname{coker}_V A] \in K(M)$ is well defined
$\Phi(H)$ is a classifying space for K)

This thus works parametrically

$$A = \begin{array}{cc} & \begin{array}{cc} \mathbb{R}^{n+k} & V \end{array} \\ \begin{array}{c} A(v)^{\perp} \\ A(v) \end{array} & \begin{pmatrix} \tilde{B} & 0 \\ \tilde{C} & \tilde{D} \end{pmatrix} \end{array}$$

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hence we can find τ again.

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Main techical goal was that if $\operatorname{Ker} A \subseteq \mathbb{R}^{n+k}$

$\operatorname{Ker} A$ is not moved during the deformations

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This proves surjectivity.

Failure of injectivity

let (X_0, A_0), (X_1, A_1) finite dimensionally framed submflds

when is $S(X_0, A_0) = S(X_1, A_1)$?

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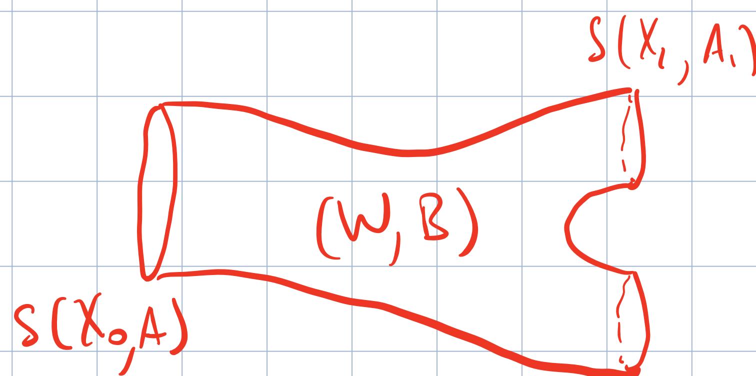

in $\mathbb{H}$

$$\mathcal{B} : \mathbb{H} \times I \longrightarrow \Phi_{n+1}(\mathbb{H})$$

- Find $n+k+1$ and $V = (\mathbb{R}^{n+k+1})^{\perp}$ as before

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$$\rightsquigarrow \quad \begin{array}{c} \widetilde{\mathrm{coker}_V B} \\ \downarrow \\ \mathbb{H}\times S^1 \end{array} = \mathrm{coker}_V B \big/ (x,0)\sim(x,1)$$

$$\mathrm{Vect}_{\mathbb{R}}(\mathbb{H} \times S^1) \cong \mathrm{Vect}_{\mathbb{R}}(S^1) \cong \mathbb{Z}/_{2\mathbb{Z}}.$$

$$\mathrm{Vect}_k(\mathbb{H} \times S^1) \cong \mathrm{Vect}_k(S^1) \cong \mathbb{Z}/2\mathbb{Z}$$

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$$\Updownarrow$$

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This describes failure

$\mathrm{coker}_V B$ not trivial

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injectivity

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Work in progress (speculative)

fin dim manifold.

Goal: Classify $\mathcal{F}_n^{prop}[M, \mathbb{H}]$ $\quad M \cong N \times \mathbb{H}$

What is the suspension operation now?

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Let $\begin{matrix} E \\ \downarrow \\ N \end{matrix}$ be a vector bundle of dim $n+k$ and

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a proper map

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a proper map

Then $$S_g : E \oplus \mathbb{H} \longrightarrow \mathbb{R}^k \oplus \mathbb{H}$$

$$S_g(x, y) = (g(x), y)$$

is proper Fredholm

But all Hilbert bundles are trivial. hence

$$E \oplus \mathbb{H} \cong N \times \mathbb{H} = M.$$

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<u>Conjecture</u>. $$\bigcup_{[E] \in K(N)} [E, \mathbb{R}^n]^{prop} \big/ \mathrm{Aut}(E) \cong \mathcal{F}^{prop}[M, \mathbb{H}]$$

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quotient of twisted stable cohomotopy of N

" Vector bundles F over $N \times S^1$ st $i_t : N \to N \times S^1$ $i_t^* F \cong E$.

Thank you for your attention

Arxiv: "Nonlinear proper Fredholm maps
and the stable homotopy groups of spheres"