Higher transgressions of the Pfaffian form

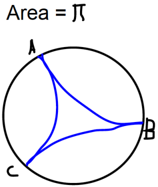
Sergiu Moroianu

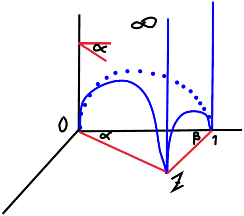
IMAR Bucharest

August 2023

Partly based on joint work with Daniel Cibotaru (Fortaleza)
Partly supported by UEFISCDI

Motivation: volume of ideal hyperbolic 4-simplex





- $\alpha + \beta + \gamma = \pi$ dihedral angles

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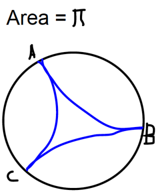

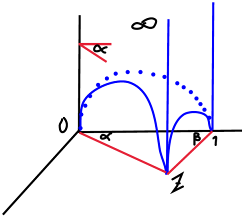

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- Volume = $Л(\alpha)+Л(\beta)+Л(\gamma)$, where Л=Lobachevski function

$$Л(x) = -\int_0^x \log|2\sin(t)|\mathrm{d}t$$

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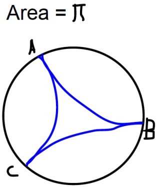

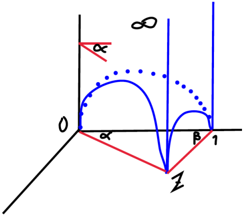

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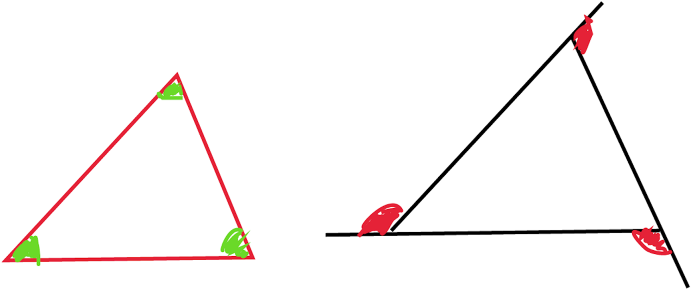
- How about dimension 4?

Euclidean (flat) geometry

- Sum of angles in a triangle = π



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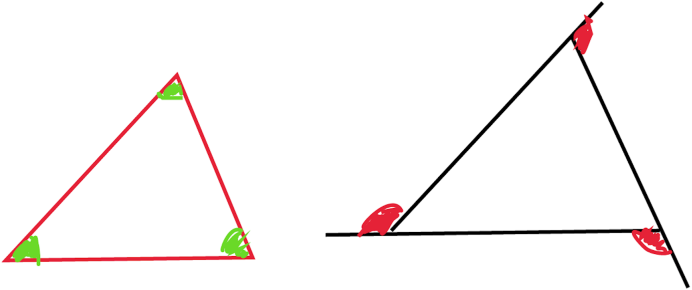
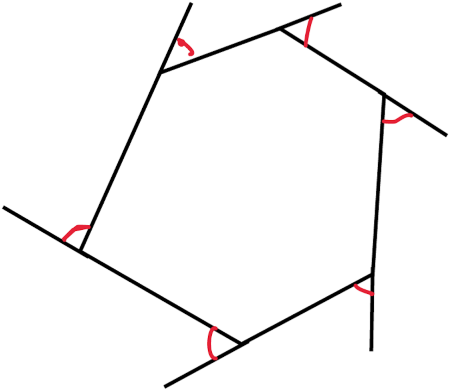


- Sum of exterior angles in every triangle = 2π

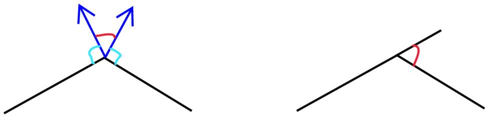
Sum of exterior angles

- Sum of exterior angles in every convex polygon $= 2\pi$



Exterior angles

- Two definitions of exterior angles



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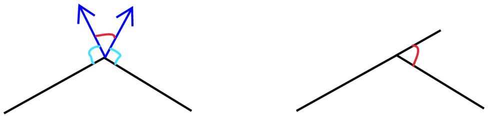

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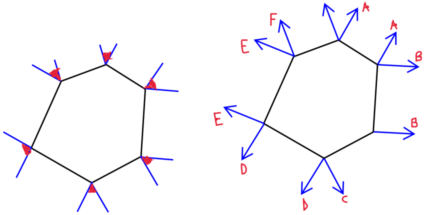

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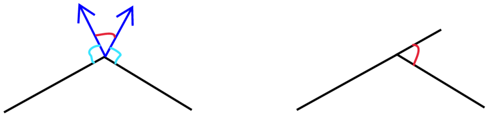

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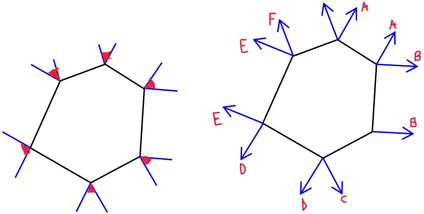
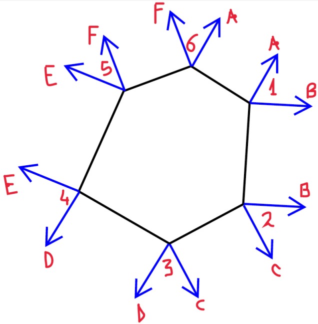
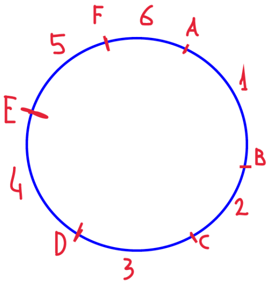


- Angle defect:= sum of ext angles -2π: In a planar polygon, the angle defect vanishes.

Plane Gauss map

- Gauss map: the unit exterior normal is constant along each side; finite number of points on the unit circle; each arc between them has measure equal to an exterior angle





Gauss theorem

- Surface $\Sigma \subset \mathbb{R}^3$, Gauss map $G : \Sigma \to \mathbb{S}^2$, $p \mapsto \nu_p$ = unit normal to Σ in p.

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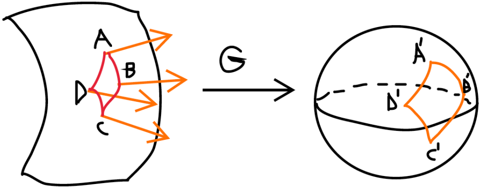

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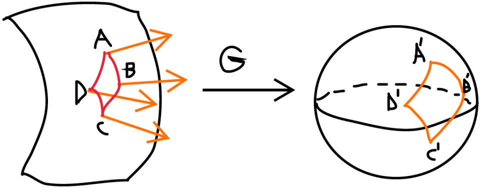

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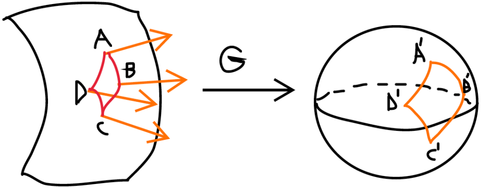

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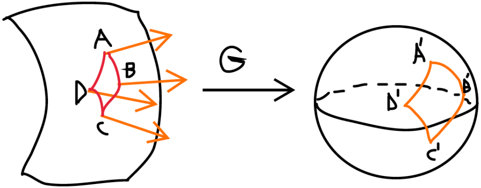

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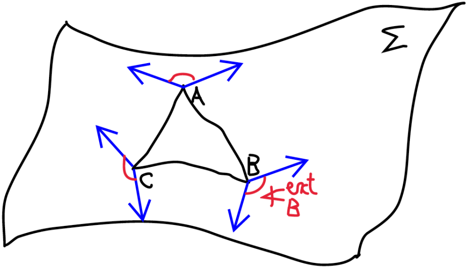
- Angle defect in a geodesic polygon on Σ:= sum of exterior angles -2π.

Gauss theorem

- Geodesic triangle ABC, exterior angles $\angle^{\text{ext}}_{A}$, etc.



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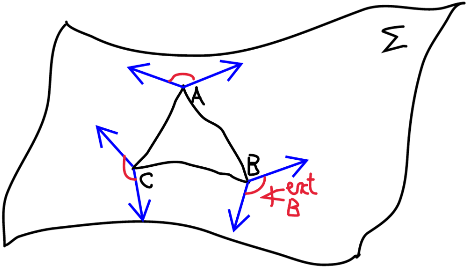

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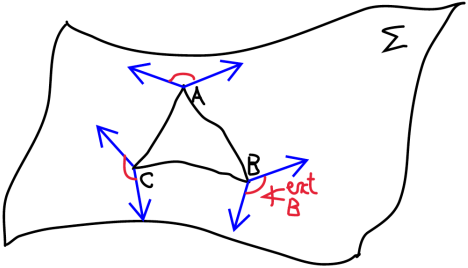

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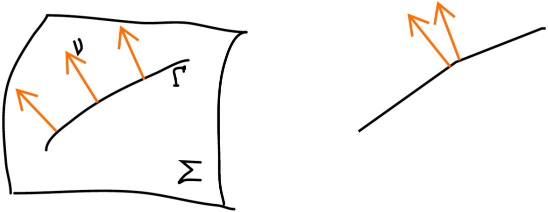
- Gauss 1827: in a geodesic triangle ABC,

$$\int_{ABC} \kappa \mathrm{d}g_\Sigma + \text{angle defect} = 0$$

Bonnet theorem

- Extrinsic curvature of a curve in a surface: infinitesimal version of the exterior angle.



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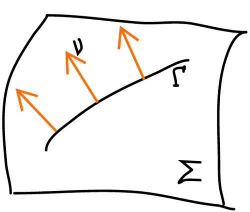

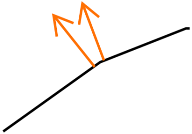

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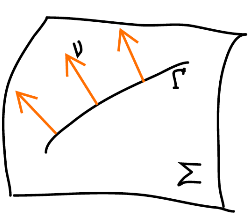

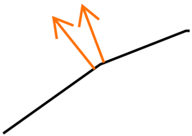

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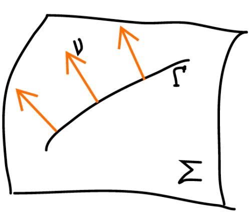

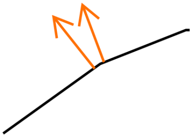

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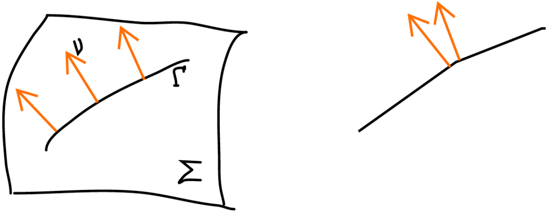

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- Bonnet 1848: $\Sigma \subset \mathbb{R}^3$ simply connected surface with polygonal boundary, not necessarily geodesic.

$$\int_\Sigma \kappa \mathrm{d}g_\Sigma + \text{angle defect} = \int_{\partial\Sigma} \text{geodesic curvature}$$

Gauss-Bonnet theorem in popular culture

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- **On doit pouvoir se rendre compte de la direction et de la distance en faisant bien attention aux détours.** *Chacun sait qu'une voiture, en tournant, fait éprouver une sensation au voyageur, surtout si le coude du voyageur est en communication avec la paroi. Deux minutes ne s'étaient pas écoulées que Vincent eut la preuve matérielle de ce fait. On tourna à droite et il en eut complétement conscience.*

Le cocher fouetta ses chevaux qui partirent au grand trot.

Vincent se disait :

— On doit pouvoir se rendre compte de la direction et de la distance en faisant bien attention aux détours.

Et il tint son esprit en arrêt.

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—Est-ce le pont des Saints-Pères ou le pont Royal? demanda-t-il.

— Voilà! fit le colonel en riant bonnement, tu calcules déjà comme un malheureux! Je parie cinquante centimes avec toi que, dans une demi-heure.

- Everyone knows that a car, when turning, makes the traveler experience a sensation, especially if their elbow is in contact with the wall. **One can calculate the direction and the distance by paying close attention to the detours.**
  Vincent had material proof of this fact. They turned right and he was fully aware of it.
  "Is it the Pont des Saints-Pères or the Pont Royal?"he asked.
  - So ! said the colonel with a good laugh, **you're already calculating like a wretch!**

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$$\begin{aligned}\int_\Sigma \kappa \mathrm{d}g_\Sigma &= 2\pi\chi(\Sigma)\\ &= 4\pi \times \frac{\chi(\Sigma)}{2}\end{aligned}$$

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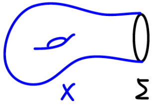

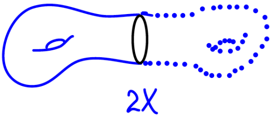

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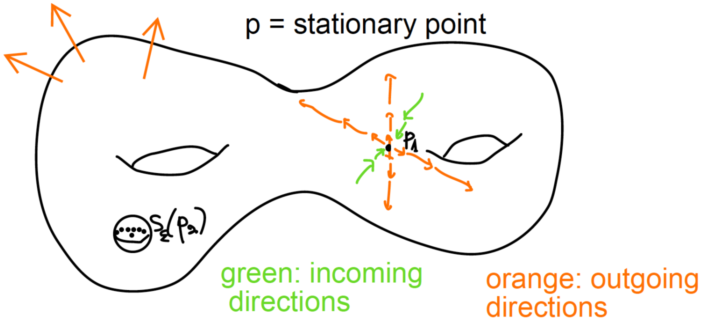
- Inclusion/exclusion principle: $\chi(2X) = 2\chi(X) - \chi(\Sigma)$.

Hypersurfaces in $\mathbb{R}^{2n+1}$

- $\Sigma \subset \mathbb{R}^{2n+1}$ hypersurface, X = the interior of Σ, $\partial X = \Sigma$.
  Heinz Hopf 1925: Extend the unit normal G to a vector field ν on X, with isolated zeros of the form $\nu = \sum_{j=1}^{2n+1} \pm x_j \partial_{x_j}$



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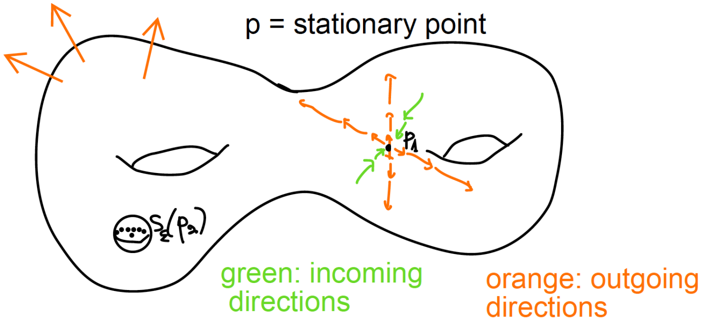

- Index of ν at p_1 in the picture is -1. Index $=$ parity of the number of incoming directions (number of $-$ signs).

Hopf's proof

- Define a unit vector field on $\Sigma \setminus Z(\nu) = \{p_1, p_2, \ldots\}$ = the (finite) zero set of ν by $V = \nu/|\nu|$. Equivalently, the function $V : \Sigma \setminus Z(\nu) \to \mathbb{S}^{2n}$ extends the Gauss map G.

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- $V^* \mathrm{d}g_{\mathbb{S}^{2n}}$ is a closed $2n$-form since $\mathrm{d}g_{\mathbb{S}^{2n}}$ is closed
- Stokes formula on $\Sigma \setminus B_\epsilon(p_j)$:

$$\begin{aligned}\int_\Sigma V^* \mathrm{d}g_{\mathbb{S}^{2n}} &= \sum \int_{\mathbb{S}_\epsilon(p_j)} V^* \mathrm{d}g_{\mathbb{S}^{2n}} \\ &= \mathrm{Vol}(\mathbb{S}^{2n}) \sum \mathrm{index}(p_j) = \mathrm{Vol}(\mathbb{S}^{2n}) \chi(X)\end{aligned}$$

($\chi(X) = \deg(G) = \sum \mathrm{index}(p_j)$, independent of the choice of ν)

The Pffafian form

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- Case $n = 1$: $\mathrm{Pf}(R^\Sigma) = G^*\mathrm{d}g_{\mathbb{S}^{2n}} = \kappa \mathrm{d}g_\Sigma$ where $\kappa = R^\Sigma_{1221}$ in any orthonormal frame e_1, e_2.

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- Jacobian of the Gauss map = the shape operator $\leftrightarrow$ the second fundamental form. So $G^*\mathrm{d}g_{\mathbb{S}^{2n}} = \det(\mathrm{I\!I})\mathrm{d}g_\Sigma$. Gauss equation for a hypersurface in $\mathbb{R}^{2n+1}$ with second fundamental form $\mathrm{I\!I}$:

$$R^\Sigma(X, Y, Z, T) = \mathrm{I\!I}(X, T)\mathrm{I\!I}(Y, Z) - \mathrm{I\!I}(X, Z)\mathrm{I\!I}(Y, T)$$

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$$R^\Sigma(X, Y, Z, T) = \mathrm{I\!I}(X, T)\mathrm{I\!I}(Y, Z) - \mathrm{I\!I}(X, Z)\mathrm{I\!I}(Y, T)$$

- Equivalently: $R^\Sigma = -\frac{1}{2}\mathrm{I\!I}^2 = -\frac{1}{2}(\mathrm{I\!I}_{ik} e_i \otimes e_k)^2 = -\frac{1}{2}\mathrm{I\!I}_{ik}\mathrm{I\!I}_{jl} e_i \wedge e_j \otimes e_k \wedge e_l$. How to get $\det(\mathrm{I\!I})$ from $\mathrm{I\!I}^2$?

Skew-symmetric matrices

- $A \in \mathrm{End}^-(\mathbb{R}^{2n})$ skew-symmetric, real, even dimension. Then

$$\det(A) \geq 0...$$

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- Example: $n = 2$, $A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & f \\ -b & -d & 0 & g \\ -c & -f & -g & 0 \end{bmatrix}$, $\mathrm{Pf}(A) = ag - bf + cd$.

The Berezin integral and the Pfaffian

- E^{2n} oriented even-dimensional real vector space endowed with a non-degenerate symmetric bilinear form. The *Berezin integral* $\mathscr{B} : \Lambda^{2n}E \to \mathbb{R}$ is the map of "erasing"the volume form:

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- If Q is a (k, h) form, then $d^\nabla Q$ is a $(k + 1, h)$ form.

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$$\mathrm{Pf}(\nabla) = \frac{1}{n!}\mathscr{B}_h(R^n) \in \Omega^{2n}(M).$$

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- (Last two properties not true for characteristic classes $\mathrm{Tr}(R^k)$.)

The generalized Gauss-Bonnet theorem

Take $E = TM$, ∇ the Levi-Civita connection of g and R its curvature.

Theorem

(M, g) compact Riemannian manifold of dimension $2n$.

$$(2\pi)^n \chi(M) = \int_M \mathrm{Pf}(R).$$

Gauss (1821), Bonnet (1848), Walther von Dyck(1888), Hopf (1925), Allendoerfer (1940), Fenchel (unpublished), Allendoerfer&Weil (1943), Chern (1944).

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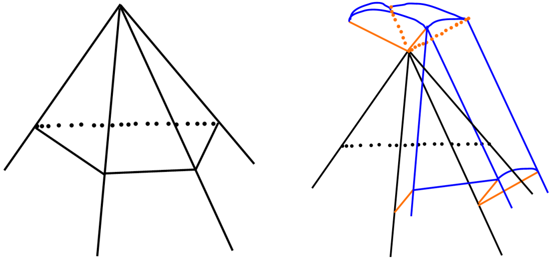

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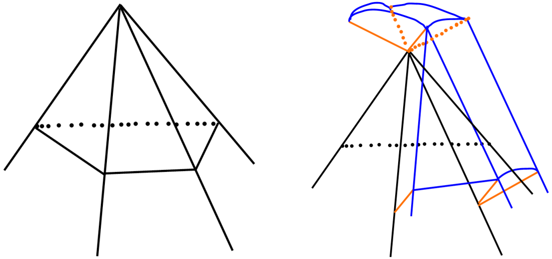

- The *outer cone* of a face Y: the set of vectors (orthogonal to Y), making an angle greater than π with every interior-pointing vector. The *outer angle*: set of unit vectors in the outer cone.

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- $R \in \Omega^{2,2}(M, E)$
- $c(n, k, l) := \frac{2^k k!}{(n-1-k)!(2k+1-l)!} \in \mathbb{Q}$

Definition

The $(l+1)^{\text{th}}$ transgression of the Pfaffian, $\mathscr{T}_V^{(l+1)} \in \Omega^{2n-l-1}(M)$

$$\mathscr{T}_V^{(l+1)} = \sum_{l \le 2k+1 \le 2n-1} \frac{c(n,k,l)}{l!} \int_X \mathscr{B}\left[V(d^X V)^l (\nabla V)^{2k+1-l} R^{n-1-k}\right]$$

For $l = 0$: Chern (1944).

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- $E \to M$ vector bundle, metric h and compatible connection ∇
- V = family of unit-length sections in $E \to M$ indexed by an oriented l-dimensional polytope X.

Theorem

$$d\mathscr{T}_X^{(l+1)} = \begin{cases} -\mathscr{T}_{\partial X}^{(l)} & \text{for } l = \dim X \geq 1, \\ -\mathrm{Pf}(\nabla) & \text{for } l = \dim(X) = 0 \end{cases}$$

$\mathscr{T}_{\partial X}^{(l)} \in \Omega^{2n-l}(M)$ is the sum of the transgression forms corresponding to the oriented hyperfaces of X.

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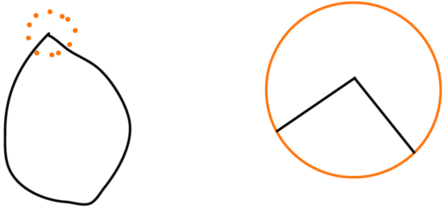
- The computation breaks down for characteristic forms of the type $\mathrm{Tr}(R^k)$ where $R \in \Omega^2(M, \mathrm{End}(E))$ (Chern, Pontriagin classes).

Polyhedral manifolds

- Compact, separated topological space M, locally homeo to some polyhedra of fixed dimension, with smooth atlas.



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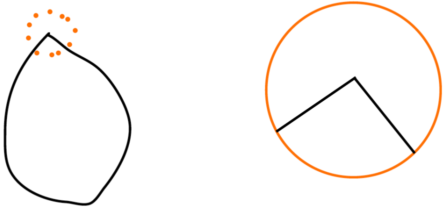

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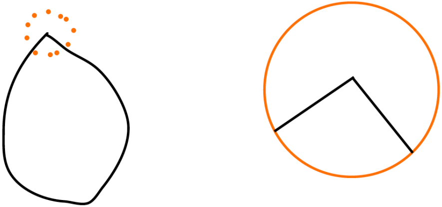
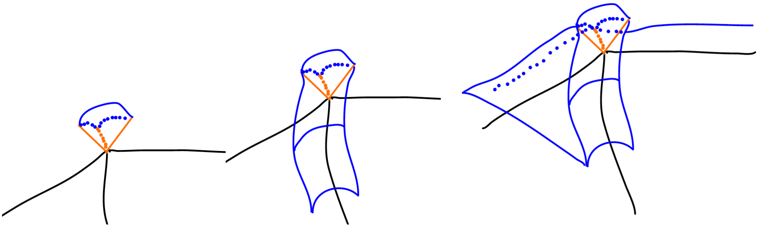


- Riemannian metric: locally it extends from the polyhedron to the ambient euclidean space
- Outer cone bundle of a boundary face Y: family of cones in TM consisting of those vectors making an angle $\geq \pi/2$ with every interior vector.

Outer angle

- Outer angle of a face Y: unit vectors in the outer cone of Y



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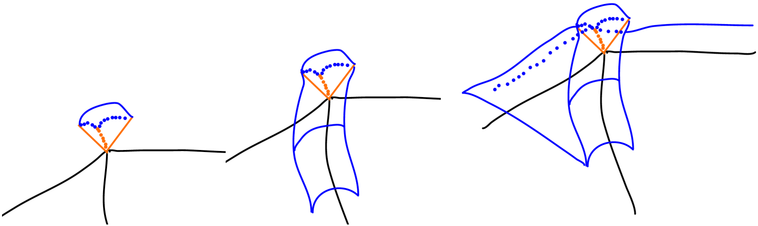

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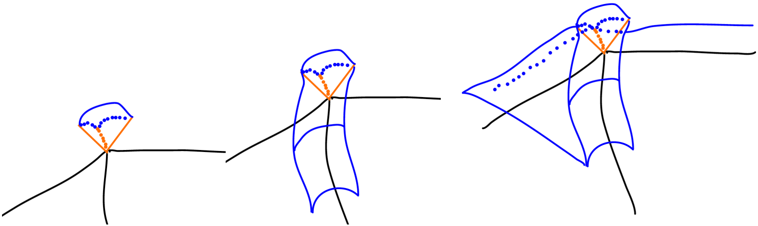

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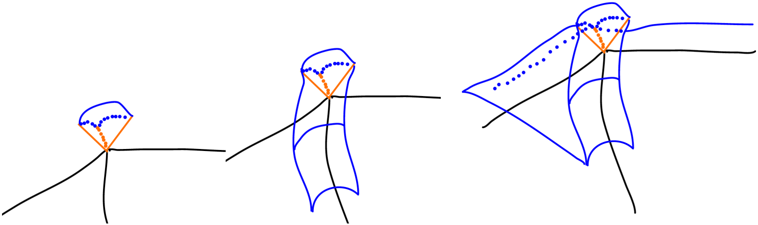

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- $\partial_{n-2} M^{\mathrm{out}} = 0$ (Wintgen's normal cycle)

Gauss-Bonnet, even dimension

Theorem

Let M^{2n} be a compact Riemannian polyhedral manifold. Then

$$(2\pi)^n\chi(M) - \int_M \mathrm{Pf}(R) = \sum_{l=1}^{2n} \sum_{k=\lceil \frac{l}{2} \rceil}^{n} \frac{(-1)^l 2^{k-1}(k-1)!}{(n-k)!(2k-l)!}$$

$$\sum_{Y\in\mathscr{F}^{(l)}(M)} \int_{S^{\mathrm{out}}Y} \mathscr{B}_Y\left[(R^Y - \tfrac{1}{2}g(A,A))^{n-k}(A^*)^{2k-l}\right]|dg|$$

- $A \in \Lambda^1 Y \otimes \Lambda^1 Y \otimes N_Y$ is the second fundamental form of $Y \hookrightarrow M$.

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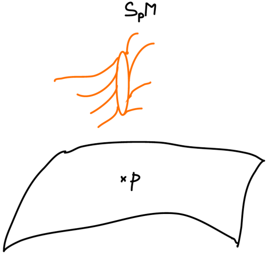

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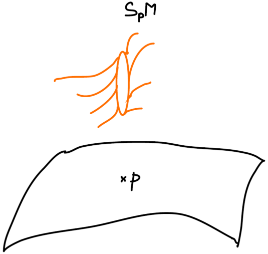
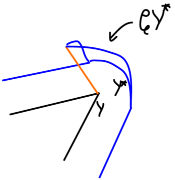


- The Pfaffian is exact along this graph, with explicit primitive $\mathscr{T}^1(V_0)$. By Stokes, its integral localizes on those spheres (yielding $\chi(M)$), and on the boundary hyperfaces, $\int_Y \mathscr{T}^1(V_0)$.

- Y^* outer cone of face Y; $\mathscr{C}Y^*$ spherical cone over Y^*



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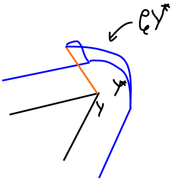

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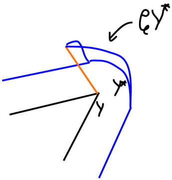

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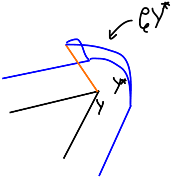

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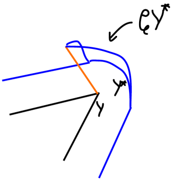

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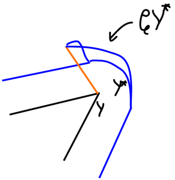

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- $\int_Z \mathscr{T}^2(\partial\mathscr{C}Z^*) - \mathscr{T}^2(Z^*)$ for all Z of codim 2, etc.
- For a face Y of codim k, can compute $\int_{\partial Y} \mathscr{T}^k(Y^*)$, indep of V_0

Gauss-Bonnet, odd dimension

Theorem

Let (N, g) be a compact Riemannian polyhedral manifold of odd dimension $2n-1$. Then

$$(2\pi)^n \chi(N) = \sum_{l=1}^{2n-1} \sum_{k=\lceil \frac{l-1}{2} \rceil}^{n-1} \frac{(-1)^{l-1}\pi(2k-1)!!}{(n-1-k)!(2k+1-l)!} \sum_{Y \in \mathcal{F}^{(l)}(N)} \int_{S^{\mathrm{out}} Y} \mathscr{B}_Y \left[(R^Y - \tfrac{1}{2} g(A, A))^{n-1-k} (A^*)^{2k+1-l} \right] |dg|$$

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- Proof: apply the even-dimensional formula to $M = N \times [0,1]$

Flat polyhedral manifolds

- M^k = flat compact polyhedral manifold with totally geodesic faces.

$$\mathrm{vol}(S^{k-1})\chi(M) = \sum_{Y=\text{vertex}} \measuredangle^{\mathrm{out}} Y.$$

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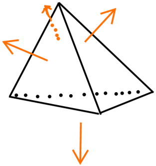
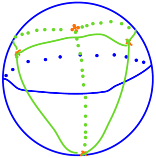
- Clear for $M^k \subset \mathbb{R}^k$ polytope.





Theorem

Let M be a d-dimensional compact polyhedral manifold of constant sectional curvature $\mathfrak{k}$, with totally geodesic faces. Then

$$\frac{\chi(M)}{2} = \sum_{j\geq 0} \sum_{Y\in\mathscr{F}^{(d-2j)}} \mathfrak{k}^j \frac{\mathrm{vol}_{2j}(Y)}{\mathrm{vol}(S^{2j})} \frac{\angle^{\mathrm{out}} Y}{\mathrm{vol}(S^{d-2j-1})}$$

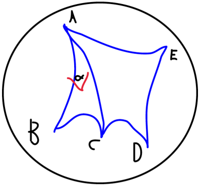
where $\mathscr{F}^{(d-2j)}$ is the set of faces of M of dimension $2j$, S^k is the standard unit sphere in $\mathbb{R}^{k+1}$, and $\angle^{\mathrm{out}} Y$ is the measure of the outer angle at the face Y.

Ideal hyperbolic 4-simplex

- M is the convex hull in $\mathbb{H}^4$ of 5 points in $\partial\mathbb{H}^4$.

$$\mathrm{vol}(M) = -2\pi^2 + \frac{\pi}{3} \sum_{Y \in \mathcal{F}^{(2)}(M)} \angle^{\mathrm{out}}(Y)$$

$\angle^{\mathrm{out}}(Y)$ is the outer dihedral angle of the ideal triangle Y in M.



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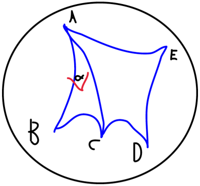

- M is an ideal polyhedron in $\mathbb{H}^4$.

$$\mathrm{vol}(M) = \frac{2\pi^2(2 - n_0(M))}{3} + \frac{\pi}{3} \sum_{Y = \text{2-face}} (n_0(Y) - 2) \angle^{\mathrm{out}}(Y)$$

Gauss-Bonnet for edge metrics

(N, h) oriented $2k-1$-dimensional Riemannian manifold. Define

$$\mathrm{Pf}^{\mathrm{odd}}(h) := \sum_{j=0}^{k-1} (-1)^{k+j} (2k-2j-3)!! \mathscr{B}_h \left(\frac{(R^h)^j \wedge h^{2k-1-2j}}{j!(2k-2j-1)!} \right) \in \Lambda^{2k-1}(N).$$

(M^{2k}, g) manifold with edge singularities: $\pi : \partial M \to B$ fibration, r boundary-defining function for the boundary; In a product collar neighborhood of $\partial M = \{r = 0\}$, the metric is

$$g = dr^2 \oplus g(r), \qquad g(r) = r^2 g^V \oplus \pi^* g^B$$

where g^B = metric on B, g^V = Riemannian metric on the fibers.

Theorem

If $\dim(B)$ *is even,*

$$(2\pi)^k \chi(M) = \int_M \mathrm{Pf}^g - \int_B \left(\mathrm{Pf}(g^B) \int_{\partial M/B} \mathrm{Pf}^{\mathrm{odd}}(g^V) \right).$$

References

Higher transgressions of the Pfaffian (2022)
Odd Pfaffian forms (with Daniel Cibotaru), (2021)