Topological properties of (tall) monotone complexity one spaces

Silvia Sabatini
University of Cologne

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Based on:

- "*On topological properties of positive complexity one spaces*", S. and Sepe, *Transformation Groups* **9** (2020).
- "*Tall and monotone complexity one spaces of dimension six*", Charton, PhD Thesis, Cologne 2021.
- "*Compact monotone tall complexity one T-spaces*" Charton, S. and Sepe, arXiv:2307.04198 [math.SG].

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Henceforth consider *positive monotone symplectic manifolds*

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(Example: $\dim_{\mathbb{C}}(Y) = 1 \implies Td(Y) = \frac{c_1}{2}[Y]$,

$\dim_{\mathbb{C}}(Y) = 2 \implies Td(Y) = \frac{c_1^2 + c_2}{12}[Y]$,

$\dim_{\mathbb{C}}(Y) = 3 \implies Td(Y) = \frac{c_1 c_2}{24}[Y]$)

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What if one assumes that (M,ω) has symmetries?

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Assume $T \circlearrowright (M, \omega)$

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Assume $T \circlearrowright (M,\omega)$ is *Hamiltonian*:

$\exists\ \psi\colon (M,\omega) \to Lie(T)^*$ (*moment map*) s.t.

- ψ is T-invariant
- $\forall \xi \in Lie(T)$

$$d\langle \psi, \xi\rangle = -\iota_{X_\xi}\omega$$

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such that

$$\psi_2\circ\Psi = a\circ\psi_1$$

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- Finitely many examples in each dimension? (Modulo equivalence)

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Remark: GKM action $\implies$ the torus acting is T^2

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If $b_2 = 1$ then there are finitely many possibilities for the cohomology rings and Chern classes that can arise, and they all come from Fano varieties.

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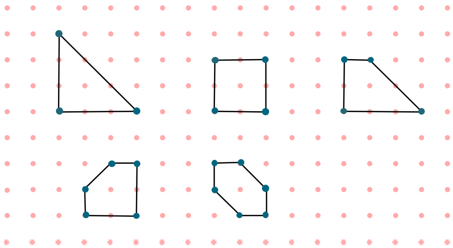
E.g. $\dim(M) = 4$, modulo $GL(2, \mathbb{Z})$, $\psi(M) =$



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Let (M, ω, ψ) be a compact Hamiltonian T-space. For any $\alpha \in \psi(M)$, $\pi_1(M) \simeq \pi_1(M_\alpha)$, where $M_\alpha = \psi^{-1}(\alpha)/T$ is the reduced space at α.

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To prove (b): prove that $\psi^{-1}(v)$ is a sphere.

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- $N_\Sigma = N_1 \oplus \cdots \oplus N_{n-1}$
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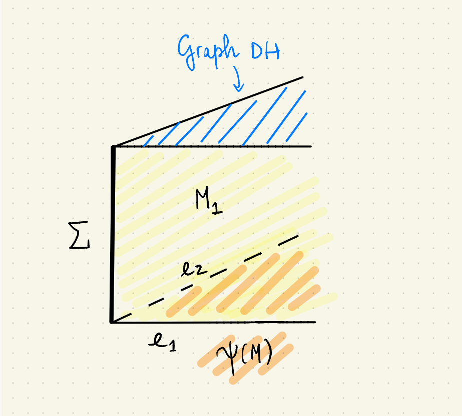
- Normal bundle to Σ in M_i is N_i



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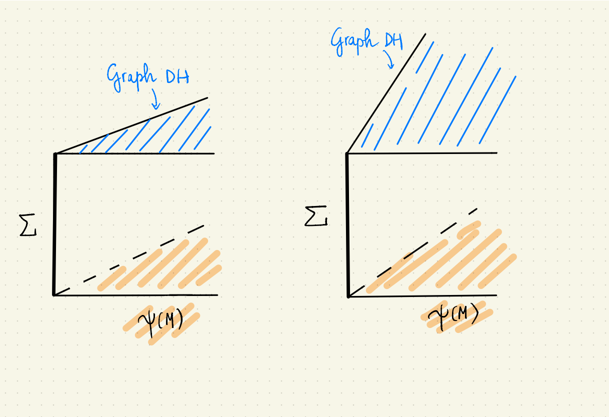

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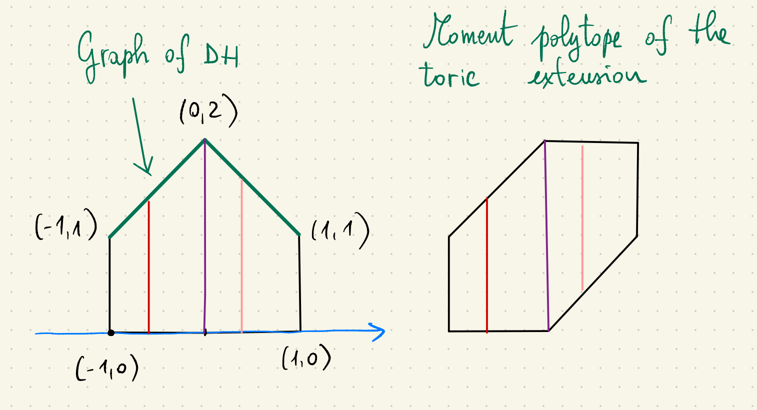
- Prove that each of those "comes" from a toric one

Example:


Graph of DH
(0,2)
(-1,1)
(1,1)
(-1,0)
(1,0)
Moment polytope of the toric extension

THANK YOU!